# AN ALEXANDROFF TOPOLOGY ON GRAPHS

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ABSTRACT. Let $G = (V, E)$ be a locally finite graph, i.e. a graph in which every vertex has finitely many adjacent vertices. In this paper, we associate a topology to $G$, called graphic topology of $G$ and we show that it is an Alexandroff topology, i.e. a topology in which intersection of each family of open sets is open. Then we investigate some properties of this topology. Our motivation is to give an elementary step toward investigation of some properties of locally finite graphs by their corresponding topology which we introduce in this paper.

## 1. Introduction

There are some publications to define a topology for discrete structures like numbers, words and graphs. The reader can refer to [5, 6, 7]. The Alexandroff topology that was introduced in [6, 7] on a graph $G$ is a topology on the vertex set $V$ of a graph $G$ by declaring subsets of $V$ as "open", so that a subset of $V$ is topologically connected if and only if it is connected in $G$ (i.e. if the induced subgraph of $G$ on $V$ is connected). There are some graphs which do not have such topology( see Corollary 4.1 and Theorem 4.2 of [7]). In this paper we introduce an Alexandroff topology on every locally finite graphs called graphic topology. The topology is defined on each graph but unlike the topology

introduced in [6, 7] there are some connected graphs like $K_n$ and $C_n$ whose corresponding topological space is not connected.

In the article, we investigate some properties of graphic topology and its relation with the corresponding graphs. Our motivation is to give an elementary step toward investigation of some properties of locally finite graphs by their corresponding topology which we introduce in this paper. In Section 2 of the paper we give some definitions and preliminaries of graph theory and topology. We also define our topology on graphs by introducing a subbasis family for the topology. Section 3 is devoted to some preliminaries results of graphic topology. In Section 4 more properties of graphic topology is discussed. In Section 5 some continuity properties of functions between graphs is investigated. Connectivity or disconectivity of graphic topology is the subject of Section 6. In Section 7 we study some necessary and sufficient conditions for topological spaces to be graphic. Finally the last section of the paper is devoted to the study of dense subsets of graphic topology. One of the main results in this section is an upper bound for chromatic number of a graph in term the cardinal of minimal dense subset in graphic topology.

## 2. **Preliminaries**

In this section we give the preliminaries. All definitions are standard and can be found for example in [2, 3, 8].

For a set $V$, by $[V]^k$ we denote the set of all $k-$element subsets of $V$. A (simple) graph $G$ is a pair $G = (V, E)$ of sets such that $E \subseteq [V]^2$. The elements of $V$ and $E$ are vertices and edges of the graph $G$, respectively. A graph with vertex set $V$ is said to be a graph on $V$. The vertex set of a graph $G$ is referred to as $V(G)$, its edge set as $E(G)$. A graph $G$ is finite, if $V(G)$ (and so $E(G)$) are finite; otherwise it is infinite. A vertex $v$ is incident with an edge $e$ if $v \in e$. An edge $\{x, y\}$ is usually written as $xy$ (or $yx$). Two vertices $x, y$ of $G$ are adjacent if $xy$ is an edge of $G$. The set of all adjacent vertices of $v$ and the set of all the edges $e \in E$ with $v \in e$ are denoted by $A_v$ and $E(v)$, respectively. The degree $d_G(v) = d(v)$ of a vertex $v$ is the number $|E(v)|$ of edges at $v$, that is equal to the number $|A_v|$ of adjacent vertices of $v$. A vertex of degree 0 is isolated. An independent set in a graph is a set of pairwise nonadjacent vertices. A clique is a set of pairwise adjacent vertices. The complement $\overline{G}$ of a (simple) graph $G$ is the (simple) graph with vertex set $V(G)$ defined by $xy \in E(\overline{G}) \Leftrightarrow xy \notin E(G)$.

If $V' \subseteq V$ and $E' \subseteq E$, then $G' = (V', E')$ is a subgraph of $G = (V, E)$, written as $G' \subseteq G$. If $G' \subseteq G$ and $G'$ contains all the edges $xy \in E$ with $x, y \in V'$, then $G'$ is the induced subgraph of $G$ on $V'$, written as $G' = G[V']$.

We use notations $K_n$, $K_{m,n}$, $P_n$ and $C_n$ for a complete graph with $n$ vertices, the complete bipartite graph when partite sets have sizes $m$ and $n$, the path on $n$ vertices and the cycle on $n$ vertices, respectively. Obviously the $n$ vertex star i.e. a tree with one vertex adjacent to all the others is $K_{1,n-1}$.

The maximum degree in a graph $G$ is $\Delta(G)$, the minimum degree is $\delta(G)$, and $G$ is regular if $\Delta(G) = \delta(G)$. It is $k-$regular if the common degree is $k$.

An infinite graph is called locally finite if all its vertices have finite degrees (see [2]).

Now, we define our topology on graphs. Suppose that $G = (V, E)$ is a (simple) graph without isolated vertex. Remember that $A_x$ is the set of all vertices adjacent to $x$. It is clear that $x \in A_y$ if and only if $y \in A_x$ for all $x, y \in V$ and $x \notin A_x$ for all $x \in V$. Define $\mathcal{S}_G$ as follows:

$$\mathcal{S}_G = \{A_x \mid x \in V\},$$

Since $G$ has no isolated vertex, we have $V = \cup_{x \in V} A_x$. Hence $\mathcal{S}_G$ forms a subbasis for a topology $\tau_G$ on $V$, called graphic topology of $G$. It is easy to see that, the graphic topologies of $K_n$ and $C_n$ are discrete, but the graphic topology of $P_n$ is not discrete because the set contains two vertices of degree one is not open. Also the graphic topology of $K_{n,m}$ is equal to $\{\phi, V, A, B\}$, where $A$ and $B$ are partite sets of $K_{n,m}$. Throughout the paper all graphs are locally finite.

