ON THE DENSENESS OF THE INVERTIBLE GROUP IN UNIFORM FRÉCHET ALGEBRAS

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ABSTRACT. We first extend the Arens-Royden theorem to unital, commutative Fréchet algebras under certain conditions. Then, we show that if A is a uniform Fréchet algebra on $X = M_A$, where M_A is the continuous character space of A, then A does not have dense invertible group, if we impose some conditions on X. On the other hand, if A has dense invertible group, then it is shown that $A = C(X)$, with certain conditions on X. Thus, the results of Dawson and Feinstein on denseness of the invertible group in Banach algebras are extended to uniform Fréchet algebras.

1. **Introduction**

Here, we assume that all algebras are unital and commutative.

Let B be a unital commutative Banach algebra (B-algebra) with the character space (maximal ideal space) $M_B = X$, and let $G(B)$ denote the group of invertible elements in B and $exp(B) = \{e^x : x \in B\}$. The multiplicative group $G(B)/\exp(B)$ is denoted by $H^1(B)$. The Arens-Royden theorem asserts that $H^1(B)$ and $H^1(C(X))$ are isomorphic [8, p. 413], or in other words, for every f in $C(X)$, which does not vanish

on X, there exists $g \in G(B)$ such that $f/\hat{g}$ has a continuous logarithm on X [5, III. Theorem 7.2].

Here, we obtain a result similar to the Arens-Royden theorem, for certain unital commutative Fréchet algebras, and then we extend some results of Dawson and Feinstein on denseness of the invertible group [3] to uniform Fréchet algebras.

Definition 1.1. *Let A be a topological algebra. Then, A is a locally multiplicatively convex algebra or an LMC algebra if there is a base of neighbourhoods (V_α) of the origin consisting of sets which are absolutely convex and multiplicative, i.e., $V_\alpha \cdot V_\alpha \subseteq V_\alpha$. Equivalently, an LMC algebra is a topological algebra whose topology is defined by a separating family (p_α) of submultiplicative seminorms, i.e., $p_\alpha(fg) \le p_\alpha(f)p_\alpha(g)$, for all $f, g \in A$.*

A Fréchet algebra (F-algebra) A is an LMC-algebra which is also a complete metrizable space. Its topology can be defined by an increasing sequence (p_n) of submultiplicative seminorms. Without loss of generality, we may assume that $p_n(1) = 1$, for all $n \in \mathbb{N}$, if A has a unit. A uniform Fréchet algebra (uF-algebra) is a Fréchet algebra A with the defining sequence (p_n) of seminorms such that, for all $f \in A$ and $n \in \mathbb{N}$, $p_n(f^2) = (p_n(f))^2$ [6, Definition 4.1.2]. For those terms concerning topological algebras or Fréchet algebras, which are not defined here, one may refer, for example, to [2], [6] and [7].

Definition 1.2. *The weak* topology on the dual space A^* is denoted by $\sigma = \sigma(A^*, A)$, so that $\varphi_\nu \to \varphi$ in (A^*, σ) if and only if $\varphi_\nu(f) \to \varphi(f)$, for all $f \in A$. The continuous character space of an F-algebra $(A, (p_n))$, denoted by M_A, is the set of all non-zero continuous complex-valued homomorphisms on A. The space M_A is taken to have the related weak* topology from A^*, which is called the Gelfand topology. We always endow M_A with the Gelfand topology.*

The Gelfand transform of an element $f \in A$ is defined by $\hat{f}(\varphi) = \varphi(f)$, for all $\varphi \in A^*$. The definition of the weak* topology shows immediately that $\hat{f}$ is continuous on (A^*, σ). We also take $\hat{A}$ to be the set of all Gelfand transforms $\hat{f}$ of elements f in A, see, for example, [2, Section 4.10].

Definition 1.3. *A Hausdorff space X is hemicompact if there exists a sequence (K_n) of compact subsets of X such that, for all $n \in \mathbb{N}$,*

$K_n \subseteq K_{n+1}$, and each compact subset K of X is contained in some K_n. The sequence (K_n) is called an *admissible exhaustion* of X.

A Hausdorff space is a *k-space* if every subset intersecting each compact subset in a closed set is itself closed.

