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MULTIPLIERS OF GENERALIZED FRAMES IN HILBERT SPACES

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ABSTRACT. In this paper, we introduce the concept of g-Bessel multipliers which generalizes Bessel multipliers for g-Bessel sequences and we study the properties of g-Bessel multipliers when the symbol $m \in \ell^1, \ell^p, \ell^\infty$. Also, we review the behavior of these operators when the parameters are changing.

Furthermore, we show that equivalent g-frames have equivalent multipliers and conversely. Finally, we specialize the results to fusion frames.

1. **Introduction**

Frames have been introduced by Duffin and Schaeffer in [11], in connection with non-harmonic Fourier series, but they have attracted more attention since the beginning of wavelet theory. Frames have many nice properties which make them very useful in the characterization of function spaces, signal processing and many other fields. We refer to [9, 12, 20] for an introduction to the frame theory and its applications.

G-frames, introduced by W. Sun in [29], are a natural generalization of frames which cover many other extensions of frames, e.g. bounded quasi-projectors [16, 17], pseudo-frames [21], frame of subspaces or fusion frames [7], outer frames [1], oblique frames [10, 14], continuous frames

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[9] and a class of time-frequency localization operators [13]. Also it was shown that g-frames are equivalent to stable spaces splitings studied in [24]. All of these concepts are proved to be useful in many applications. The perturbation of g-frames and some other properties of this family of operator have been studied in [23, 30].

Bessel multipliers are investigated by Peter Balazs [2, 3, 4] for Hilbert spaces. For Bessel sequences, the investigation of the operator $\mathbf{M} = \sum m_k \langle f, \psi_k \rangle \varphi_k$, where the analysis coefficients $\langle f, \psi_k \rangle$ are multiplied by a fix symbol (m_k) before resynthesis (with φ_k), is very natural and useful and there are numerous applications of this kind of operators. Bessel multipliers of p-Bessel sequences in Banach spaces are introduced in [27].

In this paper, by using operator theory tools, we investigate multipliers for g-Bessel sequences. We show that when the symbol $m \in c_0$, then this operator is compact and when $m \in \ell^1, \ell^2, \ell^p$ it is a trace class, Hilbert-Schmidt and Schatten p-class operator, respectively. Also, we show that equivalent g-frames have equivalent multipliers and conversely. Finally, we specialize the results to fusion frames.

2. G-Frames

Through this paper, $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $\{\mathcal{H}_i : i \in I\}$ is a sequence of Hilbert spaces, where I is a subset of $\mathbb{Z}$. $\mathcal{L}(\mathcal{H}, \mathcal{H}_i)$ is the collection of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}_i$.

Note that for any sequence $\{\mathcal{H}_i : i \in I\}$, we can assume that there exits a Hilbert space $\mathcal{K}$ such that for all $i \in I, \mathcal{H}_i \subseteq \mathcal{K}$ (for example $\mathcal{K} = \bigoplus_{i \in I} \mathcal{H}_i$).

Definition 2.1. *A sequence $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called generalized frame, or simply a g-frame, for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ if there exist constants A and B, $0 < A \leq B < \infty$, such that*

$$A\|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{2.1}$$

The numbers A and B are called g-frame bounds.

$\{\Lambda_i : i \in I\}$ is called tight g-frame if $A = B$ and Parseval g-frame if $A = B = 1$. If in (2.1) the second inequality holds, then the sequence is named g-Bessel sequence. $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a g-frame sequence if it is a g-frame for $\overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I}$.

It is easy to see that, if $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with bounds A and B, then by putting $\mathcal{H}_i = \mathbb{C}$ and $\Lambda_i(\cdot) = \langle \cdot, f_i\rangle$, the family $\{\Lambda_i : i \in I\}$ is a g-frame for $\mathcal{H}$ with bounds A and B.

Example 2.1. *Let $\mathcal{H}_i = \mathbb{C}$ and let $\Lambda_i : \mathcal{H} \to \mathbb{C}$ be defined by $\Lambda_i f = \frac{1}{i}\|f\|$, for any $f \in \mathcal{H}$. Then, $\sum_i \|\Lambda_i f\|^2 = \sum_i \frac{1}{i^2}\|f\|^2 = \frac{\pi^2}{6}\|f\|^2$, and so $\{\Lambda_i : i \in I\}$ is a tight g-frame for $\mathcal{H}$.*

Similar to g-frames, the means g-complete, g-orthonormal bases and g-Riesz bases can be defined as follows.

Definition 2.2. *The family $\{\, \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \,\}$ is called g-complete if $\{ f : \Lambda_i f = 0, \, i \in I \,\} = \{0\}$ and it is called g-orthonormal basis for $\mathcal{H}$ if it satisfies*

$$\langle \Lambda_i^* g_i, \Lambda_j^* g_j\rangle = \delta_{i,j}\langle g_i, g_j\rangle, \quad \forall i, j \in I, g_i \in \mathcal{H}_i, g_j \in \mathcal{H}_j,$$

$$\sum_{i\in I} \|\Lambda_i f\|^2 = \|f\|^2, \quad \forall f \in \mathcal{H}.$$

$\{\, \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \,\}$ is called g-Riesz basis if it is g-complete and there are positive constants A and B such that for any finite subset $J \subset I$ and $f_i \in \mathcal{H}_i, i \in J$,

$$A\sum_{i\in J} \|f_i\|^2 \le \|\sum_{i\in J} \|\Lambda_i^* f_i\|^2 \le B\sum_{i\in J} \|f_i\|^2.$$

It is clear that if $\{e_i\}_{i\in I}$ is an orthonormal basis (or ONB) for $\mathcal{H}$, then $\{\Lambda_i(\cdot) = \langle \cdot, e_i\rangle, i \in I\}$ is a $g-$ONB.

