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# TWO RESULTS ABOUT FIXED POINT OF MULTIFUNCTIONS

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ABSTRACT. We shall establish two fixed point theorems for contractive multifunctions in a non-empty and closed subset of a complete metric space with certain assumptions.

## 1. Introduction

In spite of its simplicity, the Banach fixed point theorem still seems to be the most important result in metric fixed point theory. As we know, there are many works in this field. Also, there are some works about fixed point of multifunctions (see [1], [2], [3], [5], [7] and [9]). In fact, fixed point theory of multifunctions is generalization of the metric fixed point theory in a certain sense. Here, we establish two fixed point theorems for certain multifunctions of contractive type by using some ideas of the work of Reem, Reich and Zaslavski ([8]).

## 2. Main results

Let $(X, d)$ be a metric space. Throughout this section, we suppose that $\mathcal{H}(X)$ is the set of all compact subsets of $(X, d)$ and $d_h$ stands for the Hausdorff metric with respect to $d$.

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**Theorem 2.1.** *Let $K$ be a non-empty closed subset of a complete metric space$(X,d)$. Assume that $T : K \to \mathcal{H}(X)$ satisfies*

$$d_h(Tx,Ty) \leq cd(x,y)$$

*for all $x,y \in X$, where $c \in [0,1)$ is a contractive constant. Let $K_0$ be a bounded subset of $K$ and let $\{x_n\}_{n\geq 1}$ be a sequence in $K_0$ such that $\bigcup_{i=1}^{n} T^i x_n \subseteq K$, for all $n \geq 1$. Then, $T$ has a fixed point in $K$.*

**Proof.** First, we claim that

$$d_h(T^{i+1}x, T^{i+1}y) \leq cd_h(T^i x, T^i y), \tag{2.1}$$

for all $i \geq 1$ and all $x,y \in K$ with $T^i x, T^i y \subseteq K$. This claim is obtained from a well-known remark of Nadler [6], which states

$$d_h(T(A),T(B)) \leq d_h(A,B),$$

for all compact subsets $A$ and $B$ of a metric space. Since $K_0$ is bounded, there exist $\theta \in K$ and $c_0 > 0$ such that

$$d(\theta, z) \leq c_0, \tag{2.2}$$

for all $z \in K_0$. Now, we continue the proof in several steps.

**Step I.** For each $\varepsilon > 0$, there exists $n_0 \geq 1$ such that

$$d_h(T^i x_n, T^{i+1}x_n) \leq \varepsilon, \tag{2.3}$$

for all $n > n_0$ and $n_0 \leq i < n$.

**Proof of Step I.** If (2.3) does not hold, then for each $m \geq 1$ there exist $n_m$ and $i_m$ such that $m \leq i_m < n_m$ and

$$d_h(T^{i_m} x_{n_m}, T^{i_m+1} x_{n_m}) > \varepsilon, \tag{2.4}$$

for some $\varepsilon > 0$. Choose a natural number $m$ such that

$$m > \frac{2c_0 + d_h(\{\theta\}, T\theta)}{(1-c)\varepsilon} \ .$$

Since $c < 1$, by (2.1) and (2.4) we have

$$d_h(T^i x_{n_m}, T^{i+1} x_{n_m}) > \varepsilon,$$

for all $i = 1, 2, \cdots, i_m$. Since

$$d_h(T^{i+2}x_{n_m}, T^{i+1}x_{n_m}) \leq cd_h(T^{i+1}x_{n_m}, T^i x_{n_m}),$$

for all $i = 1, 2, \cdots, i_m - 1$, by (2.4) we have

$$d_h(T^{i+2}x_{n_m}, T^{i+1}x_{n_m}) - d_h(T^{i+1}x_{n_m}, T^i x_{n_m})$$
$$\leq (c-1)d_h(T^{i+1}x_{n_m}, T^i x_{n_m}) < -(1-c)\varepsilon.$$

On the other hand,

$$\varepsilon < d_h(T^2x_{n_m}, Tx_{n_m}) \leq c d_h(\{x_{n_m}\}, Tx_{n_m}) \leq d_h(\{x_{n_m}\}, Tx_{n_m}).$$

Hence,

$$d_h(T^2x_{n_m}, Tx_{n_m}) - d_h(\{x_{n_m}\}, Tx_{n_m})$$
$$\leq (c-1)d_h(\{x_{n_m}\}, Tx_{n_m}) < -(1-c)\varepsilon.$$