Examples of k-spaces are locally compact spaces and first countable spaces [4, p. 248]. It is also known that if X is a σ-compact and locally compact space then it is hemicompact. Moreover, if X is a hemicompact k-space, then $C(X)$ is a Fréchet algebra with respect to the compact open topology [6, Remark 3.1.10]. Note that a complex-valued function f on a k-space X is continuous if and only if it is continuous on each compact subset of X. Hence, whenever X is a hemicompact k-space with the admissible exhaustion (X_n), a necessary and sufficient condition for the continuity of a complex-valued function f on X is that f is continuous on each X_n.

For an F-algebra $(A, (p_n))$, let A_n be the completion of $A/\ker p_n$ with respect to the norm $p'_n(f + \ker p_n) = p_n(f)$ $(f \in A)$. It is known that $A = \varprojlim A_n$, the projective limit of the sequence (A_n) of B-algebras. Furthermore, $M_A = \bigcup_{n=1}^{\infty} M_{A_n}$, as sets, in a natural way. In fact, M_A is a hemicompact space with the admissible exhaustion (M_{A_n}). For further information, see, for example, [6, pp. 77-80] or [2, pp. 581-582].

We now give examples of hemicompact k-spaces and uniform algebras on them.

Example 1.4. (i) *Let U be an open subset of the complex plane. Then, U is the union of a sequence K_n of compact subsets of U such that for all $n \in \mathbb{N}$, $K_n \subseteq K_{n+1}$, and each compact subset K of U is contained in some K_n [10, Theorem 13.3]. Hence, U is a hemicompact space. Since U is locally compact, it is also a k-space. Now, let $C(U)$ denote the algebra of continuous functions on U. If we define $p_n(f) = \|f\|_{K_n}$ $(f \in C(U))$, then (p_n) is an increasing sequence of submultiplicative seminorms on $C(U)$, which defines the compact-open topology on $C(U)$ and with respect to this topology, $C(U)$ is a uniform Fréchet algebra. Also, $C(U) = \varprojlim C(K_n)$, the projective limit of the sequence $(C(K_n))$ of B-algebras.*

(ii) *Let $A(U)$ denote the algebra of analytic functions on U. Then, $A(U)$ is a closed subalgebra of $C(U)$, and so it is a uniform Fréchet algebra.*

Definition 1.5. *Let X be a non-empty topological space, and let A be an algebra of complex functions on X. Then, A is a function algebra on*

X if A separates the points of X and contains the constants and if the A-topology on X is the given topology. The algebra A is a Fréchet function algebra (Ff-algebra) or a Banach function algebra (Bf-algebra) on X if A is a function algebra which is also an F-algebra or a B-algebra, respectively, with respect to some topology [2, Definition 4.1.1].

It is interesting to note that if there is a function algebra on a topological space X, then X must be Hausdorff and completely regular.

Definition 1.6. *Let A be an Ff-algebra (Bf-algebra) on X such that the evaluation homomorphisms $\varphi_x : A \longrightarrow \mathbb{C}$ are all continuous, where $\varphi_x(f) = f(x)$, for $f \in A$ and $x \in X$. Obviously, the map $x \longrightarrow \varphi_x$ from X into M_A is continuous and injective. If this map is also surjective, then it is in fact a homeomorphism from X onto M_A, since the topology on X is the A-topology. In this case, we say that A is a natural Ff-algebra (Bf-algebra) on X, and we identify X with M_A.*

Note that evaluation homomorphisms are always continuous in Bf-algebras.

2. Continuous Functions of Exponential Type

In this section, we impose some conditions on the elements of $G(C(X))$ to guarantee that they are in $\exp(C(X))$, when X is a hemicompact space.

First, we need the following elementary lemma, which can be proved by a classical argument.

Lemma 2.1. *Let X be a connected, Hausdorff space. If $f, g \in C(X)$ such that $\exp f = \exp g$ on X, then there exists an integer k such that $f = g + 2k\pi i$ on X.*

Theorem 2.2. *Let X be a hemicompact k-space with an admissible exhaustion (X_n) such that each X_n is connected. If $f \in C(X)$ and $f|_{X_n} \in \exp(C(X_n))$, for all $n \in \mathbb{N}$, then $f \in \exp(C(X))$.*