Let

$$\Big(\bigoplus_{i\in I}\mathcal{H}_i\Big)_{\ell_2} = \Big\{ \{f_i\}_{i\in I} \mid f_i \in \mathcal{H}_i, \, \forall i \in I \ \text{ and } \ \sum_{i\in I} \|f_i\|^2 < +\infty\Big\},$$

with the inner product defined by

$$\langle \{f_i\}, \{g_i\}\rangle = \sum_{i\in I} \langle f_i, g_i\rangle.$$

It is clear that $\left(\bigoplus_{i\in I}\mathcal{H}_i\right)_{\ell_2}$ is a Hilbert space. Let $E_i = (e^i_k)_{k\in K_i}$ be an orthonormal basis for $\mathcal{H}_i$. Define $F_{i,k} = (\underbrace{0, 0, ..., e^i_k}_{i-times}, 0, ...)$. Then, $F = ((F_{i,k})_{k\in K_i})_{i\in N}$ is an ONB of $\left(\bigoplus_{i\in I}\mathcal{H}_i\right)_{\ell_2}$.

Proposition 2.3. [23] *$\mathbf{\Lambda} = \{\,\Lambda_i \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in I\,\}$ is a g-Bessel sequence for $\mathcal{H}$ with bound B if and only if the operator*

$$T_{\mathbf{\Lambda}} : \Big(\bigoplus_{i\in I}\mathcal{H}_i\Big)_{\ell_2} \longrightarrow \mathcal{H},$$

defined by

$$T_{\mathbf{\Lambda}}(\{f_i\}_{i\in I}) = \sum_{i\in I}\Lambda_i^*(f_i),$$

is a well-defined and bounded operator with $\|T_{\mathbf{\Lambda}}\| \le \sqrt{B}$. Furthermore,

$$T_{\mathbf{\Lambda}}^* : \mathcal{H} \longrightarrow \Big(\bigoplus\mathcal{H}_i\Big)_{\ell_2},$$

$$T_{\mathbf{\Lambda}}^*(f) = \{\Lambda_i f\}_{i\in I}.$$

If $\mathbf{\Lambda} = \{\,\Lambda_i \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in I\,\}$ is a g-frame, then the operators $T_{\mathbf{\Lambda}}$ and $T_{\mathbf{\Lambda}}^*$ in Proposition 2.3 are called **synthesis operator** and **analysis operator** of $\mathbf{\Lambda} = \{\,\Lambda_i \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in I\,\}$, respectively.

Definition 2.4. *Let $\mathbf{\Lambda} = \{\,\Lambda_i \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in I\,\}$ be a g-frame for $\mathcal{H}$. The operator*

$$S_{\mathbf{\Lambda}} : \mathcal{H} \longrightarrow \mathcal{H}, \quad S_{\mathbf{\Lambda}} = T_{\mathbf{\Lambda}}T_{\mathbf{\Lambda}}^*$$

is called the g-frame operator of $\mathbf{\Lambda} = \{\,\Lambda_i \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in I\,\}$.

For any $f \in \mathcal{H}$, we have

$$S_{\mathbf{\Lambda}} f = \sum_{i\in I}\Lambda_i^*\Lambda_i f,$$

$$AI \le S_{\mathbf{\Lambda}} \le BI.$$

It is known [23] that $S_{\mathbf{\Lambda}}$ is a bounded, positive and invertible operator and every $f \in \mathcal{H}$ has an expansion $f = \sum_i \Lambda_i^*\Lambda_i S_{\mathbf{\Lambda}}^{-1} f$.

One of the most important advantages of g-frames is a resolution of identity $\sum_i \Lambda_i^*\Lambda_i S_{\mathbf{\Lambda}}^{-1} = I$.

Proposition 2.5. [23] *$\mathbf{\Lambda} = \{\,\Lambda_i \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in I\,\}$ is a g-frame for $\mathcal{H}$ if and only if the synthesis operator $T_{\mathbf{\Lambda}}$ is well-defined, bounded and onto.*

Definition 2.6. *If $\mathbf{\Lambda} = \{\,\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ and $\mathbf{\Theta} = \{\,\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ are g-Bessel sequences for $\mathcal{H}$ and $T_{\mathbf{\Lambda}} T^*_{\mathbf{\Theta}} = Id$, then $(\mathbf{\Lambda}, \mathbf{\Theta})$ is called a dual pair. For example, $(\mathbf{\Lambda}, S^{-1}\mathbf{\Lambda})$ is a dual pair.*

Lemma 2.7. [23] *If $\mathbf{\Lambda} = \{\,\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ and $\mathbf{\Theta} = \{\,\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ are g-Bessel sequences for $\mathcal{H}$ and $(\mathbf{\Lambda}, \mathbf{\Theta})$ is a dual pair, then both of $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ are g-frames for $\mathcal{H}$.*

3. Multipliers of Frames

Gabor multipliers [13, 15], Gabor filters [22] and some other applications of frames led Peter Balazs to introduce Bessel and frame multipliers for abstract Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. These operators are defined by a fixed multiplication pattern (the symbol) which is inserted between the analysis and synthesis operators [2, 3, 4]

Definition 3.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, $\{\psi_k\} \subseteq \mathcal{H}_1$ and $\{\phi_k\} \subseteq \mathcal{H}_2$ be Bessel sequences. Fix $m \in l^\infty$. The operator $\mathbf{M}_{m,(\phi_k),(\psi_k)} : \mathcal{H}_1 \to \mathcal{H}_2$, defined by*

$$\mathbf{M}_{m,(\phi_k),(\psi_k)}(f) = \sum_k m_k \langle f, \psi_k \rangle \phi_k \quad \forall f \in \mathcal{H},$$

is called the Bessel multiplier for the Bessel sequences $\{\psi_k\}$ and $\{\phi_k\}$. The sequence m is called the symbol of $\mathbf{M}$. For frames, we will call the resulting Bessel multiplier a frame multiplier and for Riesz sequence a Riesz multiplier.

The interested reader can find the properties of this operator in [2, 3, 4].

4. Multipliers of G-Frames

In this section, the concept of multiplier operators for g-Bessel sequences will be introduced and some of their properties will be shown.

