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A NOTE ON THE COMPARISON BETWEEN LAPLACE AND SUMUDU TRANSFORMS

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Abstract. In this paper, we discuss the existence of double Sumudu transform and study relationships between Laplace and Sumudu transforms. Further, we apply two transforms to solve linear ordinary differential equations with constant coefficients and non constant coefficients.

1. Introduction

In the literature, there are several works on the theory and applications of integral transforms such as Laplace, Fourier, Mellin, Hankel, to name a few, but very little work on the power series transformation such as Sumudu transform. This is probably, because it is not widely used yet. The Sumudu transform was recently proposed by Watugala; see [12], [13]. The properties were established in [1] and subsequently applied to partial differential equations; see [2], [4], [7] and [9].

In our study, we use the following convolution notation: double convolution between two continuous functions $F(x,y)$ and $G(x,y)$ given by

$$F_1(x,y) ** F_2(x,y) = \int_0^y \int_0^x F_1(x-\theta_1, y-\theta_2) F_2(\theta_1,\theta_2) d\theta_1 d\theta_2; \quad (1.1)$$

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for further details and properties of the double convolutions and derivatives, we refer to [8] and [10]. The single Sumudu transform is defined over the set of the functions

$$A=\left\{f(t):\exists M,\tau_1,\tau_2>0,|f(t)|<Me^{t/\tau_j},\ \text{if } t\in(-1)^j\times[0,\infty)\right\}$$

by

$$F(u)=S\left[f(t);u\right]=\frac{1}{u}\int_0^\infty e^{-t/u}f(t)\,dt,\quad u\in(-\tau_1,\tau_2).$$

A sufficient condition for the existence of the Sumudu transform of a function f is of exponential order, that is, there exist real constants $M>0$, K_1, and K_2, such that $|f(t,x)|\leq Me^{\frac{t}{K_1}+\frac{x}{K_2}}$.

2. Existence of the Sumudu Transform

Theorem 2.1. *If f is of exponential order, then its Sumudu transform $S\left[f(t,x)\right]=F(v,u)$ exists and is given by*

$$F(v,u)=\int_0^\infty\int_0^\infty e^{-\frac{t}{v}-\frac{x}{u}}f(t,x)dtdx,$$

where, $\frac{1}{u}=\frac{1}{\eta}+\frac{i}{\tau}$ and $\frac{1}{v}=\frac{1}{\mu}+\frac{i}{\xi}$. The defining integral for F exists at points $\frac{1}{u}+\frac{1}{v}=\frac{1}{\eta}+\frac{1}{\mu}+\frac{i}{\tau}++\frac{i}{\xi}$ in the right half plane $\frac{1}{\eta}+\frac{1}{\mu}>\frac{1}{K_1}+\frac{1}{K_2}$.

Proof. Using $\frac{1}{u}=\frac{1}{\eta}+\frac{i}{\tau}$ and $\frac{1}{v}=\frac{1}{\mu}+\frac{i}{\xi}$, we can express $F(v,u)$ as:

$$\begin{aligned}F(v,u) &= \int_0^\infty\int_0^\infty f(t,x)\cos\left(\frac{t}{\tau}+\frac{x}{\zeta}\right)e^{-\frac{t}{\eta}-\frac{x}{\mu}}dtdx\\ &\quad -i\int_0^\infty\int_0^\infty f(t,x)\sin\left(\frac{t}{\tau}+\frac{x}{\zeta}\right)e^{-\frac{t}{\eta}-\frac{x}{\mu}}dtdx.\end{aligned}$$