Proof. By the hypothesis, for each $n \in \mathbb{N}$, there exists $g_n \in C(X_n)$ such that $f = \exp g_n$ on X_n. Since $\exp g_{n+1} = \exp g_n$ on X_n, by Lemma 2.1, there exists an integer $k_n \in \mathbb{Z}$ such that $g_{n+1} = g_n + 2k_n\pi i$ on X_n. Taking $k_0 = 0$ and defining g on each X_n by $g = g_n - 2(\sum_{j=0}^{n-1} k_j)\pi i$,

we observe that g is a well-defined continuous function on X. It is then clear that $f = \exp g$ on X. $\hfill\square$

Theorem 2.3. *Let X be a hemicompact k-space with an admissible exhaustion (X_n) such that each X_n has finitely many components. If $f \in C(X)$ and $f|_{X_n} \in \exp(C(X_n))$, for all $n \in \mathbb{N}$, then $f \in \exp(C(X))$.*

Proof. Let $x \in X$ and $n \in \mathbb{N}$. If $x \in X_n$, take $C_n(x)$ to be the component of X_n containing x; otherwise, take $C_n(x) = \phi$. Clearly, each $C_n(x)$ is compact and $C_n(x) \subseteq C_{n+1}(x)$. For each $x \in X$, take $C(x) = \bigcup_{n=1}^{\infty} C_n(x)$. We will show that $C(x)$ is a hemicompact k-space with the admissible exhaustion $(C_n(x))$.

Obviously, any closed subset of a hemicompact k-space is again a hemicompact k-space. So, it is enough to show that $C(x)$ is closed. Let K be an arbitrary compact subset of X having nonempty intersection with $C(x)$. Since X is hemicompact, $K \subseteq X_n$, for some $n \in \mathbb{N}$. Let $Y_1, Y_2, \ldots, Y_m$ be components of X_n which have nonempty intersection with $C(x)$. For each i, $1 \le i \le m$, there exists a large enough $k_i \in \mathbb{N}$ such that $k_i \ge n$ and $Y_i \cap C_{k_i}(x) \ne \phi$. Since Y_i is connected and $C_{k_i}(x)$ is a component of X_{k_i}, it follows that $Y_i \subseteq C_{k_i}(x)$ $(1 \le i \le m)$. Hence, $\bigcup_{i=1}^{m} Y_i \subseteq C_k(x)$, where $k = \max\{k_1, k_2, \ldots, k_m\}$, and so
$$K \cap C(x) \subseteq C(x) \cap X_n = C(x) \cap (\cup_{i=1}^{m} Y_i) \subseteq C(x) \cap C_k(x) = C_k(x).$$

Therefore, $K \cap C(x) = K \cap C_k(x)$, which implies that $K \cap C(x)$ is a closed subset of X. Since X is a k-space, we conclude that $C(x)$ is a closed subset of X.

Now, let K be a compact subset of $C(x)$. By following a similar argument as in the above paragraph, we have $K = K \cap C(x) = K \cap C_k(x)$, for some $k \in \mathbb{N}$, and hence $(C_n(x))$ is an admissible exhaustion of $C(x)$.

It is easy to see that for every $x, y \in X$, either $C(x) = C(y)$ or $C(x) \cap C(y) = \phi$. Therefore, the family $\{C(x) : x \in X\}$ is, in fact, a partition of X. We may choose Y to be a subset of X such that X is disjoint union of the $C(x)$, when x runs over Y.

Now, let $f \in C(X)$ and $f|_{X_n} \in \exp(C(X_n))$, for each $n \in \mathbb{N}$. Since $f|_{C_n(x)} \in \exp(C(C_n(x)))$, for each $x \in Y$ and $n \in \mathbb{N}$ when $C_n(x) \ne \phi$, then by Theorem 2.2, for every $x \in Y$, there exists a continuous function g_x on $C(x)$ such that $f = \exp g_x$ on $C(x)$. Let g be a function defined on X by $g = g_x$ on $C(x)$, for each $x \in Y$. Clearly, g is well-defined. Since for every $x \in Y$, each component of X_n having nonempty intersection with $C(x)$, is contained in some $C_k(x)$, we conclude that g is continuous

on each component of X_n, and so it is continuous on X_n. Therefore, g is continuous on X. Clearly, $f = \exp g$, which completes the proof of the theorem.   □

In general, when X_n has infinitely many components, we do not know whether the above theorem is still valid. However, the following example shows that the result may be true even if X_n has infinitely many components.