Proposition 4.1. *If $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ are g-Bessel sequences with bounds $B_{\mathbf{\Lambda}}$ and $B_{\mathbf{\Theta}}$ and $m \in \ell^\infty$, then the operator $\mathbf{M} = \mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}} : \mathcal{H} \to \mathcal{H}$, defined by $\mathbf{M}(f) = \sum_i m_i \Lambda_i^* \Theta_i f$, for all $f \in \mathcal{H}$, is well defined.*

Proof. For any $f, g \in \mathcal{H}$,

$$\langle \sum_i m_i \Lambda_i^* \Theta_i f, g \rangle = \sum_i m_i \langle \Theta_i f, \Lambda_i g \rangle$$

and

$$\begin{aligned} |\sum_i m_i \langle \Theta_i f, \Lambda_i g \rangle| &\leq \|m\|_\infty \sum_i |\langle \Theta_i f, \Lambda_i g \rangle| \\ &\leq \|m\|_\infty (\sum_i \|\Theta_i f\|^2)^{\frac{1}{2}} (\sum_i \|\Lambda_i g\|^2)^{\frac{1}{2}} \\ &\leq \|m\|_\infty \sqrt{B_{\mathbf{\Theta}}} \|f\| \sqrt{B_{\mathbf{\Lambda}}} \|g\|. \end{aligned}$$

□

Now, we can define multiplier operators for g-Bessel sequences.

Definition 4.2. *Let $\{\mathcal{H}_i : i \in I\}$ be a family of Hilbert spaces. Let $\mathbf{\Lambda} = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\mathbf{\Theta} = \{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be g-Bessel sequences for $\mathcal{H}$ with bounds $B_{\mathbf{\Lambda}}$ and $B_{\mathbf{\Theta}}$. For $m \in \ell^\infty$, the operator*

$$\mathbf{M} = \mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}} : \mathcal{H} \to \mathcal{H},$$

$$\mathbf{M}(f) = \sum_i m_i \Lambda_i^* \Theta_i f, \tag{4.1}$$

is called the g-Bessel multiplier of $\mathbf{\Lambda}$, $\mathbf{\Theta}$ with symbol m.

It is clear that if $m = (m_i) = (1, 1, 1, ...)$ and $(\mathbf{\Lambda}, \mathbf{\Theta})$ is a dual pair, then $\mathbf{M} = Id$.

Let $\{\lambda_i\}$ and $\{\varphi_i\}$ be Bessel sequences and $m \in \ell^\infty$. Consider the corresponding g-Bessel sequences $\Lambda_i \cdot = \langle \cdot, \lambda_i \rangle$ and $\Theta_i \cdot = \langle \cdot, \varphi_i \rangle$. For any $f \in \mathcal{H}$, we have

$$\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}(f) = \mathbf{M}_{m,(\phi_k),(\lambda_k)}(f).$$

The proof of Proposition 4.1 shows that

$$\begin{aligned} \|\mathbf{M}\|_{op} = \sup\{\langle \mathbf{M}f, f \rangle : \|f\| = 1\} &\leq \sup_{\|f\|=1} \|m\|_\infty \sqrt{B_{\mathbf{\Theta}}} \sqrt{B_{\mathbf{\Lambda}}} \|f\|^2 \\ &= \|m\|_\infty \sqrt{B_{\mathbf{\Theta}}} \sqrt{B_{\mathbf{\Lambda}}}. \end{aligned}$$

Lemma 4.3. *If* $\mathbf{\Lambda}$ *and* $\mathbf{\Theta}$ *are g-Bessel sequences and* $m \in \ell^\infty$*, then the adjoint of* $\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}$ *is* $\mathbf{M}_{\overline{m},\mathbf{\Theta},\mathbf{\Lambda}}$.

Proof. For any $f, g \in \mathcal{H}$,

$$\begin{aligned}\langle \mathbf{M}f, g\rangle &= \langle \sum_i m_i \Lambda_i^* \Theta_i f, g\rangle = \sum_i m_i \langle \Lambda_i^* \Theta_i f, g\rangle \\ &= \sum_i m_i \langle \Theta_i f, \Lambda_i g\rangle = \sum_i m_i \langle f, \Theta_i^* \Lambda_i g\rangle,\end{aligned}$$

and so $\mathbf{M}^*_{m,\mathbf{\Lambda},\mathbf{\Theta}} = \sum_i \overline{m_i} \Theta_i^* \Lambda_i = \mathbf{M}_{\overline{m},\mathbf{\Theta},\mathbf{\Lambda}}$. □

Lemma 4.4. *If* $\mathbf{\Theta} = \{\Theta_i\}_{i\in I}$ *is a g-Bessel sequence with bound* $B_{\mathbf{\Theta}}$ *and* $m = (m_i) \in \ell^\infty$*, then* $\{m_i\Theta_i\}_{i\in I}$ *is a g-Bessel sequence with bound* $\|m\|_\infty B_{\mathbf{\Theta}}$.

Proof. It is easy to show that, for all $f \in \mathcal{H}$,

$$\sum_i \|m_i \Theta_i f\|^2 \leq \|m\|_\infty \sum_i \|\Theta_i f\|^2 \leq \|m\|_\infty B_{\mathbf{\Theta}} \|f\|^2.$$

□

Like weighted frames [6], $\{m_i\Theta_i\}_{i\in I}$ can be called weighted g-frame (g-Bessel). By using the synthesis and the analysis operators of $\mathbf{\Lambda}$ and $m\mathbf{\Theta}$, respectively, we can write

$$\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}} f = \sum_i m_i \Lambda_i^* \Theta_i f = \sum_i \Lambda_i^* (m_i \Theta_i) f = T_{\mathbf{\Lambda}} \{m_i \Theta_i f\} = T_{\mathbf{\Lambda}} T^*_{m\mathbf{\Theta}} f.$$

So,

$$\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}} = T_{\mathbf{\Lambda}} T^*_{m\mathbf{\Theta}}. \tag{4.2}$$

If we define the diagonal operator

$$D_m : \Big(\bigoplus \mathcal{H}_i\Big)_{\ell_2} \to \Big(\bigoplus \mathcal{H}_i\Big)_{\ell_2},$$

$$D_m((\xi_i)) = (m_i \xi_i)_{i\in I} \tag{4.3}$$

then

$$\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}} = T_{\mathbf{\Lambda}} D_m T^*_{\mathbf{\Theta}}. \tag{4.4}$$

We will use both (4.2) and (4.4) in this paper.

