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UNISERIAL MODULES OF GENERALIZED POWER SERIES

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ABSTRACT. Let R be a ring, M a right R-module and $(S, \leq)$ a strictly ordered monoid. In this paper we will show that if $(S, \leq)$ is a strictly ordered monoid satisfying the condition that $0 \leq s$ for all $s \in S$, then the module $[[M^{S,\leq}]]$ of generalized power series is a uniserial right $[[R^{S,\leq}]]$-module if and only if M is a simple right R-module and S is a chain monoid.

1. Introduction

A module is said to be *uniserial* if any two of its submodules are comparable with respect to inclusion, i.e., any two of its cyclic submodules are comparable by set inclusion. A ring R is said to be *right* (respectively, *left*) *uniserial* if R_R (respectively, ${}_RR$) is a uniserial module. Uniserial modules are also called chain modules in some literature. Let R be a ring and M a right R-module. In [7], among others, it was proved that $M[[x]]$ is a uniserial right $R[[x]]$-module if and only if M is a simple right module. In recent years, many researchers (for example, Liu [2], Varadarajan [8, 9]) have carried out an extensive study of modules of generalized power series. Motivated by these facts, in this paper, we study the uniserial condition for generalized power series modules,

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with restriction to monoids of exponents with all nonnegative elements. Our result extends Tuganbaev's result to such modules and partially generalizes the results of [5, Theorem 4.3].

Throughout this paper, all rings are associative with identity and all modules are unitary. If R is a ring, then the group of invertible elements of R is denoted by $U(R)$. Regarding ordered sets, monoids and ordered monoids we will be following the terminology in [6].

Let $(S, \leq)$ be an ordered set. Recall that $(S, \leq)$ is *artinian* if every strictly decreasing sequence of elements of S is finite, and that $(S, \leq)$ is *narrow* if every subset of pairwise order-incomparable elements of S is finite. Let S be a commutative monoid. Unless stated otherwise, the operation of S will be denoted additively, and the neutral element by 0. The following definition is due to Ribenboim (see [6]).

Assume that $(S, \leq)$ is a strictly ordered monoid, that is, $(S, \leq)$ is an ordered monoid satisfying the condition that if $s, s', t \in S$ and $s < s'$, then $s + t < s' + t$, and R is a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f : S \longrightarrow R$ such that $\mathrm{supp}(f) = \{s \in S \mid f(s) \neq 0\}$ is artinian and narrow. With pointwise addition, $[[R^{S,\leq}]]$ is an abelian additive group. For every $s \in S$ and $f, g \in [[R^{S,\leq}]]$, let

$$X_s(f, g) = \{(u, v) \in S \times S \mid u + v = s, f(u) \neq 0, g(v) \neq 0\}.$$

It follows from [6, 4.1] that $X_s(f, g)$ is finite. This fact allows one to define the operation of convolution:

$$(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)g(v).$$

With this operation, and pointwise addition, $[[R^{S,\leq}]]$ becomes an associative ring, with identity element e, namely $e(0) = 1, e(s) = 0$ for every $0 \neq s \in S$. This is called the *ring of generalized power series* with coefficients in R and exponents in S. To any $r \in R$ and $s \in S$, we associate the maps c_r, $e_s \in [[R^{S,\leq}]]$ defined by

$$c_r(x) = \begin{cases} r, & \text{if } x = 0, \\ 0, & \text{otherwise}, \end{cases} \qquad e_s(x) = \begin{cases} 1, & \text{if } x = s, \\ 0, & \text{otherwise}. \end{cases}$$

It is clear that $r \mapsto c_r$ is a ring embedding of R into $[[R^{S,\leq}]]$, $s \mapsto e_s$, is a monoid embedding of S into the multiplicative monoid of the ring $[[R^{S,\leq}]]$, and $c_r e_s = e_s c_r$.