Then, for values of $\frac{1}{\eta}+\frac{1}{\mu}>\frac{1}{K_1}+\frac{1}{K_2}$, we have

$$\begin{aligned}&\int_0^\infty\int_0^\infty|f(t,x)|\left|\cos\left(\frac{t}{\tau}+\frac{x}{\zeta}\right)\right|e^{-\frac{t}{\eta}-\frac{x}{\mu}}dtdx\\ \leq\ & M\int_0^\infty\int_0^\infty e^{\left(\frac{1}{K_1}-\frac{1}{\eta}\right)t+\left(\frac{1}{K_2}-\frac{1}{\mu}\right)x}dtdx\\ \leq\ & M\left(\frac{\eta K_1}{\eta-K_1}\right)\left(\frac{\eta K_2}{\mu-K_2}\right)\end{aligned}$$

and

$$\begin{aligned} &\int_0^\infty \int_0^\infty |f(t,x)| \left|\sin\left(\frac{t}{\tau}+\frac{x}{\zeta}\right)\right| e^{-\frac{t}{\eta}-\frac{x}{\mu}} dtdx \\ \leq\ & M \int_0^\infty \int_0^\infty e^{\left(\frac{1}{K_1}-\frac{1}{\eta}\right)t+\left(\frac{1}{K_2}-\frac{1}{\mu}\right)x} dtdx \\ \leq\ & M\left(\frac{\eta K_1}{\eta-K_1}\right)\left(\frac{\eta K_2}{\mu-K_2}\right) \end{aligned}$$

which imply that the integrals defining the real and imaginary parts of F exist for value of $\mathrm{Re}(\frac{1}{\mathrm{u}}+\frac{1}{\mu}) > \frac{1}{K_1}+\frac{1}{K_2}$, and this completes the proof. □

Thus, we note that for a function f, the sufficient conditions for the existence of the Sumudu transform are to be piecewise continuous and of exponential order.

We also note that the double Sumudu transform of function $f(t,x)$ is defined in [5], by

$$F(v,u) = S_2\left[f(t,x);(v,u)\right] = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-(\frac{t}{v}+\frac{x}{u})} f(t,x)dtdx \tag{2.1}$$

where, S_2 indicates double Sumudu transform and $f(t,x)$ is a function which can be expressed as a convergent infinite series. Now, it is well known that the derivative of convolution for two functions f and g is given by

$$\frac{d}{dx}(f*g)(x) = \frac{d}{dx}f(x)*g(x) \text{ or } f(x)*\frac{d}{dx}g(x)$$

and it can be easily proved that Sumudu transform is:

$$S\left[\tfrac{d}{dx}(f*g)(x);\ v\right] = uS\left[\tfrac{d}{dx}f(x);u\right]S\left[g(x);\ u\right]$$

or

$$uS\left[f(x);\ u\right]S\left[\tfrac{d}{dx}g(x);\ u\right].$$

The double Sumudu and double Laplace transforms have strong relationships that may be expressed either as

$$\text{(I)} \quad uvF(u,v) = \pounds_2\left(f(x,y);\left(\tfrac{1}{u},\tfrac{1}{v}\right)\right)$$

or

$$\text{(II)} \quad psF(p,s) = \pounds_2\left(f(x,y);\left(\tfrac{1}{p},\tfrac{1}{s}\right)\right)$$

where $\pounds_2$ represents the operation of double Laplace transform. In particular, the double Sumudu and double Laplace transforms interchange the image of $\sin(x+t)$ and $\cos(x+t)$. It turns out that

$$S_2\left[\sin(x+t)\right] = \pounds_2\left[\cos(x+t)\right] = \tfrac{u+v}{(1+u)^2(1+v)^2}$$

and $$S_2\left[\cos(x+t)\right] = \pounds_2\left[\sin(x+t)\right] = \tfrac{1}{(1+u)^2(1+v)^2}.$$

3. Convolution Theorem

In the next theorem, we prove double Sumudu transform of the double convolution.