Proposition 4.5. *Let $m \in \ell^\infty$, $\mathbf{\Lambda}$ be a g-Riesz base and $\mathbf{\Theta}$ be a g-Bessel sequence. The map $m \to \mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}$ is injective.*

Proof. Let $\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}} = 0$. Then, for any $f \in \mathcal{H}$, $\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}} f = 0$. Without loss of generality, we can chose $f \in \cap[ker\Theta_i]^\perp$. Since $\mathbf{\Lambda} = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-Riesz base, we have

$$A \sum \|m_i \Theta_i f\|^2 \leq \| \sum \Lambda_i^*(m_i \Theta_i f)\| = 0 \Rightarrow m_i = 0, \quad i \in I,$$

that is, the map $m \to \mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}$ is injective. □

Proposition 4.6. *Let $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ be g-Bessel sequences for $\mathcal{H}$. If $m = (m_i) \in c_0$ and $(rank\Theta_i) \in \ell^\infty$, then $\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}$ is compact.*

Proof. Since $m = (m_i) \in c_0$, for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $|m_i| < \varepsilon$, for all $i > N$. Let $m_N = (m_{N_1}, m_{N_2}, ..., m_{N_N}, 0, 0, ...)$. Then, for $f \in \mathcal{H}$,

$$\begin{aligned}\|(T^*_{m\Theta} - T^*_{m_N\Theta})f\| &= \|\{(m_i - m_{N_i})\Theta_i f\}\|_{\ell^2} \\ &\leq \|m - m_N\|_\infty \Big(\sum_i \|\Theta_i f\|^2 \Big)^{\frac{1}{2}} \\ &\leq \varepsilon \sqrt{B_{\mathbf{\Theta}}} \|f\|.\end{aligned}$$

So, $T^*_{m_N\Theta} \to T^*_{m\Theta}$. Since

$$rankT^*_{m_N\Theta} = \dim kerT_{m_N\Theta} = \prod_{i=1}^{m_N} (\dim ker\Theta_i^*) = \prod_{i=1}^{m_N} rank\Theta_i < \infty,$$

it follows that $T^*_{m_N\Theta}$ is a finite rank operator. Hence, $T^*_{m\Theta}$ is compact. Therefore, $\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}} = T_{\mathbf{\Lambda}} T^*_{m\mathbf{\Theta}}\ \mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}$ is compact. □

Recall that if T is a compact operator on a separable Hilbert space $\mathcal{H}$, then there exist orthonormal sets $\{e_n\}$ and $\{\sigma_n\}$ in $\mathcal{H}$ such that

$$Tx = \sum_n \lambda_n \langle x, e_n \rangle \sigma_n,$$

for $x \in \mathcal{H}$, where λ_n is the $n-$th singular value of T. Given $0 < p < \infty$, the Schatten p-class of $\mathcal{H}$ [28], denoted by $\mathcal{S}_p$, is the space of all compact operators T on $\mathcal{H}$ with the singular value sequence $\{\lambda_n\}$ belonging to ℓ^p. It is known that $\mathcal{S}_p$ is a Banach space with the norm

$$\|T\|_p = [\sum_n |\lambda_n|^p]^{\frac{1}{p}}. \tag{4.5}$$

$\mathcal{S}_1$ is called the *trace class* and $\mathcal{S}_2$ is called the *Hilbert-Schmidt class.* Theorem 1.4.6. in [31] shows that $T \in \mathcal{S}_p$ if and only if $T^p \in \mathcal{S}_1$. Moreover, $\|T\|_p^p = \|T^p\|_1$. Also, $T \in \mathcal{S}_p$ if and only if $|T|^p = (T^*T)^{\frac{p}{2}} \in \mathcal{S}_1$ if and only if $T^*T \in \mathcal{S}_{\frac{p}{2}}$. Moreover, $\|T\|_p^p = \|T^*\|_p^p = \||T|\|_p^p = \||T|^p\|_1 = \|T^*T\|_{\frac{p}{2}}$.

It is proved that $\mathcal{S}_p$ is a two sided $*$-ideal of $\mathcal{L}(\mathcal{H})$, that is, a Banach algebra under the norm (4.5) and the finite rank operators are dense in $(\mathcal{S}_p, \|.\|_p)$. Moreover, for $T \in \mathcal{S}_p$, one has $\|T\|_p = \|T^*\|_p$, $\|T\| \leq \|T\|_p$ and if $S \in \mathcal{L}(\mathcal{H})$, then $\|ST\|_p \leq \|S\|\|T\|_p$ and $\|TS\|_p \leq \|S\|\|T\|_p$. For more information about these operators, see [18, 25, 28, 31]. Analogously, for Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$ and for operators $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, $B \in \mathcal{L}(\mathcal{H}_4, \mathcal{H}_3)$ and $C \in \mathcal{S}_p(\mathcal{H}_3, \mathcal{H}_1)$, we have $AC \in \mathcal{S}_p(\mathcal{H}_3, \mathcal{H}_2)$ and $CB \in \mathcal{S}_p(\mathcal{H}_4, \mathcal{H}_1)$.

Theorem 4.7. *Let $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ be g-Bessel sequences for $\mathcal{H}$. If $m = (m_i) \in \ell^p$ and $(dim\mathcal{H}_i)_{i\in I} \in \ell^\infty$, then $\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}$ is a Schatten p-class operator.*

Proof. Since $\mathcal{S}_p$ is a two sided ideal of $\mathcal{L}(\mathcal{H})$, it is enough to show that the operator D_m in (4.4) is in $\mathcal{S}_p$. Let $F = \Big((F_{i,k})_{k\in K_i}\Big)_{i\in\mathbb{N}}$ be an ONB of $\Big(\bigoplus_{i\in I}\mathcal{H}_i\Big)_{\ell_2}$. Now, let $(\hat{F}_j)_{j\in\mathbb{N}}$ be the rearrangement of F given by:

(1) If $1 \leq j \leq K_1$ then $\hat{F}_j = F_{1,j}$;
(2) If $j > K_1$ then $\hat{F}_j = F_{n+1,k}$, where $n = max\{m \in \mathbb{N} : j > K_1 + ... + K_m\}$ and $k = j - (K_1 + ... + K_n)$.