Thus,

$$-d_h(\{x_{n_m}\}, Tx_{n_m}) \leq d_h(T^{i_m+1}x_{n_m}, T^{i_m}x_{n_m}) - d_h(\{x_{n_m}\}, Tx_{n_m})$$
$$\leq \sum_{i=1}^{i_m-1}[d_h(T^{i+2}x_{n_m}, T^{i+1}x_{n_m}) - d_h(T^{i+1}x_{n_m}, T^i x_{n_m})]$$
$$+[d_h(T^2x_{n_m}, Tx_{n_m}) - d_h(\{x_{n_m}\}, Tx_{n_m})]$$
$$< -(c-1)\varepsilon i_m \leq -m(1-c)\varepsilon. \tag{2.5}$$

Since $d_h$ is a metric on $\mathcal{H}(X)$, from (2.2) and (2.5) we have

$$m(1-c)\varepsilon \leq d_h(\{x_{n_m}\}, Tx_{n_m})$$
$$\leq d_h(\{x_{n_m}\}, \{\theta\}) + d_h(\{\theta\}, T\theta) + d_h(T\theta, Tx_{n_m})$$
$$\leq d(x_{n_m}, \theta) + d_h(\{\theta\}, T\theta) + cd(\theta, x_{n_m}) \leq 2c_0 + d_h(\{\theta\}, T\theta),$$

which is a contradiction. Therefore, (2.3) holds.

**Step II.** For each $\delta > 0$, there exists $n_0 \geq 1$ such that

$$d_h(T^i x_n, T^j x_n) \leq \delta, \tag{2.6}$$

for all $n > n_0$ and $n_0 \leq i, j < n$.

**Proof of Step II.** Let $\varepsilon < \frac{1}{4}\delta(1-c)$ and choose $n_0 \geq 1$ such that (2.3) holds, for all $n > n_0$ and $n_0 \leq i < n$. Let $i$, $j$ and $n$ be natural numbers so that $n_0 \leq i, j < n$. We claim that (2.6) holds. If $d_h(T^i x_n, T^j x_n) > \delta$, then

$$d_h(T^i x_n, T^j x_n)$$
$$\leq d_h(T^i x_n, T^{i+1}x_n) + d_h(T^{i+1}x_n, T^{j+1}x_n) + d_h(T^{j+1}x_n, T^j x_n)$$
$$\leq 2\varepsilon + d_h(T^{i+1}x_n, T^{j+1}x_n) \leq 2\varepsilon + cd_h(T^i x_n, T^j x_n).$$

Hence,

$$\delta < d_h(T^i x_n, T^j x_n) \leq \frac{2\varepsilon}{1-c},$$

which is a contradiction.

**Step III.** For each $\varepsilon > 0$, there exists $n_0 \geq 1$ such that

$$d_h(T^{n_0} x_{n_1}, T^{n_0} x_{n_2}) \leq \varepsilon, \tag{2.7}$$

for all $n_1, n_2 > n_0$.

**Proof of Step III.** Choose a natural number $m$ such that $m > \frac{4c_0}{\varepsilon(1-c)}$ and let $n_1, n_2 > m$. We claim that

$$d_h(T^m x_{n_1}, T^m x_{n_2}) \leq \varepsilon. \tag{2.8}$$

If (2.8) does not hold, then $d_h(T^i x_{n_1}, T^i x_{n_2}) > \varepsilon$, for all $i = 1, 2, \cdots, m$, because $c < 1$. Note that

$$\varepsilon < d_h(Tx_{n_1}, Tx_{n_2}) \leq cd(x_{n_1}, x_{n_2}) \leq d(x_{n_1}, x_{n_2}).$$

So, $-d(x_{n_1}, x_{n_2}) < -\varepsilon$. Since

$$d_h(T^{i+1} x_{n_1}, T^{i+1} x_{n_2}) \leq cd_h(T^i x_{n_1}, T^i x_{n_2}),$$

for all $i = 1, 2, \cdots, m-1$,

$$\begin{aligned} &d_h(T^{i+1} x_{n_1}, T^{i+1} x_{n_2}) - d_h(T^i x_{n_1}, T^i x_{n_2}) \\ &\leq (c-1) d_h(T^i x_{n_1}, T^i x_{n_2}) < -(1-c)\varepsilon, \end{aligned}$$

for all $i = 1, 2, \cdots, m-1$. This implies that

$$\begin{aligned} &-d(x_{n_1}, x_{n_2}) \leq d_h(T^m x_{n_1}, T^m x_{n_2}) - d(x_{n_1}, x_{n_2}) \\ &\leq \sum_{i=1}^{m-1} [d_h(T^{i+1} x_{n_1}, T^{i+1} x_{n_2}) - d_h(T^i x_{n_1}, T^i x_{n_2})] \\ &+[d_h(Tx_{n_1}, Tx_{n_2}) - d(x_{n_1}, x_{n_2})] \leq -m(1-c)\varepsilon. \end{aligned}$$

Hence,

$$m(1-c)\varepsilon \leq d(x_{n_1}, x_{n_2}) \leq d(x_{n_1}, \theta) + d(\theta, x_{n_2}) \leq 2c_0,$$

and so,

$$m \leq \frac{2c_0}{\varepsilon(1-c)}.$$

This contradiction shows that (2.8) holds.