## 3. Preliminary results

**Proposition 3.1.** *Suppose that $G = (V, E)$ is a graph. Then $(V, \tau_G)$ is an Alexandroff space.*

*Proof.* It is enough to prove that arbitrary intersection of members of $\mathcal{S}_G$ is open. Let $S \subseteq V$. If $x \in \cap_{y \in S} A_y$, then $x \in A_y$ for each $y \in S$. Hence $y \in A_x$ for each $y \in S$ and so $S \subseteq A_x$. Since $G$ is locally finite, $A_x$ and so $S$ are finite sets. This means that if $S$ is infinite, then $\cap_{y \in S} A_y$ is empty, but if $S$ is finite, then $\cap_{y \in S} A_y$ is the intersection of finitely many open sets and hence is open. $\square$

Let $G = (V, E)$ be a graph. Then by Remark 1.1, for each $x \in V$, the intersection of all open sets containing $x$ is the smallest open set containing $x$, we still call it $U_x$ and the family $\mathcal{B}_G = \{U_x \mid x \in V\}$ is the minimal basis for the topological space $(V, \tau_G)$.

**Proposition 3.2.** *Let $G = (V, E)$ be a graph. Then we have $U_x = \cap_{y \in A_x} A_y$ and so $U_x$ is finite for every $x \in V$.*

*Proof.* Since $U_x$ is the smallest open set containing $x$ and $\mathcal{S}_G$ is a subbasis of $\tau_G$, we have $U_x = \cap_{z \in S} A_z$ for some subset $S$ of $V$. This implies that $x \in A_z$ for each $z \in S$. Therefore $S \subseteq A_x$ and so $x \in \cap_{z \in A_x} A_z \subseteq U_x$. Now by definition of $U_x$, the proof is complete. $\qquad\square$

**Corollary 3.3.** *Let $G = (V, E)$ be a graph. Then for every $x, z \in V$ we have $z \in U_x$ if and only if $A_x \subseteq A_z$. Equivalently $U_x = \{z \in V \mid A_x \subseteq A_z\}$.*

*Proof.* By Proposition 2.2, $z \in U_x$ if and only if $z \in A_y$ for each $y \in A_x$ if and only if $y \in A_z$ for each $y \in A_x$. $\qquad\square$

**Remark 3.4.** *Suppose that $G = (V, E)$ is a graph. By Corollary 2.3, $(V, \tau_G)$ is a discrete topological space if and only if $A_x \nsubseteq A_y$ and $A_y \nsubseteq A_x$ for every distinct pair of vertices $x, y \in V$.*

**Remark 3.5.** *We also know from Remark 1.1 that an Alexandroff topological space $(X, \tau)$ is $T_1$ if and only if it is discrete. Now, Corollary 2.3, implies that the graph $G = (V, E)$ has $T_0$ graphic topology if and only if $A_x \neq A_y$ for every distinct pair of vertices $x, y \in V$. Let $T = (V, E)$ be a tree. Then $(V, \tau_G)$ is a $T_0$ space if and only if $A_x \neq A_y$ for every $x, y \in V$ such that $x \neq y$ and $\deg x = \deg y = 1$.*

**Corollary 3.6.** *Suppose that $G = (V, E)$ is a graph and $x \in V$. Then we have $U_x \subseteq \{x\} \cup \{y \in V \mid d(x, y) = 2\}$ and so $U_x \cap A_x = \phi$. In particular, $U_x \subseteq A_x^c$ and $A_x \subseteq U_x^c$. Moreover, If $x, y \in V$ are adjacent, then $U_x \cap U_y = \phi$.*

*Proof.* Let $z \in U_x \setminus \{x\}$. By Proposition 2.2, $z \in A_y$ for each $y \in A_x$. Hence $d(x, z) \leq 2$. If $d(x, z) = 1$, then $z \in A_x$ and so $z \in A_z$ which is a contradiction. The second part of the corollary is then obvious. Now suppose $x, y$ be adjacent then we have $y \in A_x$ and $A_x$ is an open set. So $U_y \subseteq A_x$ and by the first part of corollary, $A_x \subseteq U_x^c$. Hence $U_y \subseteq U_x^c$ or equivalently $U_x \cap U_y = \phi$. $\qquad\square$

An obvious consequence of Corollary 2.6 implies for every $x \in V$ we have $\overline{\{x\}} \subseteq \overline{U_x} \subseteq A_x^c$ and $\overline{A_x} \subseteq U_x^c$.

Finally, the following is a trivial consequence of Remark 1.1 and Corollary 2.3.

**Corollary 3.7.** *Let $G = (V, E)$ be a graph. For every $x, y \in V$, $y \in \overline{\{x\}}$ if and only if $A_y \subseteq A_x$.*

## 4. Some Properties of Graphic Topology

**Definition 4.1.** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We call $G_1$ and $G_2$ isomorphic, and write $G_1 \cong G_2$, if there exists a bijection $\varphi : V_1 \longrightarrow V_2$ with $xy \in E_1 \Leftrightarrow \varphi(x)\varphi(y) \in E_2$ for all $x, y \in V_1$. Such a map $\varphi$ is called an isomorphism; if $G_1 = G_2$, it is called an automorphism of $G_1$.*