In [2, 8, 9], the notion of generalized power series rings was extended to modules. Let M be a right R-module and $(S, \leq)$ a strictly ordered monoid. We denote by $[[M^{S,\leq}]]$ the set of all maps $\phi : S \longrightarrow M$ such that

$\mathrm{supp}(\phi)$ is artinian and narrow, where $\mathrm{supp}(\phi) = \{s \in S \mid \phi(s) \neq 0\}$. With pointwise addition, $[[M^{S,\leq}]]$ is an abelian additive group. For each $s \in S$, $f \in [[R^{S,\leq}]]$ and $\phi \in [[M^{S,\leq}]]$, let

$$X_s(\phi, f) = \{(u, v) \in S \times S \mid u + v = s, \phi(u) \neq 0, f(v) \neq 0\}.$$

Then by[2, Lemma 1], $X_s(\phi, f)$ is finite. This allows one to define the scalar multiplication as follows:

$$(\phi f)(s) = \sum_{(u,v)\in X_s(\phi,f)} \phi(u)f(v).$$

With this operation and pointwise addition, $[[M^{S,\leq}]]$ becomes a right $[[R^{S,\leq}]]$-module, which is called the *module of generalized power series* with coefficients in M and exponents in S. To any $m \in M$ and any $s \in S$, we associate the map $d_m^s \in [[M^{S,\leq}]]$ via

$$d_m^s(x) = \begin{cases} m, & \text{if } x = s, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $m \mapsto d_m^0$ is a module embedding of M into $[[M^{S,\leq}]]$.

For example, if $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the usual order, then

$$[[M^{\mathbb{N}\cup\{0\},\leq}]]_{[[R^{\mathbb{N}\cup\{0\},\leq}]]} \cong M[[x]]_{R[[x]]},$$

the right $R[[x]]$-module of formal power series over M. If $S = \mathbb{Z}$ and $\leq$ is the usual order, then $[[M^{\mathbb{Z},\leq}]]_{[[R^{\mathbb{Z},\leq}]]} \cong M[[x^{-1}, x]]_{R[[x^{-1},x]]}$, the Laurent series extension of M_R. If S is a commutative monoid and $\leq$ is the trivial order, then $[[M^{S,\leq}]]_{[[R^{S,\leq}]]} \cong M[S]_{R[S]}$, the monoid extensions of S over M_R. Further examples are given in [2].

2. **Main results**

Recall from [4] that a strictly ordered monoid $(S, \leq)$ is said to be a *positively strictly ordered* if $0 \leq s$ for all $s \in S$. Note that in this case, $(\phi f)(0) = \phi(0)f(0)$ for any $\phi \in [[M^{S,\leq}]]$ and any $f \in [[R^{S,\leq}]]$. The following result appeared in [3, Lemma 5.2].

Lemma 2.1. *Let R be a ring, $(S, \leq)$ a positively strictly ordered monoid and $f \in [[R^{S,\leq}]]$. Then $f \in U([[R^{S,\leq}]])$ if and only if $f(0) \in U(R)$.*

Following [1], a monoid S is said to be a *chain* if the ideals of S are totally ordered by set inclusion, i.e., for any $s, t \in S$, either $s+S \subseteq t+S$ or $t + S \subseteq s + S$. The following result appeared in [4, Lemma 4].

Lemma 2.2. *Let $(S, \leq)$ be a positively strictly ordered monoid. If S is a chain monoid, then $(S, \leq)$ is a totally ordered monoid.*

Remark 2.3. *The example of the monoid $S = (\mathbb{N}, \cdot)$ shows that the converse of Lemma 2.2 is false.*

Lemma 2.4. *Let M be a right R-module and $(S, \leq)$ a strictly ordered monoid. Assume that $W = \{\phi \in [[M^{S,\leq}]] \mid \phi(0) = 0\}$. If $(S, \leq)$ is positive, then W is an $[[R^{S,\leq}]]$-submodule of $[[M^{S,\leq}]]$.*

Proof. Let $\phi \in W$, $f \in [[R^{S,\leq}]]$. Then

$$(\phi f)(0) = \sum_{(u,v)\in X_0(\phi,f)} \phi(u)f(v) = \phi(0)f(0) = 0.$$

This means that $\phi f \in W$. Now it is easy to see that W is an $[[R^{S,\leq}]]$-submodule of $[[M^{S,\leq}]]$. □

Now, we are ready to prove the main result of our discussion.