So, for every $\xi = (\xi_i)_{i\in\mathbb{N}} \in \Big(\bigoplus_{i\in I}\mathcal{H}_i\Big)_{\ell_2}$,

$$D_m(\xi) = (m_i\xi_i)_{i\in\mathbb{N}} = \sum_{i\in\mathbb{N}}\sum_{k\in K_i} m_i\langle\xi, F_{i,k}\rangle F_{i,k} = \sum_{j\in\mathbb{N}} \hat{m}_j\langle\xi, \hat{F}_j\rangle\hat{F}_j,$$

where,

$$(\hat{m}_j)_{j\in\mathbb{N}} = (\underbrace{m_1, m_1, ..., m_1}_{K_1}, \underbrace{m_2, m_2, ..., m_2}_{K_2}, ..., \underbrace{m_j, m_j, ..., m_j}_{K_j}, ...).$$

Now,
$$\sum_{j\in\mathbb{N}}|\hat{m}_j|^p=\sum_{i\in\mathbb{N}}K_i|m_i|^p\leq\|(dim\mathcal{H}_i)\|_\infty\|m\|_p^p<\infty,$$
and therefore, $D_m\in\mathcal{S}_p$. The fact that $\mathcal{S}_p$ is a $*$-ideal of $\mathcal{L}(\mathcal{H})$ and (4.4) imply that $\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}$ is a Schatten p-class operator. $\square$

Corollary 4.8. *Let $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ be g-Bessel sequences for $\mathcal{H}$.*

(1) *If $m=(m_i)\in\ell^1$ and $(dim\mathcal{H}_i)_{i\in I}\in\ell^\infty$, then $\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}$ is a trace-class operator.*
(2) *If $m=(m_i)\in\ell^2$ and $(dim\mathcal{H}_i)_{i\in I}\in\ell^\infty$, then $\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}$ is a Hilbert-Schmit operator.*

5. **Perturbation of Multipliers**

Like Bessel multipliers [2], a g-Bessel multiplier $\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}$ clearly depends on the parameters m , $\mathbf{\Lambda}$ and $\mathbf{\Theta}$. So, it is natural to ask: What happens if these items are changed?

Proposition 5.1. *Let $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ be g-Bessel sequences for $\mathcal{H}$.*

(1) *If $m^{(k)}\to m$ in ℓ^∞, then*
$$\|\mathbf{M}_{m^k,\mathbf{\Lambda},\mathbf{\Theta}}-\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}\|_{op}\to 0.$$
(2) *If $m^{(k)}\to m$ in ℓ^p and $(\dim\mathcal{H}_i)\in\ell^\infty$, then*
$$\|\mathbf{M}_{m^k,\mathbf{\Lambda},\mathbf{\Theta}}-\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}\|_p\to 0.$$

Proof. The decomposition (4.4) shows that
$$\mathbf{M}_{m^k,\mathbf{\Lambda},\mathbf{\Theta}}-\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}=\mathbf{M}_{m^k-m,\mathbf{\Lambda},\mathbf{\Theta}}=T_\Lambda D_{m^k-m}T_\Theta^*.$$
It is easy to observe that
(1)
$$\begin{aligned}\|\mathbf{M}_{m^{(k)},\mathbf{\Lambda},\mathbf{\Theta}}-\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}\|_{op}&=\|T_\Lambda D_{m^k-m}T_\Theta^*\|_{op}\\&\leq\|T_\Lambda\|_{op}\|D_{m^{(k)}-m}\|_{op}\|T_\Theta^*\|_{op},\end{aligned}$$
and since
$$\begin{aligned}\|D_{m^{(k)}-m}\|_{op}&=\sup_{\|(\xi_i)\|=1}\|D_{m^{(k)}-m}(\xi_i)\|\\&=\sup_{\|(\xi_i)\|=1}\|((m_i^{(k)}-m_i)(\xi_i))\|\\&\leq\|m^{(k)}-m\|_{\ell^\infty}\to 0.\end{aligned}$$

So

$$\|\mathbf{M}_{m^k,\mathbf{\Lambda},\mathbf{\Theta}} - \mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}\|_{op} \to 0.$$

(2)

$$\begin{aligned}\|\mathbf{M}_{m^{(k)},\mathbf{\Lambda},\mathbf{\Theta}} - \mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}\|_p &= \|T_\Lambda D_{m^k-m} T_\Theta^*\|_p \\ &\leq \|T_\Lambda\|_{op}\|D_{m^{(k)}-m}\|_p\|T_\Theta^*\|_{op},\end{aligned}$$

and since

$$\|D_{m^{(k)}-m}\|_p \leq \|m^{(k)} - m\|_p \to 0,$$

so

$$\|\mathbf{M}_{m^k,\mathbf{\Lambda},\mathbf{\Theta}} - \mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}\|_p \to 0.$$

□

By an argument like Proposition 5.1, it can be proved that if $\Lambda^k \to \Lambda$ strongly, then $\mathbf{M}_{m,\mathbf{\Lambda^k},\mathbf{\Theta}} \to \mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}}$.

6. **Equivalent G-Frames and their Multipliers**

In this section we introduce equivalent g-frames and we show that the multipliers of equivalent g-frames are equivalent in the special sense. For more information about equivalent g-frames see [26].