**Step IV.** For each $\varepsilon > 0$, there exists $m(\varepsilon) \geq 1$ such that

$$d_h(T^i x_{n_1}, T^j x_{n_2}) \leq \varepsilon,$$

for all $n_1, n_2 > m(\varepsilon)$ and all natural numbers $i \in [m(\varepsilon), n_1)$ and $j \in [m(\varepsilon), n_2)$.

**Proof of Step IV.** Let $\varepsilon > 0$ be given. By using (2.7), choose $m_1 \geq 1$ such that

$$d_h(T^{m_1} x_{n_1}, T^{m_1} x_{n_2}) \leq \frac{\varepsilon}{4},$$

for all $n_1, n_2 > m_1$. Also, by using (2.6), choose $m_2 \geq 1$ such that

$$d_h(T^i x_n, T^j x_n) \leq \frac{\varepsilon}{4},$$

for all $n > m_2$ and $m_2 \leq i, j < n$. Now, let $n_1, n_2 > m(\varepsilon) := m_1 + m_2$, $i \in [m(\varepsilon), n_1)$ and $j \in [m(\varepsilon), n_2)$. Then, we have

$$d_h(T^{m(\varepsilon)} x_{n_1}, T^{m(\varepsilon)} x_{n_2}) \leq d_h(T^{m_1} x_{n_1}, T^{m_1} x_{n_2}) \leq \frac{\varepsilon}{4}.$$

Also,

$$d_h(T^{m(\varepsilon)} x_{n_1}, T^i x_{n_1}) \leq \frac{\varepsilon}{4}, \quad d_h(T^{m(\varepsilon)} x_{n_2}, T^j x_{n_2}) \leq \frac{\varepsilon}{4}.$$

Thus,

$$\begin{aligned} d_h(T^i x_{n_1}, T^j x_{n_2}) &\leq d_h(T^{m(\varepsilon)} x_{n_1}, T^i x_{n_1}) \\ &+ d_h(T^{m(\varepsilon)} x_{n_1}, T^{m(\varepsilon)} x_{n_2}) + d_h(T^{m(\varepsilon)} x_{n_2}, T^j x_{n_2}) < \varepsilon. \end{aligned}$$

This completes the proof of the step.

Now, we complete the proof of the theorem. Consider the sequences $\{T^{n-2} x_n\}_{n \geq 3}$ and $\{T^{n-1} x_n\}_{n \geq 2}$. For each $\varepsilon > 0$, take $N = m(\varepsilon) + 2$. Let $m, n \geq N$, $i = m-2$, $j = n-2$, $n_1 = m$ and $n_2 = n$. Then, $i \in [m(\varepsilon), n_1)$ and $j \in [m(\varepsilon), n_2)$. Hence, by Step IV, $d_h(T^{m-2} x_m, T^{n-2} x_n) < \varepsilon$. Thus, $\{T^{n-2} x_n\}_{n \geq 3}$ is a Cauchy sequence. A similar argument shows that $\{T^{n-1} x_n\}_{n \geq 2}$ is a Cauchy sequence and

$$\lim_{n \to \infty} d_h(T^{n-2} x_n, T^{n-1} x_n) = 0.$$

Note that the sequences $\{T^{n-2} x_n\}_{n \geq 3}$ and $\{T^{n-1} x_n\}_{n \geq 2}$ lie in $K$ and $(\mathcal{H}(K), d_h)$ is a complete metric space. Hence, there exists $A \in \mathcal{H}(K)$ such that

$$\lim_{n \to \infty} d_h(T^{n-2} x_n, A) = \lim_{n \to \infty} d_h(A, T^{n-1} x_n) = 0.$$