**Remark 4.2.** *It is easy to check, If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic graphs, then topological spaces $(V_1, \tau_{G_1})$ and $(V_2, \tau_{G_2})$ are homeomorphic. The converse is not true, in general. For example, $C_n$ and $K_n$ for $n > 4$ are not isomorphic graphs, but their corresponding graphic topologies are both discrete and hence homeomorphic. Also, we can obtain two infinite nonisomorphic graphs $G_1$ and $G_2$ with discrete graphic topologies as follows: Let $P$ be the infinite path on $x_1 - x_2 - x_3 - \cdots$, $K_n$ be the complete graph on $\{x_1, y_2, \cdots, y_n\}$ for $n \geq 5$ and $C_n$ the cycle on $\{x_1, y_2, \cdots, y_n\}$ for $n \geq 5$. Let $V = \{y_2, \cdots, y_n, x_1, x_2, \cdots\}$ and put $G_1 = (V, E(K_n) \cup E(P))$, and $G_2 = (V, E(C_n) \cup E(P))$.*

**Proposition 4.3.** *Let $G = (V, E)$ be a locally finite graph. Then $(V, \tau_G)$ is a compact topological space if and only if $V$ is finite.*

*Proof.* By Proposition 2.2, $U_x$ is finite for every $x \in V$. Hence if $V$ is infinite, then $\mathcal{B}_G$ is an open covering of $(V, \tau_G)$ which has no finite subcover. $\square$

**Proposition 4.4.** *Let $G = (V, E)$ be a graph and $T = \{x \in V \mid \deg x = \Delta\}$. Then $T \in \tau_G$.*

*Proof.* Suppose that $x \in T$ and $y \in U_x$. It follows from Corollary 2.3 that $\Delta = \deg x \leq \deg y$ and this implies that $\deg y = \Delta$ and so $y \in T$. Thus $x \in U_x \subseteq T$ and so $x$ is an interior point of $T$. $\square$

**Proposition 4.5.** *Suppose that $G = (V, E)$ is a graph $L = \{x \in V \mid \deg x = \delta\}$. Then $L$ is a closed set in $(V, \tau_G)$.*

*Proof.* By Remark 1.1, we have $\overline{L} = \bigcup_{x \in L} \overline{\{x\}}$. Suppose that $y \in \overline{L}$. Thus $y \in \overline{\{x\}}$ for some $x \in L$. It follows from Corollary 2.9 that $\deg y \leq \deg x = \delta$. Hence $\deg y = \delta$ and $y \in L$. Therefore $\overline{L} \subseteq L$ and the proof is complete. $\qquad\square$

**Definition 4.6.** *For a graph $G$, if $S \subseteq V(G)$, then we write $G - S$ for the subgraph obtained by deleting the set of vertices $S$, equivalently $G - S = G[V \setminus S]$. A cut-vertex of $G$ is a vertex whose deletion increases the number of components of $G$, i.e. a vertex $v \in V(G)$ such that $G - \{v\}$ has more components than $G$. A vertex cut of a connected graph $G$ is a set $C \subseteq V(G)$ such that $G - C$ has more than one component. A vertex cut $C$ of $G$ is said to be minimal if every proper subset of $C$ is not a vertex cut.*

It is obvious that, if $x$ be a cut-vertex in a (not necessarily connected) graph $G = (V, E)$. Then $\{x\} \in \tau_G$. It is well-known that a connected graph is a tree if and only if every vertex of degree greater than one is a cut-vertex. Therefore, if $T = (V, E)$ is a tree and $x \in V$ with $\deg x \geq 2$, then $\{x\} \in \tau_T$.

**Proposition 4.7.** *Let $G = (V, E)$ be a connected graph and $C$ is a minimal vertex cut in $G$. Then $C \in \tau_G$.*

*Proof.* Suppose that $G - C$ has $k \geq 2$ components, say $G_i = (V_i, E_i)$ for $i = 1, 2, \ldots, k$. Every vertex $x \in C$ must be adjacent to vertices of at least two different components, say $G_1$ and $G_2$, because $C$ is a minimal vertex cut. Suppose that $\{y_1, \ldots, y_{k_1}\} = A_x \cap V_1$ and $\{z_1, \ldots, z_{k_2}\} = A_x \cap V_2$, then we have $x \in \cap_{i=1}^{k_1} A_{y_i} \subseteq C \cup V_1$ and $x \in \cap_{i=1}^{k_2} A_{z_i} \subseteq C \cup V_2$ and so

$$x \in \left(\bigcap_{i=1}^{k_1} A_{y_i}\right) \cap \left(\bigcap_{i=1}^{k_2} A_{z_i}\right) \subseteq C \cup (V_1 \cap V_2) = C$$

that is $x$ is an interior point of $C$. $\qquad\square$

**Definition 4.8.** *A graph $G$ is vertex-transitive if for every pair $u, v \in V(G)$, there is an automorphism of $G$ that maps $u$ to $v$.*

It is easy to see, if $G = (V, E)$ is a vertex-transitive graph, then $(V, \tau_G)$ is a discrete topological space if and only if $U_x = \{x\}$ for some vertex $x \in V$.

Let $G = (V, E)$ be a locally finite graph. Set $\tau_G^c = \{U^c \mid U \in \tau_G\}$. Then $\tau_G^c$ is a topology on $V$. We have the following proposition only for finite graphs:

**Proposition 4.9.** *Let $G = (V, E)$ be a finite graph and $\tau_G^c = \{U^c \mid U \in \tau_G\}$. If $\overline{G}$ is a connected graph and $\tau_{\overline{G}} = \tau_G^c$, then $(V, \tau_G)$ is a discrete topological space.*

*Proof.* Let $A_x$ be the set of adjacent vertices of $x$ in $G$, as usual. Then the set of adjacent vertices of $x$ in $\overline{G}$ is $(A_x \cup \{x\})^c$, that is open in $\tau_{\overline{G}}$. Since $\tau_{\overline{G}} = \tau_G^c$, we have $(A_x \cup \{x\})^c \in \tau_G^c$ and so $A_x \cup \{x\} \in \tau_G$. Therefore $U_x \subseteq A_x \cup \{x\}$, where $U_x$ is the smallest open set containing $x$ in $\tau_G$. Hence $U_x = \{x\}$, because $U_x \subseteq A_x^c$, by Corollary 2.6. Thus $(V, \tau_G)$ is discrete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

The following example shows, the graphic topology of the complement $\overline{G}$ of a graph $G$ with discrete graphic topology can have also discrete topology.

