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A COMPOSITE EXPLICIT ITERATIVE PROCESS WITH A VISCOSITY METHOD FOR LIPSCHITZIAN SEMIGROUP IN A SMOOTH BANACH SPACE

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Abstract. We introduce a new explicit composite iteration scheme with a viscosity iteration method for approximating a common fixed point of Lipschitzian semigroup on a compact convex subset of a smooth Banach space. We show that the iterative sequence converges strongly to a common fixed point under some parameter controlling conditions. Our results extend and improve the recent results by Saeidi [S. Saeidi, *Fixed Point Theory Appl.* (2008) Art. ID 363257 17pp.], Zhang et al. [S.-S. Zhang, L. Yang and J.-A. Liu, *Appl. Math. Mech. (English Ed.)* **28** (2007) 1287–1297.] and several others.

1. Introduction

Let E be a Banach space and let E^* be the topological dual of E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^*\rangle$ or $x^*(x)$. With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x^*\|^2 = \|x\|^2\}.$$

Using the Hahn-Banach theorem, it immediately follows that $J(x) \neq \emptyset$, for each $x \in E$. A Banach space E is said to be smooth if the duality

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mapping J of E is single valued. We know that if E is smooth, then J is norm to weak-star continuous; see [7, 20].

Let C be a nonempty closed convex subset of E. A mapping $T : C \to C$ is said to be

(i) *Lipschitzian* with Lipschitz constant $L > 0$ if

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C;$$

(ii) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

(iii) *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of positive numbers satisfying the property $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - T^n y\| \leq k_n\|x - y\|, \quad \forall x, y \in C.$$

Recall that a self mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ and $x, y \in C$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|.$$

Clearly, every nonexpansive mapping T is asymptotically nonexpansive with sequence $\{1\}$. Also, every asymptotically nonexpansive mapping is uniformly L-Lipschitzian with $L = \sup_{n\in\mathbb{N}} k_n$.

In 1953, Mann [9] introduced an iterative process as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \tag{1.1}$$

where, the initial guess $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. The Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Reich [10]. In an infinite-dimensional Hilbert space, the Mann iteration can conclude only weak convergence [2]. Attempts to modify the Mann iteration method (1.1) so that strong convergence is guaranteed have recently been made. Saeidi [16] considered an implicit iteration process of Mann's type for (asymptotically) quasi-nonexpansive affine mappings in normed and Banach spaces and prove the weak and strong convergence of the process to a fixed point of the mappings.

On the other hand, let C be a nonempty closed convex subset of a Banach space E. Then, $\{T(s) : s \in \mathbb{R}^+\}$ is called a *strongly continuous semigroup of Lipschitzian mappings* from C into itself if it satisfies the following conditions:

(i) for each $s > 0$, there exists a function $k(\cdot) : (0, \infty) \to (0, \infty)$ such that

$$\|T(s)x - T(s)y\| \leq k(s)\|x - y\|, \quad \forall x, y \in C;$$

(ii) $T(0)x = x$ for each $x \in C$;
(iii) $T(s_1 + s_2)x = T(s_1)T(s_2)x$ for any $s_1, s_2 \in \mathbb{R}^+$ and $x \in C$;
(iv) for each $x \in C$, the mapping $T(\cdot)x$ from $\mathbb{R}^+$ into C is continuous.

If $k(s) = L$ for all $s > 0$ in (i), then $\{T(s) : s \in \mathbb{R}^+\}$ is called a *strongly continuous semigroup of uniformly L-Lipschitzian mappings*. If $k(s) = 1$ for all $s > 0$ in (i), then $\{T(s) : s \in \mathbb{R}^+\}$ is called a *strongly continuous semigroup of nonexpansive mappings* (see [12]). For a semigroup S, we can define a partial preordering $\prec$ on S by $a \prec b$ if and only if $aS \supset bS$. If S is a *left reversible semigroup* (i.e., $aS \cap bS \neq \emptyset$ for $a, b \in S$), then it is a directed set. (Indeed, for every $a, b \in S$, applying $aS \cap bS \neq \emptyset$, there exist $a', b' \in S$ with $aa' = bb'$; by taking $c = aa' = bb'$, we have $cS \subseteq aS \cap bS$, and then $a \prec c$ and $b \prec c$.) If a semigroup S is left amenable, then S is left reversible [5].