Definition 6.1. *Let $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ be g-frames for $\mathcal{H}$.*

(1) *If Q is an invertible bounded operator and $\Theta_i = \Lambda_i Q$, for $i \in I$, then we say that $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ are Q-equivalent.*
(2) *We say they are unitarily equivalent if they are Q-equivalent for a unitary operator Q.*
(3) *If Q is a bounded operator (not necessarily invertible) and $\Theta_i = \Lambda_i Q$, then we say that $\mathbf{\Lambda}$ is Q-partial equivalent with $\mathbf{\Theta}$.*

Proposition 6.2. [26] *Let $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ be g-frames with synthesis operators T_Λ and T_Θ, respectivly. Then, $\mathcal{R}_{T_\Theta^*} \subseteq \mathcal{R}_{T_\Lambda^*}$ if and only if there exists a bounded and one to one operator U on $\mathcal{H}$ such that $\Lambda_i = \Theta_i U$, for any $i \in I$. Furthermore, $kerU^* = T_\Lambda(\mathcal{R}_{T_\Lambda^*} \cap (\mathcal{R}_{T_\Theta^*})^\perp)$.*

Corollary 6.3. *Under the assumption of Proposition 6.2, $\mathcal{R}_{T_\Theta^*} = \mathcal{R}_{T_\Lambda^*}$ if and only if U is a bounded and invertible operator.*

Lemma 6.4. *If $\mathbf{\Lambda}$ is a g-Bessel sequence with bound B and $U : \mathcal{H} \to \mathcal{H}$ is a bounded operator, then $\mathbf{\Lambda U} = \{ \Lambda_i U \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-Bessel sequence.*

Proof. It is easy to see that for any $f \in \mathcal{H}$,

$$\sum_i \|\Lambda_i U f\|^2 \leq B\|Uf\|^2 \leq BK^2\|f\|^2.$$

□

Let $U \in \mathcal{L}(\mathcal{H})$ and $\mathbf{\Lambda} = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ be a g-Bessel sequence. Lemma 6.4 shows that $\mathbf{\Lambda U} = \{ \Lambda_i U \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-Bessel sequence, and by a simple calculation we have

$$T_{\mathbf{\Lambda U}} = U^* T_{\mathbf{\Lambda}} \quad \textit{and} \quad T^*_{\mathbf{\Lambda U}} = T^*_{\mathbf{\Lambda}} U. \tag{6.1}$$

(4.2), (4.4) and (6.1) imply that for bounded operators U, $V \in \mathcal{L}(\mathcal{H})$,

$$\mathbf{M}_{m,\mathbf{\Lambda U},\mathbf{\Theta}} = T_{\mathbf{\Lambda U}} D_m T^*_{\mathbf{\Theta}} = U^* T_{\mathbf{\Lambda}} D_m T^*_{\mathbf{\Theta}} = U^* \mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}},$$
$$\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta V}} = T_{\mathbf{\Lambda}} D_m T^*_{\mathbf{\Theta V}} = T_{\mathbf{\Lambda}} D_m T^*_{\mathbf{\Theta}} V = \mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}} V,$$

and

$$\mathbf{M}_{m,\mathbf{\Lambda U},\mathbf{\Theta V}} = T_{\mathbf{\Lambda U}} D_m T^*_{\mathbf{\Theta V}} = U^* T_{\mathbf{\Lambda}} D_m T^*_{\mathbf{\Theta}} V = U^* \mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}} V.$$

The following theorems may be used as criteria for equivalent g-frames.

Theorem 6.5. *Let $m \in \ell^\infty$, $\mathbf{\Lambda} = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$, and $\mathbf{\Theta} = \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ and $\mathbf{\Lambda}' = \{ \Lambda'_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ be g-Bessel sequences. $\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}} = U\mathbf{M}_{m,\mathbf{\Lambda}',\mathbf{\Theta}}$ if and only if $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$ are U^*-equivalent on $\mathcal{R}_{T^*_{m\mathbf{\Theta}}}$.*

Proof. $\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}} = U\mathbf{M}_{m,\mathbf{\Lambda}',\mathbf{\Theta}} \Rightarrow T_{\mathbf{\Lambda}} T^*_{m\mathbf{\Theta}} = U T_{\mathbf{\Lambda}'} T^*_{m\mathbf{\Theta}}$. So, on $\mathcal{R}_{T^*_{m\mathbf{\Theta}}}$ we have $T_{\mathbf{\Lambda}} = U T_{\mathbf{\Lambda}'}$. Therefore, $\mathbf{\Lambda} = \mathbf{\Lambda}' U^*$ and conversely. □

Similarly, we can state the following assertion.

Theorem 6.6. *Let $m \in \ell^\infty$, $\mathbf{\Lambda} = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$, and $\mathbf{\Theta} = \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ and $\mathbf{\Theta}' = \{ \Theta'_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ be g-Bessel sequences. $\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}} = \mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}'} V$ if and only if $\mathbf{\Theta}$ and $\mathbf{\Theta}'$ are V-equivalent on $\mathcal{R}_{T_{\mathbf{\Lambda}} D_m}$.*

Proof. $\mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}} = \mathbf{M}_{m,\mathbf{\Lambda},\mathbf{\Theta}'}V \Rightarrow T_{\mathbf{\Lambda}} D_m T^*_{\mathbf{\Theta}} = T_{\mathbf{\Lambda}} D_m T^*_{\mathbf{\Theta}'}V$. So, on $\mathcal{R}_{T_{\mathbf{\Lambda}} D_m}$ we have $T^*_{\mathbf{\Theta}} = T^*_{\mathbf{\Theta}'}V = T^*_{\mathbf{\Theta}'\mathbf{V}}$. Therefore, $\mathbf{\Theta} = \mathbf{\Theta}'\mathbf{V}$ and conversely. □

7. Fusion Frames Multipliers

Fusion frames (or frames of subspaces) are recent developments of frames that provide a natural mathematical framework for two-stage (or, more generally, hierarchical) data processing. The notion of a fusion frame was introduced in [8] with the main ideas already presented in [7]. A fusion frame is a frame-like collection of subspaces in a Hilbert space. In frame theory, a signal is represented by a collection of scalars, which measure the amplitudes of the projections of the signal onto the frame vectors, whereas in fusion frame theory the signal is represented by the projections of the signal onto the fusion frame subspaces. In a two-stage data processing setup, these projections serve as locally processed data, which can be combined to reconstruct a signal of interest.