**Example 1.** *Suppose $G = C_5$ then $\overline{G} = C_5$ and both of them have discrete topology.*

## 5. **On Functions Between Graphs**

In section 3, we saw that an isomorphism of graphs, induces a homeomorphism of topological spaces. In this section, we generalize this fact.

It is known that a function $\varphi : (X_1, \tau_1) \longrightarrow (X_2, \tau_2)$ between topological spaces is continuous if and only if $\varphi(\overline{A}) \subseteq \overline{\varphi(A)}$ for every subset $A$ of $X_1$ and is closed if and only if $\overline{\varphi(A)} \subseteq \varphi(\overline{A})$ for every subset $A$ of $X_1$ (see [3] pages 80,87).

**Theorem 5.1.** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs and $\tau_{G_1}$ and $\tau_{G_2}$ be the corresponding graphic topologies. Suppose that $\varphi : V_1 \longrightarrow V_2$ is a function. Consider $\varphi$ as a function between topological spaces $(V_1, \tau_{G_1})$ and $(V_2, \tau_{G_2})$. Then we have the following:*
*1. $\varphi$ is continuous if and only if $A_y \subseteq A_x$ implies $A_{\varphi(y)} \subseteq A_{\varphi(x)}$ for every $x, y \in V_1$.*
*2. If $\varphi$ is closed and injective, then $A_{\varphi(y)} \subseteq A_{\varphi(x)}$ implies $A_y \subseteq A_x$ for every $x, y \in V_1$. Conversely, if $\varphi$ is surjective and $A_{\varphi(y)} \subseteq A_{\varphi(x)}$ implies $A_y \subseteq A_x$ for every $x, y \in V_1$, then $\varphi$ is closed.*

*Proof.* (i) By Corollary 2.3 and Remark 1.1, it is enough to show that $\varphi$ is continuous if and only if $y \in \overline{\{x\}}$ implies $\varphi(y) \in \overline{\{\varphi(x)\}}$ for every $x, y \in V_1$. Let $\varphi$ be continuous and $y \in \overline{\{x\}}$. Then $\varphi(y) \in \varphi(\overline{\{x\}})$. By continuity of $\varphi$, $\varphi(\overline{\{x\}}) \subseteq \overline{\{\varphi(x)\}}$ and so $\varphi(y) \in \overline{\{\varphi(x)\}}$. For the converse, let $A$ be a subset of $V_1$ and $y \in \overline{A}$. Note that $A = \bigcup_{x \in A} \{x\}$ and so

$\overline{A} = \overline{\bigcup_{x \in A} \{x\}} = \bigcup_{x \in A} \overline{\{x\}}$ by Remark 1.1. Hence there exist an element $x \in A$ such that $y \in \overline{\{x\}}$. By the assumption, $\varphi(y) \in \overline{\{\varphi(x)\}} \subseteq \overline{\varphi(A)}$. Therefore $\varphi(\overline{A}) \subseteq \overline{\varphi(A)}$ and so $\varphi$ is continuous.

(ii) If $\varphi$ is closed and injective, then $\varphi^{-1}$ is continuous on $\varphi(V_1)$. Therefore, by (i) for all $x, y \in V_1$ $A_{\varphi(y)} \subset A_{\varphi(x)}$ implies $A_{\varphi^{-1}(\varphi(y))} \subset A_{\varphi^{-1}(\varphi(x))}$, then $A_y \subset A_x$. Suppose $\varphi$ is surjective and $\psi$ is a right inverse of $\varphi$. For each $x, y \in V_1$, suppose that $A_y \subset A_x$. Hence $A_{\varphi(\psi(x))} \subset A_{\varphi(\psi(y))}$, because $\varphi o \psi = id_{V_2}$. By assumption $A_{\psi(x)} \subset A_{\psi(y)}$. By (i) $\psi$ is continuous. If $\varphi(x) = \varphi(y)$, then by assumption $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$. Therefore, if $x \in C \subset V_1$ and $C$ is closed, then $y \in C$. Hence, $\varphi(C) = \psi^{-1}(C)$ which is closed, because $\psi$ is continuous. $\qquad\square$

**Corollary 5.2.** *By notation of Theorem 4.1, $\varphi$ is a homeomorphism if and only if it is bijective and $A_y \subseteq A_x$ if and only if $A_{\varphi(y)} \subseteq A_{\varphi(x)}$ for every $x, y \in V_1$.*

## 6. Connectedness of Graphic Topology

It is easy to see that, every disconnected graph has disconnected graphic topology. Also the graphic topology of a bipartite graph $G = (V, E)$, is disconnected. Since every tree is a bipartite graph, then graphic topology of a tree is disconnected. In this section, we investigate some other conditions which guarantee connectedness or disconnectedness of a graphic topology for a connected graph $G$.

**Definition 6.1.** *Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is clopen, if it is both open and closed in $(X, \tau)$.*

It is easy to see that if $G = (V, E)$ is a disconnected graph, then $(V, \tau_G)$ is a disconnected topological space.