Example 2.4. *Let $M = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, and $X = \mathbb{R} \times M$. For each $n \in \mathbb{N}$, take $X_n = [-n, n] \times M$. Then, (X_n) is an admissible exhaustion of X, i.e., X is a hemicompact space. Moreover, X is a k-space [4, p. 249]. Since for each $n \in \mathbb{N}$, $\mathbb{C} \setminus X_n$ is connected, $H^1(C(X_n))$ is trivial, for all $n \in \mathbb{N}$ ([1] and [3]). Now, it is clear from Theorem 2.2 that the group $H^1(C(X)) = G(C(X))/\exp(C(X))$ is also trivial.*

Clearly, if $f \in C(X)$ and $f|_{X_n} \in \exp(C(X_n))$, for each $n \in \mathbb{N}$, then $f(x) \neq 0$, for every $x \in X$, and hence $f \in G(C(X)) = \exp(C(X))$. So, Theorem 2.3 is valid, although, for each $n \in \mathbb{N}$, X_n has infinitely many components.

To present the main result of this section, we need the following preliminaries.

If A is a uF-algebra, then $\hat{A}$ is a natural Ff-algebra on $X = M_A$. Since the Gelfand map is a topological and algebraical isomorphism on a uF-algebra, we can consider each uF-algebra A, as a pointseparating complete unital subalgebra of $C(X)$, endowed with the compact-open topology, where $X = M_A$ is a hemicompact space [6, Theorem 4.1.3]. On the other hand, if we take $X_n = M_{A_n}$ then $\hat{A} = \varprojlim \hat{A}_{X_n}$, the projective limit of the dense projective system $\ldots \longrightarrow \hat{A}_{X_{n-1}} \xrightarrow{r_n} \hat{A}_{X_n} \longrightarrow \ldots$, where (X_n) is the admissible exhaustion of $X = M_A$, and $\hat{A}_{X_n}$ is the completion of $\hat{A}|_{X_n}$ with respect to the supremum norm $\|\cdot\|_{X_n}$, and the r_n are the restriction mappings. Since $\hat{A}_{X_n}$ is algebraically and topologically isomorphic with A_n ($n \in \mathbb{N}$), it is then a natural uniform (Banach) algebra on X_n. Hence, identifying A with $\hat{A}$, each uF-algebra A is the projective limit of a sequence of natural uniform Banach algebras. In this case, we say that A is a uF-algebra on $X = M_A$.

Theorem 2.5. *Let A be a uF-algebra on $X = M_A$, where X is a k-space, and for each $n \in \mathbb{N}$, $X_n = M_{A_n}$ has finitely many components. If $f \in A$ and $f|_{X_n} \in \exp(A_{X_n})$, for all $n \in \mathbb{N}$, then $f \in \exp(A)$.*

Proof. By the hypothesis, for each $n \in \mathbb{N}$, there exists $g_n \in A_{X_n}$ such that $f|_{X_n} = \exp g_n$. By Theorem 2.3, there exists an element $g \in C(X)$ such that $f = \exp g$. Since $f|_{X_n} \in A_{X_n}$, and A_{X_n} is a natural Bf-algebra on X_n, we can apply the Implicit Function Theorem [8, Theorem 3.5.12] to the function $F(w, z) = e^w - z$, to obtain $h_n \in A_{X_n}$ such that $f|_{X_n} = \exp h_n$ and moreover, $h_n = g|_{X_n}$, for each $n \in \mathbb{N}$. Since $A = \varprojlim A_{X_n}$ and $g|_{X_n} \in A_{X_n}$, it follows that $g \in A$, and this completes the proof of the theorem. $\qquad\square$

3. **Extension of the Arens-Royden Theorem to Fréchet Algebras**

Let X be a topological space, and let T be the unit circle in the plane. The group of homotopy classes of maps $X \longrightarrow T$, which is denoted by $\pi^1(X)$, is called the first cohomotopy group of X. If X is a compact Hausdorff space, then it is known that $H^1(C(X)) = G(C(X))/\exp(C(X))$ is isomorphic with $\pi^1(X)$. See, for example, [8, p. 411]. Following the same technique as in [8, p. 411], we apply Theorem 2.3 to extend the above result.

Theorem 3.1. *Let X be a hemicompact k-space with an admissible exhaustion (X_n) such that each X_n has finitely many components. Then, $\pi^1(X)$ is isomorphic with $H^1(C(X))$.*

Proof. Let L denote the set of all continuous functions from X into the unit circle T. For an element $g \in L$, the homotopy class of g is denoted by $[g]$. Let $\theta : G(C(X)) \longrightarrow \pi^1(X)$ be the group homomorphism, defined by $\theta(f) = [\frac{f}{|f|}]$. Obviously, θ is surjective. Now, we show that $\ker \theta = \exp(C(X))$.