Theorem 3.1. *Let $f(t,x)$ and $g(t,x)$ have double Sumudu transform. Then, double Sumudu transform of the double convolution of f and g,*

$$(f**g)(t,x) = \int_0^t \int_0^x f(\zeta,\eta)g(t-\zeta,x-\eta)d\zeta d\eta,$$

exists and is given by

$$S_2\left[(f**g)(t,x);v,u\right] = uvF(v,u)G(v,u). \tag{3.1}$$

Proof. By using the definition of double Sumudu transform and double convolution, we have

$$S_2\left[(f**g)(t,x);v,u\right] = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-(\frac{t}{v}+\frac{x}{u})}(f**g)(t,x)dtdx$$

$$= \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-(\frac{t}{v}+\frac{x}{u})}\left(\int_0^t \int_0^x f(\zeta,\eta)g(t-\zeta,x-\eta)d\zeta d\eta\right)dtdx.$$

Let $\alpha = t-\zeta$ and $\beta = x-\eta$, and using the valid extension of upper bound of integrals to $t\to\infty$ and $x\to\infty$, we have

$$S_2\left[(f**g)(t,x);v,u\right] = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-(\frac{\zeta}{v}-\frac{\eta}{u})}d\zeta d\eta \int_{-\zeta}^\infty \int_{-\eta}^\infty e^{-(\frac{\alpha}{v}-\frac{\beta}{u})}g(\alpha,\beta)d\alpha d\beta.$$

Since both functions $f(t,x)$ and $g(t,x)$ have are zero, for $t<0$, and $x<0$, it follows with respect to lower limit of integrations that

$$S_2\left[(f**g)(t,x);v,u\right] = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-(\frac{\zeta}{v}-\frac{\eta}{u})}d\zeta d\eta \int_0^\infty \int_0^\infty e^{-(\frac{\alpha}{v}-\frac{\beta}{u})}g(\alpha,\beta)d\alpha d\beta.$$

Then, it is easy to see that

$$S_2\left[(f**g)(t,x);v,u\right] = uvF(v,u)G(v,u).$$

Thus, the relation between double Sumudu of convolution and double Laplace transform of convolution is given by

$$S_2\left[(f**g)(t,x);v,u\right] = \frac{1}{uv}\pounds_2\left[(f**g)(t,x))\right];$$

see [11]. In fact, one can easily prove the following implications for existence:

$$S_2(f(t)) \Rightarrow \pounds_2(f(t)).$$

Note that the inverse of implication need not be true; see [9].

□

Now, we let $P(x) = \sum_{k=0}^{n} \frac{a_k}{x^k}$, where, $n \geq 0$ and $a_n \neq 0$. Then, we define $M_P(x)$ to be a $(1 \times n)$ matrix in the following matrix product:

$$M_P(x) = \left(\frac{1}{x}\ \frac{1}{x^2}\ \frac{1}{x^3} \cdots \frac{1}{x^{n-1}}\right) \begin{pmatrix} a_1 & a_2 & . & . & . & a_n \\ a_2 & a_3 & . & . & a_n & 0 \\ a_3 & . & . & a_n & 0 & 0 \\ . & . & . & . & . & . \\ a_n & 0 & . & . & . & 0 \end{pmatrix}. \tag{3.2}$$

Thus, $M_P(x)$ defines a linear mapping of $\mathbb{C}^n$ into $\mathbb{C}$ in an obvious way. We shall write vectors y in $\mathbb{C}^n$ as row vectors or column vectors interchangeably, whichever is convenient, although when $M_P(x)y$ is to be computed and the matrix representation by Eq(3.2) of $M_P(x)$ is used, then, of course, y is written as a column vector:

$$M_P(x)y = \sum_{i=1}^{n} \frac{1}{x^i} \sum_{k=0}^{n-i} a_{i+k} y_k, \tag{3.3}$$

for any $y = (y_0, y_1, ..., y_{n-1}) \in \mathbb{C}^n$. If $n = 0$, $M_P(x)$ defines a unique linear mapping of $\{0\} = \mathbb{C}^0$ into $\mathbb{C}$ (empty matrix). In general, if $n > 0$ and f is $n-1$ times differentiable on an interval (a, b), with $a < b$, we shall write

$$\varphi(f; a; n) = \left(f(a+),\ f'(a+), ..., f^{(n-1)}(a+)\right) \in \mathbb{C}^n$$

and

$$\phi(f; b; n) = \left(f(b-),\ f'(b-), ..., f^{(n-1)}(b-)\right) \in \mathbb{C}^n.$$

If $a = 0$, we write $\varphi(f; n)$ for $\varphi(f; 0; n)$. If $n = 0$, we define

$$\varphi(f; a; 0) = \phi(f; a; 0) = 0 \in \mathbb{C}^0.$$

In the next section, we give the Sumudu transform of higher derivatives.