**Proposition 6.2.** *Let $G = (V, E)$ be a graph with $n$ vertices. If $G$ has $k$ vertices of degree $n - k$ that form an independent set, then the set of these vertices is clopen in $(V, \tau_G)$. In particular, $(V, \tau_G)$ is disconnected.*

*Proof.* Let $\{x_1, \ldots, x_k\}$ be an independent set in $G$ such that $\deg x_i = n - k$ for each $i \in \{1, \ldots, k\}$. Hence $A_{x_i} = V \setminus \{x_1, \ldots, x_k\}$ for each $i \in \{1, \ldots, k\}$, because $\deg x_i = n - k$ and $x_i$ is not adjacent to $x_1, \ldots, x_k$. It follows from Corollary 2.6 that $U_{x_i} \subseteq A_{x_i}^c = \{x_1, \ldots, x_k\}$ for each $i \in \{1, \ldots, k\}$. Therefore, on one hand $\{x_1, \ldots, x_k\}$ is a closed set, and

on the other hand

$$\bigcup_{i=1}^{k} U_{x_i} = \{x_1, \ldots, x_k\}$$

which implies that $\{x_1, \ldots, x_k\}$ is open, too. □

The following corollary is an immediate consequence of Proposition 5.2.

**Corollary 6.3.** *Let $G = (V, E)$ be a graph with $n$ vertices. If $x \in V$ is of degree $n - 1$, then $\{x\}$ is both open and closed in $(V, \tau_G)$ and so $(V, \tau_G)$ is disconnected.*

**Proposition 6.4.** *Let $X$ be an Alexandroff space. If there is a point $x \in X$ such that $U_x$ is both maximal and minimal in $\mathcal{B} = \{U_y| \ y \in X\}$, then $U_x$ is clopen and $X$ is disconnected.*

*Proof.* It is enough to show that $U_x$ is closed or equivalently $\overline{U_x} = U_x$. Suppose that $y \in \overline{U_x}$. Then there exists an element $z \in U_x$ such that $y \in \overline{\{z\}}$. By using definition of $U_z$ and Remark 1.1, we obtain $U_z \subseteq U_x$ and $U_z \subseteq U_y$. By minimality of $U_x$ we get $U_z = U_x$, so we have $U_x \subseteq U_y$ and by maximality of $U_x$ we get $U_x = U_y$. Hence $y \in U_x$. □

**Remark 6.5.** *It is a consequence of Corollary 2.3 that for a graph $G = (V, E)$, $U_x$ is minimal in $\mathcal{B}_G$ if and only if $A_x$ is maximal in $\mathcal{S}_G$ and $U_x$ is maximal in $\mathcal{B}_G$ if and only if $A_x$ is minimal in $\mathcal{S}_G$. Therefore by Proposition 5.4, If there is a vertex $x \in V$ such that $A_x$ is both maximal and minimal in $\mathcal{S}_G$, then $(V, \tau_G)$ is disconnected. In particular, every $k-$regular graph is disconnected.*

**Proposition 6.6.** *Let $G = (V, E)$ be a graph such that for every $x, y \in V$ we have $x \in A_y$ or $A_x \subseteq A_y$ or $A_y \subseteq A_x$. Then $(V, \tau_G)$ is disconnected.*

*Proof.* By Corollaries 2.3,2.6 and by the assumption of the proposition, we have $U_x \cap U_y = \phi$ or $U_x \subseteq U_y$ or $U_y \subseteq U_x$ for every $x, y \in V$. Let $x \in V$ be a vertex with $A_x$ minimal in $\mathcal{S}_G$. Then $U_x$ is maximal in $\{U_y| \ y \in V\}$ by Remark 5.5. Let $W = \bigcup_{y \notin U_x} U_y$, then $W$ is an open set. We prove that $U_x \cup W = V$ and $U_x \cap W = \phi$. First let $y \in V \setminus U_x$. By definition of $W$, we have $U_y \subseteq W$ and so $y \in W$. Secondly, Suppose that $z \in U_x \cap W$. Since $z \in W$, there exists a vertex $y \in V$ such that $y \notin U_x$ and $z \in U_y$. Therefore $z \in U_x \cap U_y$. By assumption, we have $U_x \subseteq U_y$ or $U_y \subseteq U_x$. We claim that in both cases we will have $y \in U_x$, which is a contradiction. If $U_x \subseteq U_y$, then by maximality of $U_x$, we get

$U_x = U_y$ and hence $y \in U_x$, and if $U_y \subseteq U_x$, then it is immediate that $y \in U_x$. Therefore $(U_x, W)$ is a separation for $(V, \tau_G)$. $\qquad \square$

**Example 2.** *Let $G = (V, E)$ be a graph. Suppose that $P = \{x \in V \,|\, \deg x = 1\}$ has at least two elements that are adjacent to two distinct vertices of $V \setminus P$. If $V \setminus P$ is a clique with at least three elements, then $(V, \tau_G)$ is connected. To prove this, assume that $P = \{a_1, \ldots, a_k\}$ and $k \geq 2$. We claim that $V = \cup_{i=1}^{k} U_{a_i}$ and $U_{a_i} \cap U_{a_j} \neq \phi$ for every $i, j \in \{1, \ldots, k\}$. Suppose that $x \in V \setminus P$, then there are $y \in V$ and $a_i \in P$ such that $x \in A_y$ and $y \in A_{a_i}$ and so $x \in A_y = U_{a_i}$, because $\deg a_i = 1$. It implies the first claim. Now suppose that $a_i, a_j$ are elements of $P$ such that there are distinct vertices $y_i, y_j \in V \setminus P$ with $y_i \in A_{a_i}$ and $y_j \in A_{a_j}$. Therefore $U_{a_i} = A_{y_i}$ and $U_{a_j} = A_{y_j}$. Since $V \setminus P$ is a clique with at least three elements, there is a $z \in V \setminus P$ such that $z \in A_{y_i} \cap A_{y_j}$ and so $z \in U_{a_i} \cap U_{a_j}$ and $U_{a_i} \cap U_{a_j} \neq \phi$. Now let by contrary $V = A \cup B$ with $A$ and $B$ open sets such that $A \cap B = \phi$ and let $a_i \in A$ and $a_j \in B$. Hence $U_{a_i} \subseteq A$ and $U_{a_j} \subseteq B$ which implies $U_{a_i} \cap U_{a_j} = \phi$. This is a contradiction.*