Since

$$d_h(T^{n-1} x_n, T(A))$$

$$= \max\{\sup_{a\in T^{n-2}x_n, a_1\in Ta} \inf_{b\in A, b_1\in Tb} d(a_1,b_1), \sup_{d_1\in Td, d\in A} \inf_{c_1\in Tc, c\in T^{n-2}x_n} d(c_1,d_1)\}$$

$$= \max\{\sup_{a\in T^{n-2}x_n} \inf_{b\in A} \sup_{a_1\in Ta} \inf_{b_1\in Tb} d(a_1,b_1), \sup_{d\in A} \inf_{c\in T^{n-2}x_n} \sup_{d_1\in Td} \inf_{c_1\in Tc} d(c_1,d_1)\}$$

$$\leq \max\{\sup_{a\in T^{n-2}x_n} \inf_{b\in A} d_h(Ta,Tb), \sup_{d\in A} \inf_{c\in T^{n-2}x_n} d_h(Tc,Td)\}$$

$$\leq c\max\{\sup_{a\in T^{n-2}x_n} \inf_{b\in A} d(a,b), \sup_{d\in A} \inf_{c\in T^{n-2}x_n} d(c,d)\}$$

$$= cd_h(T^{n-2}x_n, A) \to 0,$$

we obtain that $A = T(A)$. Thus, $T|_A : A \to \mathcal{H}(A)$ is a contraction multifunction and $(A, d|_{A\times A})$ is a complete metric space. Therefore, by [4; Theorem 6], $T|_A$ has a fixed point in $A$, that is, there exists $x_0 \in A \subseteq K$ such that $x_0 \in Tx_0$. □

Now, for clarity of the matter, we give two examples concerning Theorem 2.1.

**Example 2.2.** Let $X = [0,\infty)$ with the Euclidean norm, $m \geq 4$ a fixed natural number, $K = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\} \cup \{0,1\}$ and $K_0 = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$. Define $T : K \to \mathcal{H}(X)$ by

$$Tx = \{\frac{x}{2}, \frac{x}{3}, \cdots, \frac{x}{m}\},$$

for all $x \in K$. Note that the values of $T$ are compact and

$$d_h(Tx,Ty) \leq \frac{1}{2}|x-y|,$$

for all $x, y \in K$. Since $Tx \subseteq K$, for all $x \in K_0$, it is easy to see that if $\{x_n\}_{n\geq1}$ is a sequence in $K_0$, then $\bigcup_{i=1}^n T^i x_n \subseteq K$, for all $n \geq 1$. Thus, $T$ satisfies the conditions of Theorem 2.1. Finally, $x_0 = 0$ is the unique fixed point of $T$.

**Example 2.3.** Let $X = C[0,1]$ with the supremum norm, $K = \{f \in X : f \geq 0\}$ and $K_0 = \{f \in K : 0 < \|f\|_\infty < 1\}$. Define $T : K \to \mathcal{H}(X)$ by

$$T(f)(t) = \bigcup_{n=1}^{\infty} \{1 - \frac{1}{2}\int_{\frac{t}{n}}^{1} f(x)dx\} \bigcup \{1 - \frac{1}{2}\int_0^1 f(x)dx\},$$

for all $t \in [0,1]$. Note that the values of $T$ are compact and

$$d_h(T(f),T(g)) \leq \frac{1}{2}\|f-g\|_\infty,$$

for all $f, g \in K$. If $f \in K_0$, then $0 \leq f(x) < 1$ and so

$$0 < 1 - \frac{1}{2}\int_{\frac{t}{n}}^{1} f(x)dx < \frac{1}{2},$$

for all $t \in [0,1]$ and $n \geq 1$. Hence, $Tf \subseteq K$, for all $f \in K_0$. Thus, if $\{f_n\}_{n\geq 1}$ is a sequence in $K_0$, then $\bigcup_{i=1}^{n} T^i f_n \subseteq K$, for all $n \geq 1$. Therefore, $T$ satisfies the conditions of Theorem 2.1. Now, consider the function $g_0(t) = e^{\frac{1}{2}(t-1)}$ in $X$. Since

$$1 - \frac{1}{2}\int_{\frac{t}{1}}^{1} g_0(x)dx = g_0(t),$$

for all $t \in [0,1]$, $g_0 \in T(g_0)$, that is, $g_0$ is a fixed point of $T$.

We will use the following result of Lassonde ([4]) in our second result.