If $g \in C(X)$, then $\exp g/|\exp g| = \exp(iImg)$. Hence, the function $G(x, t) = \exp(i(1 - t)Img(x))$, for $x \in X$ and $t \in [0, 1]$, defines a homotopy between 1 and $\exp(iImg)$ so that $\theta(\exp g)$ is the identity element in $\pi^1(X)$. Hence, $\exp(C(X)) \subseteq \ker \theta$.

For the reverse inclusion, let $g \in G(C(X))$ and $\theta(g) = 1$. Then, there exists a homotopy $F : X \times [0, 1] \longrightarrow T$ with $F(x, 0) = \frac{g(x)}{|g(x)|}$ and $F(x, 1) = 1$. For each $n \in \mathbb{N}$, take $F_n = F|_{X_n \times [0,1]}$ and $g_n = g|_{X_n}$. Clearly, $t \longrightarrow F_n(\cdot, t)$ is a continuous path from 1 to $\frac{g_n}{|g_n|}$ in $G(C(X_n))$, and $\exp(C(X_n))$ is the principal component of $G(C(X_n))$. Therefore,

$\frac{g_n}{|g_n|} \in \exp(C(X_n))$. On the other hand, the function $I(x,t) = \frac{g(x)}{1-t+t|g(x)|}$ defines a homotopy between g and $\frac{g}{|g|}$. If we take $I_n = I|_{X_n \times [0,1]}$, then $t \longrightarrow I_n(.,t)$ is a continuous path connecting g_n to $\frac{g_n}{|g_n|}$. Thus, $g_n \in \exp(C(X_n))$, for each $n \in \mathbb{N}$. Now, Theorem 2.3 implies that $g \in \exp(C(X))$. Therefore, $\ker \theta = \exp(C(X))$, that is, θ induces a group isomorphism between $H^1(C(X))$ and $\pi^1(X)$. $\square$

Before presenting next results, let us recall some elementary notions, which can be found in [6]. Let $(A,(p_n))$ be a unital, commutative F-algebra, and let A_n denote the completion of $A/\ker p_n$ with respect to the norm $p'_n(f + \ker p_n) = p_n(f)$. For each $n \in \mathbb{N}$, let $\pi_{n+1,n} : A_{n+1} \longrightarrow A_n$ be the extension of the norm decreasing homomorphism $A/\ker p_{n+1} \longrightarrow A/\ker p_n$, $x + \ker p_{n+1} \mapsto x + \ker p_n$. Then, $\pi_{n+1,n}$ is a continuous, dense range homomorphism. Therefore, $\pi^*_{n+1,n}$ is a homeomorphism between M_{A_n} and $\pi^*_{n+1,n}(M_{A_n}) \subseteq M_{A_{n+1}}$. Identifying M_{A_n} with $\pi^*_{n+1,n}(M_{A_n})$, for each $x \in A_{n+1}$, we have $\hat{x}|_{M_{A_n}} = \widehat{\pi_{n+1,n}(x)} \in \hat{A}_n$. Now, by considering the function $\theta_n : H^1(A_{n+1}) \longrightarrow H^1(A_n)$, $x \cdot \exp(A_{n+1}) \mapsto \pi_{n+1,n}(x) \cdot \exp(A_n)$ $(x \in G(A_{n+1}))$, one can easily show that $\{H^1(A_n), \theta_n\}$ is a projective system of groups so that $\varprojlim H^1(A_n)$ is a subgroup of $\prod_{n=1}^\infty H^1(A_n)$.

In particular, whenever A is a uF-algebra on X, we can define θ_n by $f \cdot \exp(A_{n+1}) \mapsto r_n(f) \cdot \exp(A_n)$ $(f \in G(A_{n+1}))$, where r_n is the restriction map $A_{n+1} \longrightarrow A_n$, $f \mapsto f|_{X_n}$ $(f \in A_{n+1})$.