**Problem 1.** *What are the necessary and sufficient conditions for connectivity of graphic topology?*

## 7. **Conditions on Topological Spaces to be Graphic**

**Definition 7.1.** *A topological space $(V, \tau)$ is called graphic, if there is some (locally finite) graph $G$ with vertex set $V$ and without isolated vertex, such that $\tau_G = \tau$.*

In this section, we show that the property of being graphic is a topological one, i.e. it is invariant under homeomorphisms, and then investigate some necessary or sufficient conditions for the topological spaces to be graphic.

**Proposition 7.2.** *Let $G = (V, E)$ be a graph and $\tau_G$ be the corresponding graphic topology. If $(V', \tau)$ is a topological space homeomorphic to $(V, \tau_G)$, then $(V', \tau)$ is graphic. In particular, being graphic is a topological property.*

*Proof.* Let $\varphi : (V, \tau_G) \longrightarrow (V', \tau)$ be a homeomorphism. Then $(V', \tau)$ is also an Alexandroff space. We construct a graph structure $G' = (V', E')$ on $V'$ in the way that $\varphi(x)$ is adjacent to $\varphi(y)$ in $G'$ if and only if $x$ is adjacent to $y$ in $G$ for every $x, y \in V$. Then we have $\varphi(A_x) = A'_{\varphi(x)}$ for

every $x \in V$, where $A_x$ and $A'_{\varphi(x)}$ are the sets of adjacent vertices to $x$ in $G$ and to $\varphi(x)$ in $G'$, respectively. Hence $G'$ is also a locally finite graph. We prove that $\tau = \tau_{G'}$. For this purpose, let $U_x$ be the smallest open set containing $x$ in $(V, \tau_G)$ and $U'_y$ (resp. $W_y$) be the smallest open set containing $y$ in $(V', \tau_{G'})$ (resp. $(V', \tau)$). We prove that $U'_y = W_y$. Since $\varphi$ is a homeomorphism, $\varphi(U_x) = W_{\varphi(x)}$. But $\varphi$ is also an isomorphism of graphs $G$ and $G'$, therefore $\varphi(U_x) = U'_{\varphi(x)}$ and the proof is complete. $\square$

We know that in a graphic topology $(V, \tau_G)$, $U_x$ is finite for each $x \in V$. Therefore if an Alexandroff space $(X, \tau)$ is graphic, then $U_x$ is finite. In the following proposition we investigate another necessary condition for a topological space to be graphic:

**Proposition 7.3.** *Let $(V, \tau)$ be a graphic topological space and let $U_x$ be the smallest open set containing $x$ for every $x \in V$. Then $\overline{U_x} \neq V$. In particular, $\overline{\{x\}} \neq V$ for every $x \in V$.*

*Proof.* Let $G$ be a graph on $V$ such that $\tau_G = \tau$ and let $x \in V$. Since $G$ has no isolated vertex, we have $A_x \neq \phi$ and so $A_x^c \neq V$. Hence $\overline{U_x} \neq V$ by Corollary 2.6. $\square$

The following example shows that the condition in Proposition 6.3 is not sufficient:

**Example 3.** *Let $V = \{1, 2, 3, 4\}$ and*

$$\tau = \{\phi, V, \{1\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}.$$

*here we have $U_1 = \{1\}$, $U_2 = \{1, 2\}$, $U_3 = \{1, 3\}$ and $U_4 = \{4\}$. By Remark 1.1, we know that $y \in \overline{\{x\}}$ if and only if $x \in U_y$, and hence $\overline{\{1\}} = \{1, 2, 3\}$, $\overline{\{2\}} = \{2\}$, $\overline{\{3\}} = \{3\}$ and $\overline{\{4\}} = \{4\}$. Therefore $\overline{U_1} = \overline{U_2} = \overline{U_3} = \{1, 2, 3\}$ and $\overline{U_4} = \{4\}$. Hence $(V, \tau)$ satisfies Proposition 6.3, but it is not graphic.*

The following proposition gives a sufficient condition for a finite topological space to be graphic.

**Proposition 7.4.** *Let $(V, \tau)$ be a finite topological space and $W_x$ be the smallest open set containing $x$ for every $x \in V$. If for every $x, y \in V$, $W_x = W_y$ or $W_x \cap W_y = \phi$, then $(V, \tau)$ is graphic.*

*Proof.* We construct a graph $G = (V, E)$ as follows

$$(7.1) \qquad \{x, y\} \in E \Leftrightarrow W_x \cap W_y = \phi, \text{ for every } x, y \in V.$$

For every $x \in V$ let $U_x$ and $A_x$ be the smallest set containing $x$ in $\tau_G$ and the set of all adjacent vertices to $x$ in $G$, respectively. We prove that $\tau_G = \tau$. Let $x \in V$. It is enough to show that $U_x = W_x$. By (7.1), we have $A_y = \{z \in V \mid W_y \cap W_z = \phi\}$ for every $y \in V$. Therefore $y \in U_x$ if and only if $A_x \subseteq A_y$ if and only if $\{z \in V \mid W_x \cap W_z = \phi\} \subseteq \{z \in V \mid W_y \cap W_z = \phi\}$. Suppose that $y \in U_x \setminus W_x$. Hence $W_x \cap W_y = \phi$, otherwise $W_x = W_y$ and so $y \in W_x$ which is a contradiction. Therefore $y \in A_x$, but $A_x \subseteq A_y$ and this implies that $y \in A_y$ which is a contradiction. So $U_x \subseteq W_x$. Conversely, if $y \in W_x$, then $W_x = W_y$. So if $z \in A_x$ for some $z \in V$, then $W_x \cap W_z = \phi$ and so $W_y \cap W_z = \phi$. This implies that $z \in A_y$ and so $A_x \subseteq A_y$. Therefore $y \in U_x$ and $W_x \subseteq U_x$. This completes the proof.                                                                                        □

The next example shows that the condition in Proposition 6.4 is not necessary.