Let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of a left reversible semigroup S as Lipschitzian mappings on C with Lipschitz constants $\{k(s) : s \in S\}$. We shall say that $\mathcal{S}$ is an *asymptotically nonexpansive semigroup* on C, if there holds the uniform Lipschitzian condition $\lim_s k(s) \leq 1$ on the Lipschitz constants. (Note that a left reversible semigroup is a directed set.) It is worth mentioning that there is a notion of asymptotically nonexpansive defined depending on left ideals in a semigroup in [4] and [6].

In 2007, Lau et al. [8] introduced the following Mann's explicit iteration process:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T(\mu_n)x_n, \quad \forall n \geq 1 \tag{1.2}$$

for a semigroup $\mathcal{S} = \{T(s) : s \in S\}$ of nonexpansive mappings on a compact convex subset C of a smooth and strictly convex Banach space.

Extending the above results to the nonexpansive semigroup case, Zhang et al. [17] introduce the following composite iteration scheme:

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)T(t_n)x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \end{cases} \tag{1.3}$$

where, $\{T(t) : t \geq 0\}$ is a nonexpansive semigroup from C to C, u is an arbitrary (but fixed) element in C, $\{\alpha_n\} \subset (0, 1)$ *and* $\{\beta_n\} \subset [0, 1], \{t_n\} \subset \mathbb{R}^+$, and proved some strong convergence theorems of explicit composite iteration scheme for nonexpansive semigroups in the framework of a reflexive Banach space with a uniformly Gâteaux differentiable norm, uniformly smooth Banach space and uniformly convex Banach space with a weakly continuous normalized duality mapping.

Saeidi [13] introduced the following viscosity iterative scheme,

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)x_n, \forall n \geq 1 \tag{1.4}$$

for a representation of S as Lipschitzian mappings on a compact convex subset C of a smooth Banach space E with respect to a left regular sequence $\{\mu_n\}$ of means defined on an appropriate invariant subspace of $l^\infty(S)$; for some related results, we refer the readers to [7, 20].

Here, motivated and inspired by the idea of Zhang et al. [17] and Saeidi [13], we introduce the composite explicit viscosity iterative schemes as follows:

$$\begin{cases} y_n = \delta_n x_n + (1-\delta_n)T(\mu_n)x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \forall n \geq 1 \end{cases} \tag{1.5}$$

for a semigroup $\mathcal{S} = \{T(s) : s \in S\}$ on a compact convex subset C of a smooth Banach space E with respect to a left regular sequence $\{\mu_n\}$ of means defined on an appropriate invariant subspace of $l^\infty(S)$. Then, we prove that the sequence $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$, which is the unique solution of the variational inequality,

$$\langle (f-I)z, J(p-z)\rangle \leq 0, \forall p \in F(\mathcal{S}).$$

Equivalently, we have $z = Pfz$, where P is the unique sunny nonexpansive retraction of C onto $F(\mathcal{S})$. Our results improve and extend the recent results of Saeidi [13] and Zhang Shi-Sheng et al. [17] to Lipschitzian semigroup mapping.

2. Preliminaries

Let E be a Banach space and let C be a closed convex subset of E. Then,

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle \tag{2.1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.2}$$

for all $x, y \in E$ and $\lambda \in [0,1]$.

Let S be a semigroup. We denote by $l^\infty(S)$ the Banach space of all bounded real valued functions on S with the supremum norm. For each $s \in S$, we define l_s and r_s on $l^\infty(S)$ by $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for each $t \in S$ and $f \in l^\infty(S)$. Let X be a subspace of $l^\infty(S)$

containing 1 and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$, instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (resp. right invariant), i.e., $l_s(X) \subset X$ (resp. $r_s(X) \subset X$), for each $s \in S$. A mean μ on X is said to be left invariant (resp. right invariant) if $\mu(l_s f) = \mu(f)$ (resp. $\mu(r_s f) = \mu(f)$) for each $s \in S$ and $f \in X$. X is said to be left (resp. right) *amenable* if X has a left (resp. right) invariant mean. X is amenable if X is both left and right amenable. A net $\{\mu_\alpha\}$ of means on X is said to be *strongly left regular* if

$$\lim_{\alpha} \|l_s^* \mu_\alpha - \mu_\alpha\| = 0$$

for each $s \in S$, where l_s^* is the adjoint operator of l_s. Let C be a nonempty closed and convex subset of E. Throughout this paper, S will always denote a semigroup with an identity e. S is called left reversible if any two right ideals in S have nonvoid intersection, i.e., $aS \cap bS \neq \emptyset$ for $a, b \in S$. In this case, we can define a partial ordering $\prec$ on S by $a \prec b$ if and only if $aS \supset bS$. It is easy to see $t \prec ts, (\forall t, s \in S)$. Furthermore, if $t \prec s$, then $pt \prec ps$ for all $p \in S$. If a semigroup S is left amenable, then S is left reversible. But the converse is not true.