**Lemma 2.4.** *Let $E$ be a locally convex topological vector space, $X$ a convex subset of $E$ and $T \in \mathcal{K}_c(X, X)$ a compact multifunction. Then, $T$ has a fixed point in $X$.*

**Theorem 2.5.** *Let $G$ be a non-empty subset of a normed space $(Y, \|.\|)$ such that $0 \in IntG$, where $IntG$ is the interior of $G$. Assume that $T : \overline{G} \to Y$ is a nonexpansive and compact multifunction which satisfies the condition*

$$Tx \cap \{\lambda x : \lambda > 1\} = \emptyset, \tag{2.9}$$

*for all $x \in \partial G$, where $\partial G$ is the boundary of $G$. Also, let $Tx$ be a closed convex subset of $Y$, for all $x \in X$. Then, $T$ has a fixed point in $\overline{IntG}$.*

**Proof.** Suppose that $U = IntG$ and put $T_1 = T|_{\overline{U}} : \overline{U} \to Y$. We show that $T_1$ is upper semi-continuous. Let $x_0$ be an element of $\overline{U}$, $V$ an open set in $Y$ with $T_1x_0 \subseteq V$ and $T_1x_0 = \{z_i : i \in I\}$. For each $i \in I$, there exists $r_i > 0$ such that $N_{r_i}(z_i) \subseteq V$. Clearly, $T_1x_0 \subseteq \bigcup_{i\in I} N_{\frac{r_i}{2}}(z_i)$, $T_1x_0$ is closed and $T$ is compact. Hence, $T_1x_0$ is compact and so there exist $z_{i_1}, \cdots, z_{i_n} \in T_1x_0$ such that $T_1x_0 \subseteq \bigcup_{k=1}^{n} N_{\frac{r_{i_k}}{2}}(z_{i_k})$. Put $\varepsilon = \min\{\frac{r_{i_1}}{2}, \cdots, \frac{r_{i_n}}{2}\}$. Now, let $d$ be the metric induced by the norm, $d_h$ the metric induced by $d$, $\|x_0 - y\| < \varepsilon$ and $z \in T_1y$. Then, $d_h(T_1x_0, T_1y) \leq \|x_0 - y\| < \varepsilon$ and there exists $t_0 \in T_1x_0$ such that

$\|t_0 - z\| < \varepsilon$. Take $j \in \{1, \cdots, n\}$ such that $t_0 \in N_{\frac{r_{i_j}}{2}}(z_{i_j})$. Then, $\|t_0 - z_{i_j}\| < \frac{r_{i_j}}{2}$ and $\|z - z_{i_j}\| \le \|z - t_0\| + \|t_0 - z_{i_j}\| < r_{i_j}$. Hence, $z \in V$ and so $T_1 y \subseteq V$. Thus $T_1$ is upper semi-continuous.

Now, let $p$ be the Minkowski semi-norm of $U$. Define $r : Y \to \overline{U}$ by

$$r(x) = \begin{cases} x & x \in \overline{U} \\ \frac{x}{p(x)} & x \notin \overline{U}. \end{cases}$$

Then, $r$ is continuous because $p(x) = 1$, whenever $x \in \overline{U}$. Put

$$f = r|_{co(\overline{TU})} \ .$$

Note that $f$ is continuous and if $F = T_1 f$, then $F$ is upper semi-continuous, compact, convex-valued and compact-valued multifunction. Thus, $F$ is a Kakutani multifunction. Hence, $F \in \mathcal{K}_c(co(\overline{TU}), co(\overline{TU}))$ and by Lemma 2.4, $F$ has a fixed point, say $z_0$, in $co(\overline{TU})$. We show that $z_0 \in \overline{U}$. If $z_0 \in Y \backslash \overline{U}$, then $z_0 \in T_1(\frac{z_0}{p(z_0)})$ and $p(z_0) > 1$. Hence,

$$z_0 \in T_1(\frac{z_0}{p(z_0)}) \cap \{\lambda(\frac{z_0}{p(z_0)}) : \lambda > 1\}.$$

Since $p(\frac{z_0}{p(z_0)}) = 1$, $\frac{z_0}{p(z_0)} \in \partial U$. This contradicts (2.9) and so $z_0 \in \overline{U}$. Since $f(z_0) = z_0$, $z_0 \in T z_0$. This completes the proof. $\square$

Now, we give the following example for Theorem 2.5.

**Example 2.6.** Let $Y = \mathbb{R}$ with the usual norm and $G = (-\pi, \pi)$. Define $T : \overline{G} \to Y$ by

$$Tx = [\min\{\sin x, \cos x\}, \max\{\sin x, \cos x\}],$$

for all $x \in \overline{G}$. Note that $T$ is nonexpansive and compact multifunction and the values of $T$ are convex and compact. Since

$$T(-\pi) \cap \{-\lambda\pi, \lambda > 1\} = \emptyset \text{ and } T(\pi) \cap \{\lambda\pi, \lambda > 1\} = \emptyset,$$

$T$ satisfies in the condition (12). Also, note that $x_0 = 0$ is a fixed point of $T$.

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