**Example 4.** *If $V = \{1, 2, 3, 4\}$ and*

$$\tau = \{\phi, V, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}\},$$

*then $(V, \tau)$ does not satisfy the condition of Proposition 6.4, but it is graphic.*

**Problem 2.** *What is the necessary and sufficient condition for an Alexandroff space to be graphic?*

## 8. Density in Graphic Topologies

In this section, we investigate some conditions for dense subsets in the graphic topology associated to a graph $G = (V, E)$. For every $x \in V$, we have $x \in U_x$ and so $U_x^c \neq V$. Therefore $\overline{A_x} \neq V$ by Corollary 2.6. Then $\overline{A_x} \neq V$ for every $x \in V$.

**Remark 8.1.** *Let $(V, \tau)$ be a topological space and $A$ be a subset of $V$. It is a well-known fact that $A$ is dense in $(V, \tau)$ if and only if $A \cap U \neq \phi$ for every nonempty open subset $U$ of $V$. Specially, if $(V, \tau)$ is an Alexandroff topological space and $U_x$ is the smallest open set containing $x$ for every $x \in V$, then $A$ is dense in $(V, \tau)$ if and only if $A \cap U_x \neq \phi$ for every $x \in V$. In particular, $\{x \in V \mid \{x\} \in \tau\} \subseteq A$. Since $A_x$ is a nonempty open set in $(V, \tau_G)$, we have $A_x \cap A \neq \phi$. Let $y \in A_x \cap A$, then $x \in A_y$.*

**Proposition 8.2.** *Let $G = (V, E)$ be a connected graph which is not a star. Then the set of all vertices of $G$ with degree greater than one is dense in $(V, \tau_G)$.*

*Proof.* Let $A = \{x \in V \mid \deg x > 1\}$. By Remark 7.1, it is enough to show that $U_x \cap A \neq \phi$ for all $x \in V \setminus A$. If $x \in V \setminus A$, Then there exists some $y \in V$ such that $A_x = \{y\}$ and hence $U_x = A_y$, because $\deg x = 1$. Since $G$ is not a star, $\deg y > 1$ and there exists some $z \in A$ such that $z \in A_y$. Therefore $z \in A \cap A_y = A \cap U_x$. $\qquad\square$

**Corollary 8.3.** *Let $T = (V, E)$ be a tree which is not a star and $A$ a subset of $V$ and $B = \{x \in V \mid \deg x > 1\}$. Then $A$ is dense in $(V, \tau_T)$ if and only if $B \subseteq A$.*

*Proof.* $(\Rightarrow)$ If $A$ is dense in $(V, \tau_T)$, then by Remark 7.1, we have $\{x \in V \mid \{x\} \in \tau\} \subseteq A$. On the other hand, if $x \in B$, then $\{x\} \in \tau$, by proposition 3.7. This proves the necessity.
$(\Leftarrow)$ By Proposition 7.2 we have $\overline{B} = V$. So if $B \subseteq A$, then $\overline{A} = V$ and this is the sufficiency condition. $\qquad\square$

The following theorem which we could not find any proof for it elsewhere, is true in every Alexandroff topological space and specially in every graphic topological space.

**Theorem 8.4.** *Let $(V, \tau)$ be an Alexandroff topological space, $\mathcal{B}_\tau = \{U_x \mid x \in V\}$ and $\mathcal{B} = \{U_x \mid x \in V, U_x$ is minimal in $\mathcal{B}_\tau\}$.*
*1. If $A \subseteq V$ is a minimal dense subset in $(V, \tau)$, then there exists a surjective function $f : \mathcal{B} \longrightarrow A$ such that $f(U_x) \in U_x$ for every $U_x \in \mathcal{B}$. In particular, $A \subseteq \{x \in V \mid U_x$ is minimal in $\mathcal{B}_\tau\}$.*
*2. Conversely, if $f : \mathcal{B} \longrightarrow V$ is a function such that $f(U_x) \in U_x$ for every $U_x \in \mathcal{B}$, then $f(\mathcal{B})$ is a minimal dense subset in $(V, \tau)$.*
*Specially, if $A$ and $A'$ are minimal dense subsets in $(V, \tau)$, then we have $|A| = |A'|$.*

*Proof.* (i) By minimality of elements of $\mathcal{B}$, the intersection of every pair of distinct elements of $\mathcal{B}$ is empty. We claim that $U \cap A$ has a single element for each $U \in \mathcal{B}$. Since $\overline{A} = V$, there exists some $x \in U \cap A$, so $U_x \subseteq U$ and by minimality of $U$ we have $U_x = U$. Suppose by contrary that $y \in U \cap A \setminus \{x\}$. Then $U_y = U_x = U$. Therefore $z \in \overline{\{x\}}$ if and only if $z \in \overline{\{y\}}$. Hence $\overline{A \setminus \{y\}} = V$, which contradicts minimality of $A$. Now define $f(U)$ to be the single element of $U \cap A$ for every $U \in \mathcal{B}$. Suppose that $a \in A$. We show that $U_a \in \mathcal{B}$ which implies $f(U_a) = a$ and this will prove that $f$ is surjective. Suppose by contrary that $U_a \notin \mathcal{B}$, so there exists $U_x \in \mathcal{B}$ such that $U_x \subsetneq U_a$. If $b \in U_x \cap A$, then $U_x \cap A = \{b\}$ by above claim and so $U_x = U_b \subsetneq U_a$. Therefore $y \in \overline{\{a\}}$ implies that $y \in \overline{\{b\}}$ for every $y \in V$. Thus $\overline{A \setminus \{a\}} = V$, which is a contradiction.