Theorem 2.5. *Let $(S, \leq)$ be a positively strictly ordered monoid and M a nonzero right R-module. Then the following conditions are equivalent:*
(1) *$[[M^{S,\leq}]]$ is a uniserial right $[[R^{S,\leq}]]$-module.*
(2) *M is a simple right R-module and S is a chain monoid.*
(3) *For any $0 \neq \varphi \in [[M^{S,\leq}]]$, there exists an $s \in S$ such that $\varphi[[R^{S,\leq}]] = [[M^{S,\leq}]]e_s$.*

Proof. (1)$\Longrightarrow$(2). First we show that S is a chain monoid. Let s, $t \in S$. For any $0 \neq m \in M$, since $[[M^{S,\leq}]]$ is a uniserial right $[[R^{S,\leq}]]$-module, without loss of generality, we can assume $d_m^s[[R^{S,\leq}]] \subseteq d_m^t[[R^{S,\leq}]]$. Then there exists an $f \in [[R^{S,\leq}]]$ such that $d_m^s = d_m^t f$. Thus from

$$0 \neq m = d_m^s(s) = (d_m^t f)(s) = \sum_{(u,v)\in X_s(d_m^t,f)} d_m^t(u)f(v),$$

it follows that $t + v = s$ for some $v \in S$. Consequently $s \in t + S$. Hence S is a chain monoid.

Next we show that M is a simple right R-module. Let $0 \neq m \in M$. We will show that $M = mR$. Set $A = d_m^0[[R^{S,\leq}]]$. Then for any $\phi \in A$, $\phi = d_m^0 f$ for some $f \in [[R^{S,\leq}]]$. Thus, for any $s \in S$, $\phi(s) = (d_m^0 f)(s) = mf(s) \in mR$. Let

$$B = \{\phi \in [[M^{S,\leq}]] | \phi(0) = 0\}.$$

Then B is a submodule of $[[M^{S,\leq}]]$ by Lemma 2.4 and $B \subseteq A$ since $[[M^{S,\leq}]]$ is a uniserial right $[[R^{S,\leq}]]$-module and $0 \neq m \in M$. Let $0 \neq s \in$

S and $0 \neq n \in M$. Then $d_n^s \in B$, and so $d_n^s \in A$. Hence $n = d_n^s(s) \in mR$. This implies that $M = mR$. Hence M is a simple right R-module.

(2)$\Longrightarrow$(3). Since $(S, \leq)$ is a positively strictly ordered monoid and S is a chain monoid, $(S, \leq)$ is strictly totally ordered by Lemma 2.2. Hence, for any $0 \neq \phi \in [[M^{S,\leq}]]$, $\mathrm{supp}(\phi)$ contains a minimal element, which we denote by $\pi(\phi)$.

Let φ be a nonzero element of $[[M^{S,\leq}]]$. Assume that $\pi(\varphi) = s_0$. We will show that $\varphi[[R^{S,\leq}]] = [[M^{S,\leq}]]e_{s_0}$.

First we show that there exists a $\varphi' \in [[M^{S,\leq}]]$ such that $\varphi = \varphi' e_{s_0}$. Since $(S, \leq)$ is a strictly totally ordered monoid, it is easy to see that for the element $\varphi' : S \longrightarrow M$ defined via $\varphi'(s) = \varphi(s + s_0)$, we have $\varphi' \in [[M^{S,\leq}]]$.

Let $s_0 \leq s$. Since $(S, \leq)$ is a positively ordered chain monoid, there exists $s' \in S$ such that $s = s_0 + s'$. Otherwise, $s_0 = s + v$ for some $0 \neq v \in S$ and we get $s + v = s_0 \leq s$. Since $0 < v$ we get a contradiction. Hence

$$\begin{aligned}\varphi(s) &= \varphi(s_0 + s') = \varphi'(s') = \varphi'(s')e_{s_0}(s_0) \\ &= \sum_{(u,v)\in X_{s'+s_0}(\varphi', e_{s_0})} \varphi'(u)e_{s_0}(v) = (\varphi' e_{s_0})(s' + s_0) = (\varphi' e_{s_0})(s).\end{aligned}$$

This means that $\varphi = \varphi' e_{s_0}$.