Definition 7.1. *Let I be some index set, and $\{w_i\}_{i\in I}$ be a family of weights, i.e., $w_i > 0$, for all $i \in I$. A family of closed subspaces $\{W_i\}_{i\in I}$ of a Hilbert space $\mathcal{H}$ is a fusion frame (or frame of subspaces) with respect to $\{w_i\}_{i\in I}$ for $\mathcal{H}$, if there exist constants C and D, $0 < C \leq D < \infty$, such that*

$$C\|f\|^2 \leq \sum_{i\in I} w_i^2 \|\pi_{W_i}(f)\|^2 \leq D\|f\|^2 \quad \forall f \in \mathcal{H}, \tag{7.1}$$

where π_{W_i} is the projection on the subspace W_i. The constants C and D are called the fusion frame bounds. When the upper bound inequality in (7.1) holds, for all $f \in \mathcal{H}$, $\{W_i\}_{i\in I}$ is called a fusion Bessel sequence with respect to $\{w_i\}_{i\in I}$ with Bessel bound D.

The following is an example of fusion frames.

Example 7.1. *Given a function $g \in L^2(\mathbb{R})$ and $a, b > 0$, the Gabor system determined by g and a, b is defined by*

$$G(g, a, b) = \{E_{ma}T_{nb}g \mid m, n \in \mathbb{Z}\},$$

where E_a and T_a are unitary operators, defined on $L^2(\mathbb{R})$, by

$$E_a f(x) = e^{2\pi iax} f(x), \quad T_a f(x) = f(x - a).$$

$G(g,a,b)$ is unitarily equivalent to $G(g,\frac{1}{q},1)$, where $ab=\frac{1}{q}$, and also

$$G(g,\frac{1}{q},1)=\bigcup_{j=0}^{q-1}\{E_{\frac{1}{q}(mq+1)}T_{na}\}_{m,n\in\mathbb{Z}}.$$

By letting $W_j:=\overline{span}_{m,n\in\mathbf{Z}}\{E_{\frac{1}{q}(mq+1)}T_{na}\}_{m,n\in\mathbb{Z}}, j=0,1,...q-1$, $\{W_j\}_{j=0}^{q-1}$ is a frame of subspaces (for more details, see [7]).

The above example can be related to Gabor multipliers introduced in [19].

It is clear that by putting $\Lambda_i=w_i\pi_{W_i}$ and $\mathcal{H}_i=W_i$ in (2.1), we conclude that fusion frames are a special g-frame, so the results in sections 4 and 5 hold for fusion frames.

Definition 7.2. *Let $\mathcal{W}=(W_i,w_i)_{i\in I}$ and $\mathcal{V}=(V_i,v_i)_{i\in I}$ be fusion Bessel sequences with bounds $B_{\mathcal{W}}$ and $B_{\mathcal{V}}$, respectively and let $m\in\ell^\infty$. The operator $\mathbf{M}_{m,\mathcal{W},\mathcal{V}}:\mathcal{H}\to\mathcal{H}$ defined by*

$$\mathbf{M}_{m,\mathcal{W},\mathcal{V}}f:=\sum_i m_iv_iw_i\pi^*_{W_i}\pi_{V_i}f \tag{7.2}$$

for all $f\in\mathcal{H}$ is called Bessel fusion multiplier of $\mathcal{W}=(W_i,w_i)_{i\in I}$ and $\mathcal{V}=(V_i,v_i)_{i\in I}$.

Note that we consider $\pi_{V_i}:\mathcal{H}\to\mathcal{H}_i$ so its adjoint $\pi^*_{V_i}:\mathcal{H}_i\to\mathcal{H}$ is just the inclusion.

The results shown in Section 4 can be specialized for fusion frames to get:

Theorem 7.3. *Let $\mathcal{W}=(W_i,w_i)_{i\in I}$ and $\mathcal{V}=(V_i,v_i)_{i\in I}$ be fusion Bessel sequences with bounds $B_{\mathcal{W}}$ and $B_{\mathcal{V}}$, respectively.*

1. *If $m\in\ell^\infty$, then the operator $\mathbf{M}_{m,\mathcal{W},\mathcal{V}}$ defined by (7.2) is well defined and $\|\mathbf{M}_{m,\mathcal{W},\mathcal{V}}\|_{op}\le\sqrt{B_{\mathcal{W}}B_{\mathcal{V}}}\|m\|_\infty$.*
2. *$(\mathbf{M}_{m,\mathcal{W},\mathcal{V}})^*=\mathbf{M}_{\overline{m},\mathcal{V},\mathcal{W}}$.*
3. *If $m=(m_i)\in\mathbf{c_0}$ and $(dimV_i)\in\ell^\infty$, then $\mathbf{M}_{m,\mathcal{W},\mathcal{V}}$ is compact.*
4. *If $m=(m_i)\in\ell^p$ and $(dimV_i)\in\ell^\infty$, then $\mathbf{M}_{m,\mathcal{W},\mathcal{V}}$ is a Schatten p-class operator for all $p\ge1$. In particular also for $p=1$ (trace class) and $p=2$ (Hilbert-Schmidt class).*

Using the fact that for frame sequences, $\sum_k\|\psi_k\|^2<\infty$ if and only if its span is finite-dimensional [5], we can rephrase some items of the above theorem:

Corollary 7.4. *Let $\mathcal{W} = (W_i, w_i)_{i\in I}$ and $\mathcal{V} = (V_i, v_i)_{i\in I}$ be fusion Bessel sequences with bounds $B_{\mathcal{W}}$ and $B_{\mathcal{V}}$, respectively. Let $(\phi_k^{(i)})_{k\in K}$ be frames for V_i.*

(1) *If $m = (m_i) \in \mathbf{c_0}$ and $\sup_i \sum_k \|\phi_k^{(i)}\| < \infty$, then $\mathbf{M}_{m,\mathcal{W},\mathcal{V}}$ is compact.*
(2) *If $m = (m_i) \in \ell^p$ and $\sup_i \sum_k \|\phi_k^{(i)}\| < \infty$, then $\mathbf{M}_{m,\mathcal{W},\mathcal{V}}$ is a Schatten p-class operator, for all $p \geq 1$.*

For fusion frames, we can state (7.2) in the following form. Let $\{\psi_k^{(i)}\}_k$ be a frame for W_i and let $\{\phi_k^{(i)}\}_k$ be a frame for V_i, for all $i \in I$. It is easy to show that

$$\mathbf{M}_{m,\mathcal{W},\mathcal{V}} f = \sum_{i,k,l} m_i v_i w_i \langle f, \phi_l^{(i)} \rangle \langle \tilde{\phi}_l^{(i)}, \psi_k^{(i)} \rangle \tilde{\psi}_k^{(i)}$$

for any $f \in \mathcal{H}$,. Where $\{\tilde{\psi}_k^{(i)}\}_k$ is a dual of $\{\psi_k^{(i)}\}_k$ and $\{\tilde{\phi}_k^{(i)}\}_k$ is a dual of $\{\phi_k^{(i)}\}_k$.