The next statement, is an obvious consequence of the first one.

(ii) For every $x \in V$, there exists an element $a \in V$ such that $U_a \subseteq U_x$ and $U_a \in \mathcal{B}$. Therefore $f(U_a) \in U_x \cap f(\mathcal{B})$ and so $f(\mathcal{B})$ is dense in $V$. Now suppose that $\overline{A} = V$ and $A \subseteq f(\mathcal{B})$. Let $U_x \in \mathcal{B}$ and $f(U_x) \notin A$. Then there exists $U_y \in \mathcal{B}$ such that $f(U_y) \in U_x \cap A$. On one hand, $f(U_x) \notin A$ implies that $f(U_x) \in U_x \setminus A$ and on the other hand, we have $f(U_y) \in U_x \cap U_y$ which implies $U_x = U_y$ and so $f(U_x) = f(U_y) \in A$ which is a contradiction.

To prove the last statement, if $A$ is a minimal dense subset in $(V, \tau)$, then there exists a surjective function $f : \mathcal{B} \longrightarrow A$ such that $f(U_x) \in U_x$ for every $U_x \in \mathcal{B}$, by part (i). Hence $f$ must be injective, because if $f(U_x) = f(U_y)$ for some $U_x, U_y \in \mathcal{B}$, then $U_x \cap U_y \neq \phi$ and so $U_x = U_y$ by their minimality. Therefore $|A| = |\mathcal{B}|$ is constant. $\square$

**Corollary 8.5.** *Let $G = (V, E)$ be a graph and $A \subseteq V$. If $A$ is a minimal dense set in $(V, \tau_G)$, then $A \subseteq \{x \in V \,|\, A_x \text{ is maximal in } \mathcal{S}_G\}$. More precisely, if $A_{x_0}$ is maximal in $\mathcal{S}_G$ and $V_0 = \{x \in V \,|\, A_x = A_{x_0}\}$, then $|A \cap V_0| = 1$.*

*Proof.* By Remark 5.5, we have

$$\{x \in V \,|\, U_x \text{ is minimal in } \mathcal{B}_G\} = \{x \in V \,|\, A_x \text{ is maximal in } \mathcal{S}_G\}.$$

Now the first statement is obvious by part (i) of Theorem 7.4. To prove the next statement, note that $A_{x_0}$ is maximal if and only if $U_{x_0}$ is minimal. Hence $A_x = A_{x_0}$ if and only if $U_x = U_{x_0}$ if and only if $x \in U_{x_0}$. Therefore $V_0 = U_{x_0}$ and so $A \cap V_0 \neq \phi$. Now if $x \neq y$, $x, y \in A \cap V_0$ then similar to the proof of Theorem 7.4, we have $\overline{A \setminus \{y\}} = V$ and this contradicts minimality of $A$. $\square$

**Definition 8.6.** *A $k-$coloring of a graph $G$ is a labeling $c : V(G) \longrightarrow S$, where $|S| = k$. A $k-$coloring is proper if adjacent vertices have different labels, i.e., $xy \in E(G)$ implies $c(x) \neq c(y)$ for every $x, y \in V(G)$. A graph is $k-$colorable if it has a proper $k-$coloring. The chromatic number $\chi(G)$ is the least $k$ such that $G$ is $k-$colorable.*

**Proposition 8.7.** *Let $G = (V, E)$ be a finite graph and $A \subseteq V$. If $A$ is a minimal dense set in $(V, \tau_G)$, then $\chi(G) \leq |A|$.*

*Proof.* Let $A = \{a_1, \ldots, a_k\}$ and note that $A_{a_i}$ is maximal in $\mathcal{S}_G$, by Corollary 7.5. We define a coloring $c : V \longrightarrow \{1, \ldots, k\}$. For every $1 \leq i \leq k$, set $c(a_i) = i$ and for every $x \in V \setminus A$ there exists some $x_0 \in V$ such that $A_x \subseteq A_{x_0}$ and $A_{x_0}$ is maximal in $\mathcal{S}_G$. By Corollary 7.5, there

exists some $1 \leq i \leq k$ such that $A_{a_i} = A_{x_0}$. Therefore $A_x \subseteq A_{a_i}$. We choose such $i$ and put $c(x) = i$. Suppose that $c(x) = c(y) = i$. Then we have $A_x \subseteq A_{a_i}$ and $A_y \subseteq A_{a_i}$. This implies $U_{a_i} \subseteq U_x \cap U_y$. Hence $U_x \cap U_y \neq \phi$ and so $x$ is not adjacent to $y$, by Corollary 2.6. $\qquad\square$

**Example 5.** *Suppose that $A$ be a minimal dense set of graphic topology of $K_{n,m}$, then $\chi(K_{n,m}) = 2$ which is equal to $|A|$.*

**Proposition 8.8.** *Let $G = (V, E)$ be a finite connected graph and $A \subseteq V$. If $\overline{A} = V$, then the induced subgraph on $A$ is connected.*

*Proof.* Let $H = G[A]$ be the induced subgraph on $A$. If $H$ is not connected, then we can choose a minimal vertex cut $C \subseteq V \setminus A$. By Proposition 3.7, $C$ is an open subset of $V$. However, $C \cap A = \phi$ and this contradicts density of $A$. $\qquad\square$

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