Next we show that $\varphi[[R^{S,\leq}]] = [[M^{S,\leq}]]e_{s_0}$. Let $m_0 = \varphi'(0)$. Since M is a simple right R-module, $M = m_0 R$. Thus $[[M^{S,\leq}]] = [[(m_0R)^{S,\leq}]] = d_{m_0}^0[[R^{S,\leq}]]$, and, for any $s \in \mathrm{supp}(\varphi')$, there exists an $r_s \in R$ such that $\varphi'(s) = m_0 r_s$. Define $f : S \longrightarrow R$ via:

$$f(s) = \begin{cases} 1, & s = 0, \\ r_s, & 0 \neq s \in \mathrm{supp}(\varphi'), \\ 0, & s \notin \mathrm{supp}(\varphi'). \end{cases}$$

Clearly $f \in [[R^{S,\leq}]]$. For any $s \in S$,

$$\varphi'(s) = m_0 r_s = m_0 f(s) = \sum_{(u,v)\in X_s(d_{m_0}^0, f)} d_{m_0}^0(u) f(v) = (d_{m_0}^0 f)(s).$$

Thus $d_{m_0}^0 f = \varphi'$. Since $f(0) = 1$, by Lemma 2.1, $f \in U([[R^{S,\leq}]])$. Hence

$$\begin{aligned}[[M^{S,\leq}]]e_{s_0} &= d_{m_0}^0[[R^{S,\leq}]]e_{s_0} = d_{m_0}^0 f[[R^{S,\leq}]]e_{s_0} \\ &= d_{m_0}^0 f e_{s_0}[[R^{S,\leq}]] = \varphi' e_{s_0}[[R^{S,\leq}]] \\ &= \varphi[[R^{S,\leq}]].\end{aligned}$$

(3)$\Longrightarrow$(1). First we show that S is a chain monoid. Let $u \neq v \in S$. Fix any $0 \neq m \in M$. Then $0 \neq d_m^u + d_m^v \in [[M^{S,\leq}]]$. By (3), there exists an $s \in S$ such that

$$d_m^u + d_m^v = \phi e_s \quad \text{and} \quad d_m^s = (d_m^u + d_m^v)f$$

for some $\phi \in [[M^{S,\leq}]]$ and some $f \in [[R^{S,\leq}]]$. From $0 \neq m = (d_m^u + d_m^v)(v) = (\phi e_s)(v)$ it follows that $v \in \text{supp}(\phi e_s) \subseteq s + S$ and a similar argument shows that $u \in s + S$. On the other hand,

$$0 \neq m = d_m^s(s) = [(d_m^u + d_m^v)f](s) = (d_m^u f)(s) + (d_m^v f)(s),$$

and thus $(d_m^u f)(s) \neq 0$ or $(d_m^v f)(s) \neq 0$. In the first case, we obtain $s \in u + S$ and so $v \in u + S$ follows. Similarly, in the second case we have $u \in v + S$. Hence S is a chain monoid.

Secondly, we show that $[[M^{S,\leq}]]$ is a uniserial right $[[R^{S,\leq}]]$-module. Let $0 \neq \varphi, \psi \in [[M^{S,\leq}]]$. By (3), there exist $s, t \in S$ such that $\varphi[[R^{S,\leq}]] = [[M^{S,\leq}]]e_s$, $\psi[[R^{S,\leq}]] = [[M^{S,\leq}]]e_t$. Since S is a chain, $s \in t + S$ or $t \in s + S$. Assume that $s = t + u$, $u \in S$. Then

$$\varphi[[R^{S,\leq}]] = [[M^{S,\leq}]]e_s = [[M^{S,\leq}]]e_u e_t \leq [[M^{S,\leq}]]e_t = \psi[[R^{S,\leq}]].$$

Hence, $[[M^{S,\leq}]]$ is a uniserial right $[[R^{S,\leq}]]$-module. □

Let M be a right R-module. Recall that M is a *serial* module if M is a direct sum of uniserial modules. A ring R is called a *right* (respectively, *left*) *serial* ring, if R_R (respectively, ${}_RR$) is a serial module. A ring R is called a *serial* ring, if R is both a right and a left serial ring.

Lemma 2.6. *Let M be a right R-module, $(S, \leq)$ a positively strictly ordered monoid and S a chain monoid. If M is a semisimple artinian module, then $[[M^{S,\leq}]]$ is a serial right $[[R^{S,\leq}]]$-module.*

Proof. Assume that $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$, where M_i is a simple right R-module, $i = 1, 2, \ldots, n$, $n \in \mathbb{N}$. Then $[[M^{S,\leq}]] \cong [[M_1^{S,\leq}]] \oplus [[M_2^{S,\leq}]] \oplus \cdots \oplus [[M_n^{S,\leq}]]$. By Theorem 2.5, $[[M_i^{S,\leq}]]$ is a uniserial right $[[R^{S,\leq}]]$-module. Therefore, $[[M^{S,\leq}]]$ is a serial right $[[R^{S,\leq}]]$-module. □