4. **Sumudu Transform of Higher Derivatives**

Theorem 4.1. *Let f be n times differentiable on $(0,\infty)$ and let $f(t)=0$, for $t<0$. Suppose that $f^{(n)}\in L_{loc.}$ Then, $f^{(k)}\in L_{loc}$, for $0\le k\le n-1$, $dom(Sf)\subset dom(Sf^{(n)})$, and for any polynomial P of degree n,*

$$P(u)S(y)(u)=S\left(f\right)(u)+M_P(u)\varphi(y,n), \tag{4.1}$$

for $u\in dom(Sf)$. Thus,

$$(Sf^{(n)})(u)=\frac{1}{u^n}(Sf)(u)-\left(\frac{1}{u^n},\frac{1}{u^{n-1}},...,\frac{1}{u}\right)\varphi(f;n). \tag{4.2}$$

In particular, if $n=2$, then we have

$$(Sf'')(u)=\frac{1}{u^2}(Sf)(u)-\frac{1}{u^2}f(0+)-\frac{1}{u}f'(0+). \tag{4.3}$$

Proof. We use induction on n. The result is trivially true if the case $n=0$ and $n=1$. Suppose now the result is true for some $n\ge 1$ and let $P(x)=\sum\limits_{k=0}^{n+1}\frac{a_k}{x^k}$ have degree $n+1$. Then, the first two statements follow by putting $z=f'$ and using the induction hypothesis. Now, write $P(x)=a_0+\frac{1}{x}W(x)$, where, $W(x)=\sum\limits_{k=0}^{n}\frac{a_{k+1}}{x^k}$. Then, $P(\dot{D})f=a_0f+W(\dot{D})z$ and therefore,

$$\begin{aligned} S\left[P(\dot{D})f\right](u) &= a_0S\left[f\right](u)+S\left[W(\dot{D})z\right](u)-M_W(u)\varphi(z;n)\\ &= a_0S\left[f\right](u)+W(u)\left[\frac{1}{u}S\left[f\right](u)-\frac{1}{u}f(0+)\right]\\ &\quad -\sum_{i=1}^{n}\frac{1}{u^i}\sum_{k=0}^{n-i}a_{i+k+1}f^{\ (k+1)}(0+), \end{aligned} \tag{4.4}$$

using Eq. (3.3) and $z^{(k)} = f^{(k+1)}$. The summation above can be written in the form:

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{u^i} \sum_{k=1}^{n-i+1} a_{i+k} f^{\ (k)}(0+) &= \sum_{i=1}^{n} \frac{1}{u^i} \sum_{k=0}^{n-i+1} a_{i+k} f^{\ (k)}(0+) - \sum_{i=1}^{n} \frac{1}{u^i} a_i f(0+) \\
&= \sum_{i=1}^{n+1} \frac{1}{u^i} \sum_{k=0}^{n-i+1} a_{i+k} f^{\ (k)}(0+) \\
&\quad -\frac{1}{u}\left[\frac{1}{u^n} a_{n+1} f(0+) + \sum_{i=1}^{n} \frac{1}{u^{i-1}} a_i f(0+)\right] \\
&= M_P(u)\varphi(f;n) - \frac{1}{u} W(u) f(0+).
\end{aligned}$$

Thus,

$$\begin{aligned}
S\left[P(\dot{D})f\right](u) &= \left[a_0 + W(u)\frac{1}{u}\right] S\left[f\right](u) \\
&\quad -\frac{1}{u} W(u) f(0+) - M_P(u)\varphi(f;n) + \frac{1}{u} W(u) f(0+) \\
&= P(u) S\left(f\right)(u) - M_P(u)\varphi(f;n).
\end{aligned}$$