Theorem 3.2. *Let X be a hemicompact k-space with an admissible exhaustion (X_n) such that each X_n has finitely many components. Then, $H^1(C(X))$ is isomorphic with $\varprojlim H^1(C(X_n))$.*

Proof. Define the function $\varphi : H^1(C(X)) \longrightarrow \varprojlim H^1(C(X_n))$, by

$$\varphi(f \cdot \exp(C(X)) = (f|_{X_n} \cdot \exp(C(X_n)))_n \ ; \ (f \in G(C(X))).$$

Clearly, φ is a group homomorphism. If $f \in G(C(X))$ such that $f|_{X_n} \in \exp(C(X_n))$, for all $n \in \mathbb{N}$, then $f \in \exp(C(X))$, by Theorem 2.3. Hence, φ is injective. For the surjectivity of φ, let $(f_n \cdot \exp(C(X_n)))_n \in \varprojlim H^1(C(X_n))$. For each $n \in \mathbb{N}$, there exists $g_n \in C(X_n)$ such that $f_{n+1}|_{X_n} = f_n \cdot \exp g_n$. Since X is a normal space [6, Remarks 3.1.10], each g_n has a continuous extension to X, which is again denoted by g_n. Adopting a similar method as in the proof of Theorem 2.2, set $g_0 = 0$

and $h_n = f_n \cdot \exp(-\sum_{i=0}^{n-1} g_i)$ on X_n. Since $f_{n+1}|_{X_n} = f_n \cdot \exp g_n$, we conclude that $h_{n+1}|_{X_n} = h_n$. This shows that the function h, defined by h_n on each X_n, is well-defined, and since X is a k-space, $h \in C(X)$. In fact, $h \in G(C(X))$. Moreover, $\varphi(h \cdot \exp C(X)) = (f_n \cdot \exp(C(X_n)))_n$, and hence φ is surjective. $\qquad \square$

Theorem 3.3. *Let A be a unital commutative F-algebra. Then $\varprojlim H^1 A_n)$ is isomorphic with $\varprojlim H^1(C(M_{A_n}))$.*

Proof. Let $(x_n \cdot \exp(A_n))_n \in \varprojlim H^1(A_n)$. For each $n \in \mathbb{N}$, there exists an element y_n in A_n such that $\pi_{n+1,n}(x_{n+1}) = x_n \cdot \exp y_n$, and hence $\hat{x}_{n+1}|_{M_{A_n}} = \widehat{\pi_{n+1,n}(x_{n+1})} = \hat{x}_n \cdot \exp \hat{y}_n$. Therefore, $(\hat{x}_n \cdot \exp(C(M_{A_n})))_n \in \varprojlim H^1(C(M_{A_n}))$. This shows that the function $\psi : \varprojlim H^1(A_n) \longrightarrow \varprojlim H^1(C(M_{A_n}))$, defined by $(x_n \cdot \exp(A_n))_n \longrightarrow (\hat{x}_n \cdot \exp(C(M_{A_n})))_n$, is well-defined. Clearly, ψ is a group homomorphism.

Let $(x_n \cdot \exp(A_n))_n \in \varprojlim H^1(A_n)$ be such that $\hat{x}_n \in \exp(C(M_{A_n}))$, for all $n \in \mathbb{N}$. By the Implicit Function Theorem, for each $n \in \mathbb{N}$, there exists $c_n \in A_n$ such that $x_n = \exp c_n$. Thus, ψ is injective. For the surjectivity of ψ, let $(f_n \cdot \exp(C(M_{A_n})))_n \in \varprojlim H^1(C(M_{A_n}))$. For each $n \in \mathbb{N}$, there exists $h_n \in C(M_{A_n})$ such that $f_{n+1}|_{M_{A_n}} = f_n \cdot \exp h_n$. On the other hand, by the Arens-Royden Theorem, for each $n \in \mathbb{N}$, there exist $x_n \in G(A_n)$ and $g_n \in C(M_{A_n})$ such that $\hat{x}_n = f_n \cdot \exp g_n$. Therefore, for each $n \in \mathbb{N}$,

$$\widehat{\pi_{n+1,n}(x_{n+1})} = \hat{x}_{n+1}|_{M_{A_n}} = \hat{x}_n \cdot \exp(-g_n + h_n + g_{n+1}).$$

By applying the Implicit Function Theorem, $\pi_{n+1,n}(x_{n+1}) \cdot x_n^{-1} \in \exp(A_n)$, for all $n \in \mathbb{N}$, i.e.,

$$(x_n \cdot \exp(A_n))_n \in \varprojlim H^1(A_n).$$

Obviously, $\psi((x_n \cdot \exp(A_n))_n) = (f_n \cdot \exp(C(M_{A_n})))_n$.