In the case that $\{\tilde{\phi}_k^{(i)}\}_k$ and $\{\phi_k^{(i)}\}_k$ are biorthogonal, we have

$$\mathbf{M}_{m,\mathcal{W},\mathcal{V}} f = \sum_{i,k} m_i v_i w_i \langle f, \phi_k^{(i)} \rangle \tilde{\psi}_k^{(i)},$$

which corresponds to a frame multiplier in the sense of [2].

As fusion frames were investigated to 'string together' frames of subspaces, it is natural to look at a 'combination' of fusion frame multipliers and frame multipliers on the related subspaces.

Definition 7.5. *Let $\mathcal{W} = (W_i, w_i)_{i\in I}$ and $\mathcal{V} = (V_i, v_i)_{i\in I}$ be fusion frames and let $m \in \ell^\infty$. Furthermore, let $n^{(i)} = (n_k^{(i)})_{k\in K} \in \ell^\infty$ and $(\psi_k^{(i)})_{k\in K}$ and $(\phi_k^{(i)})_{k\in K}$ be frames for W_i and V_i, respectively. Then, we can define the* combined multiplier $\mathbf{M}_{m,\mathcal{W},\mathcal{V}} \star \left(\mathbf{M}_{n^{(i)},\psi_k^{(i)},\phi_k^{(i)}}\right)_{i\in I}$ *by*

$$\left[\mathbf{M}_{m,\mathcal{W},\mathcal{V}} \star \left(\mathbf{M}_{n^{(i)},\psi_k^{(i)},\phi_k^{(i)}}\right)_{i\in I}\right] f := \sum_i m_i w_i v_i \pi_{W_i} \mathbf{M}_{n^{(i)},\psi_k^{(i)},\phi_k^{(i)}} \pi_{V_i} f.$$

To shorten the notation, we use $\mathbb{M}$ to denote the combined multiplier.

Let $\xi_k^{(i)} = \pi_{W_i}^* \psi_k^{(i)}$ and $\eta_k^{(i)} = \pi_{V_i}^* \phi_k^{(i)}$, i.e. the extension of the frames of the subspaces to frame sequences of the whole space. Then, we can write

$$\mathbb{M}f = \sum_i m_i w_i v_i \pi_{W_i}^* \mathbf{M}^{(i)} \pi_{V_i} f = \sum_i m_i w_i v_i \pi_{W_i}^* \sum_k n_k^{(i)} \left\langle \pi_{V_i} f, \phi_k^{(i)} \right\rangle \psi^{(i)}$$

$$= \sum_{i,k} m_i w_i v_i \sum_k n_k^{(i)} \left\langle f, \pi_{V_i}^* \phi_k^{(i)} \right\rangle \pi_{W_i}^* \psi^{(i)}.$$

So

$$\mathbb{M}f = \sum_{i,k} m_i n_k^{(i)} \langle f, v_i \eta_k^{(i)} \rangle w_i \xi_k^{(i)}.$$

Under this condition, $\{v_i \eta_k^{(i)}\}_{i,k}$ and $\{w_i \xi_k^{(i)}\}_{i,k}$ are frames for $\mathcal{H}$ [7], so this is again a frame multiplier. Therefore, using the above results, we can formulate the following proposition.

Proposition 7.6. *Let $m \in \ell^\infty$ and $n^{(i)} = (n_k^{(i)}) \in \ell^\infty$. Let $\mathcal{W} = (W_i, w_i)_{i \in I}$ and $\mathcal{V} = (V_i, v_i)_{i \in I}$ be fusion frames with upper bounds $B_{\mathcal{W}}$ and $B_{\mathcal{V}}$, respectively. Let $(\psi_k^{(i)})_{k \in K}$ and $(\phi_k^{(i)})_{k \in K}$ be frames for W_i and V_i; respectively. Let $\sup_i \sum_k \|\phi_k^{(i)}\| < \infty$.*

(1) *If either*
- $m = (m_i) \in \mathbf{c_0}$ *and* $n^{(i)} = (n_k^{(i)})_{k \in K} \in \mathbf{c_0}$ *or*
- $n = (n_k^{(i)})_{(k,i) \in K \times I} \in \mathbf{c_0}$,

then $\mathbf{M}_{m,\mathcal{W},\mathcal{V}} \star \left(\mathbf{M}_{n^{(i)}, \psi_k^{(i)}, \phi_k^{(i)}} \right)_{i \in I}$ *is compact.*

(2) *Let $p \geq 1$. If either*
- $m = (m_i) \in \ell^p$ *and* $n = (n_k^{(i)})_{(k,i) \in K \times I} \in l^{(\infty,p)}$ *or*
- $n = (n_k^{(i)})_{(k,i) \in K \times I} \in l^{(p,p)}$,

then $\mathbf{M}_{m,\mathcal{W},\mathcal{V}} \star \left(\mathbf{M}_{n^{(i)}, \psi_k^{(i)}, \phi_k^{(i)}} \right)_{i \in I}$ *is a Schatten p-class operator.*

(3) *Let $p, p' \geq 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. If $m = (m_i) \in \ell^p$ and $n = (n_k^{(i)})_{(k,i) \in K \times I} \in l^{(1,p')}$, then $\mathbf{M}_{m,\mathcal{W},\mathcal{V}} \star \left(\mathbf{M}_{n^{(i)}, \psi_k^{(i)}, \phi_k^{(i)}} \right)_{i \in I}$ is a trace class operator.*

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