In general, if f be differentiable on (a, b) with $a < b$, and $f(t) = 0$, for $t < a$ or $t > b$ and $f^{(n)} \in L_{loc}$, then for all u,

$$S\left[P(\dot{D})f\right](u) = P(u)(Sf)(u) - M_P(u)\left[e^{-\frac{a}{u}}\varphi(f;a;n) - e^{-\frac{b}{u}}\phi(f;b;n)\right]. \tag{4.5}$$

□

In particular if we consider $y(t) = \sin(t)$. Clearly, $y'' + y = 0$, and so if we put $f = yH$, then $f'' + f = 0$. Thus,

$$\left(D^2 + 1\right) f = 0.$$

Since $dom(Sf)$ is contained in $(0, \infty)$, we have from Eq. (4.1), and Eq. (4.2), with $n = 2$ and $P(x) = x^2 + 1$, for $u > 0$,

$$0 = \left(\frac{1}{u^2} + 1\right) S(f) - \begin{pmatrix} \frac{1}{u} & \frac{1}{u^2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $\varphi(y, 2) = (f(0),\ f\,'(0)) = (0, 1)$. Thus, from the above equation, we get

$$S\left[\sin(t)H(t)\right] = \frac{u}{u^2 + 1}.$$

Now let y be n times differentiable on $(0,\infty)$, zero on $(-\infty,0)$ and satisfy

$$P(\dot{D})y=\sum_{i=1}^{n}f_i*g_i, \tag{4.6}$$

under the initial conditions:

$$y(0)=y_0,\ y'(0)=y_1,...,\ y^{(n-1)}(0)=y_{n-1}.$$

Then, $y^{(k)}$ is locally integrable and Sumudu transformable for $0\leq k\leq n$, and thus Sumudu transform of Eq. (4.6) is given by

$$M_P(u)\varphi(y,n)=\left(\frac{1}{u}\frac{1}{u^2}\cdots\frac{1}{u^n}\right)\begin{pmatrix} a_1 & a_2 & . & . & . & a_n \\ a_2 & a_3 & . & . & a_n & 0 \\ a_3 & . & . & a_n & 0 & 0 \\ . & . & . & . & . & . \\ a_n & 0 & . & . & . & 0 \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \\ . \\ . \\ . \\ y_{n-1} \end{pmatrix}, \tag{4.7}$$

and the non homogeneous term is a single convolution. In particular, if $n=2$, we have

$$\left(\frac{a_2}{u^2}+\frac{a_1}{u}+a_0\right)S(y)(u)=S\left(f_1*g_1+f_2*g_2\right)(u)+\left(\frac{1}{u}\frac{1}{u^2}\right)\begin{pmatrix} a_1 & a_2 \\ a_2 & 0 \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}. \tag{4.8}$$

On taking the inverse Sumudu transform for Eq(4.8), we have the solution of Eq. (4.6) as follows:

$$y(t)=S^{-1}\left[\frac{u\left[S\left(f_1\right)S\left(g_1\right)+S\left(f_2\right)S\left(g_2\right)\right]}{P(u)}\right]+S^{-1}\left[\frac{a_1y_0u+a_2y_0+a_2y_1u}{u^2}\right], \tag{4.9}$$

provided that the inverse transform exists for each term in the right hand side of Eq. (4.9). Now, let us multiply the right hand side of Eq. (4.6) by a polynomial $\Psi(t)=\sum_{k=0}^{n}t^k$. Then, we obtain an equation with non constant coefficients in the form of

$$\Psi(t)*\left[P(\dot{D})y\right]=\sum_{i=1}^{n}f_i*g_i, \tag{4.10}$$

under the same initial conditions as given above. By taking the Sumudu transform for Eq. (4.10) and using the initial conditions, we have

$$\begin{aligned} S\left[y\right](u) &= \frac{F(u)G(u)}{k!u^k P(u)} \\ &+ \frac{1}{P(u)}\left(\frac{1}{u}\ \frac{1}{u^2}\cdots\frac{1}{u^{n+1}}\right)\begin{pmatrix} a_1 & a_2 & . & . & . & a_n \\ a_2 & a_3 & . & . & a_n & 0 \\ a_3 & . & . & a_n & 0 & 0 \\ . & . & . & . & . & . \\ a_n & 0 & . & . & . & 0 \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \\ . \\ . \\ . \\ y_{n-1} \end{pmatrix}. \end{aligned}$$

By taking inverse of the Sumudu transform, we have the solution provided that the inverse exist.