$\qquad \square$

As a consequence of the above two theorems we obtain the following result, which is an extension of the Arens-Royden theorem.

Corollary 3.4. *Let A be a unital commutative F-algebra such that M_A is a k-space, and M_{A_n} has finitely many components, for each $n \in \mathbb{N}$. Then $H^1(C(M_A))$ is isomorphic with $\varprojlim H^1(A_n)$.*

Corollary 3.5. *Let A be a uF-algebra on $X = M_A$ such that X is a k-space, and each $X_n = M_{A_n}$ has finitely many components. If $\pi^1(X)$ is trivial, or equivalently, if $G(C(X)) = \exp(C(X))$, then $G(A) = \exp(A)$.*

Proof. By Corollary 3.4 and Theorem 3.1, $\pi^1(X)$ is isomorphic with $\varprojlim H^1(A_{X_n})$. Obviously, if $f \in G(A)$, then $f_n = f|_{X_n} \in G(A_{X_n})$, for all $n \in \mathbb{N}$. By considering the continuous functions θ_n, which were defined before Theorem 3.2, it follows that $(f_n \cdot \exp(A_{X_n}))_n \in \varprojlim H^1(A_{X_n})$. Since $\pi^1(X)$ is trivial, $f_n \in \exp(A_{X_n})$, for all $n \in \mathbb{N}$. Now, the result follows by Theorem 2.5. $\qquad\square$

Remark 3.6. *(i) Recall that if X is a compact plane set, then $\pi^1(X)$ is trivial, i.e., X has trivial fundamental group if and only if $\mathbb{C}\backslash X$ is connected [1].*

(ii) If X is a hemicompact k-space with the admissible exhaustion (X_n) such that $\mathbb{C}\backslash X_n$ is connected for each $n \in \mathbb{N}$, then each $\pi^1(X_n)$ is trivial by (i). If, moreover, each X_n has finitely many components, then by Theorem 2.3 $\pi^1(X)$ is trivial.

Remark 3.7. *In Corollary 3.5, the fact that $\pi^1(X)$ is trivial does not pass to compact subsets of X, i.e., $\pi^1(X_n)$ may not be trivial, in general, as the following example shows.*

Example 3.8. *Consider the admissible exhaustion of the open unit disc X consisting of an expanding sequence (X_n) of compact subsets, each of which consists of the union of a closed disc centred on the origin together with a disjoint closed annulus with two slightly larger radii. Each X_n has then 2 components, and since $\mathbb{C}\backslash X_n$ is not connected, none of the X_n have trivial fundamental group, i.e., $G(C(X_n) \neq exp(C(X_n))$, by Remark 3.6(i), although X does, i.e., $G(C(X)) = exp(C(X))$, by Remark 3.6(ii) (it is enough to take the exhaustion (K_n) for X as the increasing closed discs with centres at the origin). Nevertheless, the projective limit of the fundamental groups $\varprojlim H^1(C(X_n))$ is trivial by theorems 3.1 and 3.2.*

It is interesting to note that $H^1(C(X))$ may be trivial even if X is disconnected (consider, for example, a union of two disjoint closed intervals or discs).

Remark 3.9. *Let A be a uF-algebra on $X = M_A$, which is a k-space, with each X_n having finitely many components. If $A|_{X_n}$ is closed in $C(X_n)$, that is, $A_{X_n} = A|_{X_n}$, for each $n \in \mathbb{N}$, then, by following the same technique as in Theorem 3.2, we can obtain an isomorphism between $H^1(A)$ and $\varprojlim H^1(A_{X_n})$. Hence, in this case, $H^1(C(M_A))$ is isomorphic with $H^1(A)$. However, in general, we do not yet know whether there exists an isomorphism between $H^1(C(M_A))$ and $H^1(A)$.*

4. On the Denseness of Invertible Elements in a UF-Algebra

In 2003, Dawson and Feinstein obtained some results on the denseness of the invertible group in Banach function algebras [3]. Following the same technique and making use of the theorems proved in Section 3, we extend some of these results to *uF*-algebras. For the definition of topological dimension, refer to [9].