Lemma 2.7. *Let R be a ring and $(S, \leq)$ a strictly totally ordered monoid. Then R is a semiprime ring if and only if $[[R^{S,\leq}]]$ is a semiprime ring.*

Proof. $\Longrightarrow$) Assume the contrary. Then there exists a nonzero $f \in [[R^{S,\leq}]]$ such that $([[R^{S,\leq}]]f[[R^{S,\leq}]])^2 = 0$. Thus $f[[R^{S,\leq}]]f = 0$. Let $\pi(f) = s_0$. Then $f(s_0)Rf(s_0) = 0$. Set $I = Rf(s_0)R$. Then I is a nonzero ideal of R and $I^2 = 0$, which is contradict to the fact that R is a semiprime ring.

$\Longleftarrow$) Let I be an ideal of ring R with $I^2 = 0$. Then $[[I^{S,\leq}]]$ is an ideal of the ring $[[R^{S,\leq}]]$. For any $f, g \in [[I^{S,\leq}]]$ and any $s \in S$,

$$(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)g(v) = 0.$$

Thus $fg = 0$, which implies that $[[I^{S,\leq}]]^2 = 0$. Hence $[[I^{S,\leq}]] = 0$ since $[[R^{S,\leq}]]$ is a semiprime ring. Consequently, $I = 0$, and so R is a semiprime ring. □

Theorem 2.8. *Let R be a ring, $(S, \leq)$ a positively strictly ordered monoid and S a chain monoid. Then the following are equivalent:*
(1) R is a semisimple artinian ring,
(2) $[[R^{S,\leq}]]$ is a right serial ring,
(3) $[[R^{S,\leq}]]$ is a serial semiprime ring.

Proof. (3)$\Longrightarrow$(2) is obvious.

(1)$\Longrightarrow$(3). Note that semisimple artinian rings are semiprime rings. Thus by Lemma 2.2 and Lemma 2.7, $[[R^{S,\leq}]]$ is a semiprime ring. On the other hand, by Lemma 2.6 and its left version, $[[R^{S,\leq}]]$ is a serial ring.

(2)$\Longrightarrow$(1). Since right serial rings are right finite-dimensional semiperfect rings, $[[R^{S,\leq}]]$ is a right finite-dimensional semiperfect ring. Thus R, as a quotient ring of $[[R^{S,\leq}]]$, is a right finite-dimensional semiperfect ring. Hence, there exists a complete set $\{e_1, e_2, \ldots, e_n\}$ of pairwise orthogonal primitive idempotents in R such that $R = e_1R + e_2R + \cdots + e_nR$. Thus

$$\begin{aligned}[[R^{S,\leq}]] &\cong [[(e_1R)^{S,\leq}]] \oplus [[(e_2R)^{S,\leq}]] \oplus \cdots \oplus [[(e_nR)^{S,\leq}]] \\ &= c_{e_1}[[R^{S,\leq}]] \oplus c_{e_2}[[R^{S,\leq}]] \oplus \cdots \oplus c_{e_n}[[R^{S,\leq}]],\end{aligned}$$

and c_{e_i} is a primitive idempotent of $[[R^{S,\leq}]]$, for all $i = 1, 2, \ldots, n$. In fact, if there exist $f^2 = f$, $g^2 = g \in [[R^{S,\leq}]]$ such that $c_{e_i} = f + g$, then $e_i = c_{e_i}(0) = f(0) + g(0)$, and $f(0)^2 = f(0)$, $g(0)^2 = g(0)$. Since e_i is primitive, either $f(0) = 0$ or $g(0) = 0$. If $f(0) = 0$, then $f = 0$. In fact, if $f \neq 0$, then $\mathrm{supp}(f)$ is a nonempty set. Set $\pi(f) = s$. Since $(S, \leq)$ is a positively strictly ordered monoid, $X_s(f, f) = \{(0, s), (s, 0)\}$. Thus

$$0 \neq f(s) = f^2(s) = \sum_{(u,v)\in X_s(f,f)} f(u)f(v) = 0,$$

which is a contradiction. Hence $c_{e_i}[[R^{S,\leq}]]$ is a uniserial right $[[R^{S,\leq}]]$-module, $i = 1, 2, \ldots, n$. Thus, by Theorem 2.5, each e_iR is a simple right R-module. Therefore, R is a semisimple artinian ring. □

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