In particular, consider the differential equation in the form of

$$\begin{aligned} y''' - y'' + 4y' - 4y &= 2\cos(2t) - \sin(2t) \\ y(0) &= 1,\ y'(0) = 4,\ y''(0) = 1. \end{aligned} \tag{4.11}$$

On using Eq. (4.1), we have

$$\left(\frac{1}{u^3} - \frac{1}{u^2} + \frac{4}{u} - 4\right)S(y)(u) = S\left[\cos 2t\ \right] + S\left[\sin(2t)\right] + M_P(u)\varphi(y,4) \tag{4.12}$$

and

$$\begin{aligned} M_P(u)\varphi(y,4) &= \left(\frac{1}{u}\ \frac{1}{u^2}\ \frac{1}{u^3}\right)\begin{pmatrix} 4 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \\ &= \frac{1}{u} + \frac{3}{u^2} + \frac{1}{u^3}. \end{aligned}$$

Then, we obtain:

$$Y(u) = \frac{u^3\,(2u+1)}{(4u^2+1)(1-u+4u^2-4u^3)} + \frac{(u^2+3u+1)}{(1-u+4u^2-4u^3)}. \tag{4.13}$$

Replacing the complex variable u by $\frac{1}{s}$, Eq. (4.13) turns to:

$$Y\left(\frac{1}{s}\right) = \frac{s(s+2)}{(s^2+4)(s^2+4)(s-1)} + \frac{s(s^2+3s+1)}{(s^2+4)(s-1)}. \tag{4.14}$$

Now in order to obtain the inverse Sumudu transform for Eq.(4.14), we use

$$S^{-1}(Y(s)) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}Y\left(\frac{1}{s}\right)\frac{ds}{s} = \sum \text{residues}\left[e^{st}\frac{Y(\frac{1}{s})}{s}\right].$$

Thus, the solution of Eq. (4.11) is given by:

$$y(t) = \frac{13}{8}\sin(2t) - \frac{1}{4}t\cos(2t) + e^t.$$

Now, if we consider to multiply the left hand side of Eq. (4.11) by the non constant coefficient t^2, then Eq. (4.11) becomes

$$\begin{aligned} t^2 * \left(y''' - y'' + 4y' - 4y\right) &= 2\cos(2t) - \sin(2t) \\ y(0) &= 1,\ y'(0) = 4,\ y''(0) = 1. \end{aligned} \tag{4.15}$$

By applying a similar method, we obtain the solution of Eq. (4.15) in the form:

$$y_1(t) = \cos(2t) - t\sin(2t) + \frac{3}{2}\sin(2t).$$

Remark 4.2. A very interesting fact about the Sumudu transform is that the original function and its Sumudu transform have the same Taylor coefficients except for the factor $n!$. Thus, if $f(t) = \sum_{n=0}^{\infty} a_n t^n$, then $F(u) = \sum_{n=0}^{\infty} n! a_n t^n$; see [6]. Furthermore, Laplace and Sumudu transforms of the Dirac delta function and the Heaviside function satisfy:

$$S_2\left[H(x,t)\right] = \pounds_2\left[\delta(x,t)\right] = 1,$$

and
$$S_2\left[\delta(x,t)\right] = \pounds_2\left[H(x,t)\right] = \tfrac{1}{uv};$$

for details, see [3], where the concept of the Sumudu transform was generalized to distributions.

By using discontinuous or impulsive forcing terms, the Sumudu transform can be used to solve ODE(s) as well as PDE(s) in engineering problems.

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