Theorem 4.1. *Let A be a uF-algebra on $X = M_A$ such that X is a k-space, and each $X_n = M_{A_n}$ is locally connected. Let $\pi^1(X)$ be trivial, or equivalently, let $G(C(X)) = \exp(C(X))$. Then,*

(i) *If X contains a closed subspace E whose topological dimension is at least 2, and if $E_n = E \cap X_n$ is locally connected, for all $n \in \mathbb{N}$, then A does not have a dense invertible group.*

(ii) *If A has a dense invertible group, then $A = C(X)$.*

Proof. We first note that each X_n has finitely many components.

(i) By Corollary 3.5, $G(A) = \exp(A)$. Suppose on the contrary that A has a dense invertible group. Let $B = A|_E$, and $I = \{f \in A : f|_E = 0\}$. It is easy to see that B is algebraically isomorphic with A/I. Since I is a closed ideal of A, A/I is a Fréchet algebra with respect to the sequence of seminorms $(q_k)_k$, defined by

$$q_k(f + I) = \inf\{\|f + g\|_{X_k} : g \in I\}, \quad (f \in A).$$

If we endow B with this sequence of seminorms, then B is an Ff-algebra on E. Note that E is a hemicompact k-space with the admissible exhaustion $(E_n)_n$. Hence, $C(E)$ is an F-algebra.

Let $F = \exp(A)|_E$, and let C be the closure of B in $C(E)$ with respect to the compact-open topology. Therefore, C is a uF-algebra on E. By our assumption, A has a dense invertible group, and so F is dense in C.

Clearly, we may write $C = \varprojlim C_{E_n}$, where C_{E_n} is the completion of $C|_{E_n}$ with respect to the norm $\|\cdot\|_{E_n}$, for each $n \in \mathbb{N}$. Obviously, $F|_{E_n}$ is dense in C_{E_n} and each element in $F|_{E_n}$ has a square root in C_{E_n}. Since each E_n is locally connected, it follows from Čirka's Theorem [11, pp. 131-134] that $C_{E_n} = C(E_n)$, and hence $C = C(E)$. Since E is normal and the invertible elements are dense in $C(E)$, by [9, Theorem 3.2, p. 128], we have dim $E \leq 1$, which is a contradiction. Therefore, A does not have a dense invertible group.

(ii) Since $\pi^1(X)$ is trivial, Corollary 3.5 implies that $G(A) = exp(A)$. By the hypothesis, $G(A)$ is dense in A. Hence, $exp(A_{X_n})$ is dense in A_{X_n}, for all $n \in \mathbb{N}$. Since each X_n is locally connected, again by Čirka's Theorem [11, pp. 131-134], we conclude that $A_{X_n} = C(X_n)$, for all $n \in \mathbb{N}$. Therefore, $A = \varprojlim A_{X_n} = \varprojlim C(X_n) = C(X)$. $\square$

The following result for Fréchet algebras is also similar to Corollary 1.8 in [3].

Corollary 4.2. *Let A be a natural uF-algebra on $\mathbb{R}$. If A has a dense invertible group, then $A = C(\mathbb{R})$.*

Proof. Set $I_n = [-n, n]$, then $C(\mathbb{R}) = \varprojlim C(I_n)$. Since $\mathbb{C}\backslash I_n$ is connected, $\pi^1(I_n)$ is trivial. Hence, $H^1(C(I_n))$ is trivial. By theorems 3.1 and 3.2, $\pi^1(\mathbb{R})$ is isomorphic with $\varprojlim H^1(C(I_n))$. Therefore, $\pi^1(\mathbb{R})$ is trivial, and hence $A = C(\mathbb{R})$, by Theorem 4.1 (ii). $\square$

Note that we can also deduce the triviality of $\pi^1(\mathbb{R})$ and $\pi^1(\mathbb{C})$ from Remark 3.6(ii).

Remark 4.3. *One may think that the above corollary is also true if we replace $\mathbb{R}$ by $\mathbb{C}$. Actually this case cannot arise, since $C(\mathbb{C})$ does not have a dense invertible group. To see this, it is interesting to note that for a hemicompact k-space X, it is known that the uF-algebra $C(X)$ has a dense invertible group if and only if $dim(X) \leq 1$, where $dim(X)$ is the topological dimension of X. Since $dim(\mathbb{R}) = 1$ and $dim(\mathbb{C}) = dim(\mathbb{R}^2) = 2$ [9, Theorem 3.2.7], it follows that $C(\mathbb{R})$ has a dense invertible group, whereas $C(\mathbb{C})$ does not have a dense invertible group.*

Acknowledgments

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