$\mathcal{S} = \{T(s) : s \in S\}$ is called a representation of S as Lipschitzian mappings on C if for each $s \in S$, the mapping $T(s)$ is Lipschitzian mapping on C with Lipschitz constant $k(s)$, and $T(st) = T(s)T(t)$ for $s, t \in S$. We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, and by C_a the set of almost periodic elements in C, i.e., all $x \in C$ such that $\{T(s)x : s \in S\}$ is relatively compact in the norm topology of E. We will call a subspace X of $l^\infty(S)$, $\mathcal{S} -$ *stable* if the functions $s \mapsto \langle T(s)x, x^* \rangle$ and $s \mapsto \|T(s)x - y\|$ on S are in X for all $x, y \in C$ and $x^* \in E^*$. We know that if μ is a mean on X and if for each $x^* \in E^*$, the function $s \mapsto \langle T(s)x, x^* \rangle$ is contained in X and C is weakly compact, then there exists a unique point x_0 of E such that

$$\mu_s \langle T(s)x, x^* \rangle = \langle x_0, x^* \rangle$$

for each $x^* \in E^*$. We denote such a point x_0 by $T(\mu)x$. Note that $T(\mu)z = z$ for each $z \in F(\mathcal{S})$; see [3, 14, 19]. Let D be a subset of B, where B is a subset of a Banach space E and let P be a retraction of B onto D. Then, P is said to be sunny [11] if for each $x \in B$ and $t \geq 0$, with $Px + t(x - Px) \in B$,

$$P(Px + t(x - Px)) = Px.$$

A subset D of B is said to be a sunny nonexpansive retract of B if there exists a sunny nonexpansive retraction P of B onto D. We know that if E is smooth and P is a retraction of B onto D, then P is sunny and nonexpansive if and only if for each $x \in B$ and $z \in D$,

$$\langle x - Px, J(z - Px)\rangle \leq 0. \tag{2.3}$$

For more details see [7, 20].

We need the following lemmas to prove our main results.

Lemma 2.1. ([15]) *Let S be a left reversible semigroup and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as Lipschitzian mappings from a nonempty weakly compact convex subset C of a Banach space E into C, with the uniform Lipschitzian condition* $\lim_s k(s) \leq 1$ *on the Lipschitz constants of the mappings. Let X be a left invariant $\mathcal{S}-stable$ subspace of $l^\infty(S)$ containing 1, and μ be a left invariant mean on X. Then, $F(\mathcal{S}) = F(T(\mu)) \cap C_a$.*

Corollary 2.2. ([13]) *Let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X. If $z \in C_a$ and* $\liminf_{n\to\infty} \|T(\mu_n)z - z\| = 0$, *then z is a common fixed point for $\mathcal{S}$.*

Lemma 2.3. ([13]) *Let S be a left reversible semigroup and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as Lipschitzian mappings from a nonempty weakly compact convex subset C of a Banach space E into C, with the uniform Lipschitzian condition* $\lim_s k(s) \leq 1$ *on the Lipschitz constants of the mappings. Let X be a left invariant subspace of $l^\infty(S)$ containing 1 such that the mappings $s \mapsto \langle T(s)x, x^*\rangle$ be in X for all $x \in X$ and $x^* \in E^*$, and $\{\mu_n\}$ be a strongly left regular sequence of means on X. Then,*

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T(\mu_n)x - T(\mu_n)y\| - \|x - y\|) \leq 0.$$

Remark 2.1. Taking in Lemma 2.3,

$$c_n = \sup_{x,y\in C} (\|T(\mu_n)x - T(\mu_n)y\| - \|x - y\|), \forall n, \tag{2.4}$$

we obtain $\limsup_{n\to\infty} c_n \leq 0$. Moreover,

$$\|T(\mu_n)x - T(\mu_n)y\| \leq \|x - y\| + c_n, \forall x, y \in C. \tag{2.5}$$

Corollary 2.4. ([13]) *Let S be a left reversible semigroup and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as Lipschitzian mappings from a nonempty compact convex subset C of a Banach space E into C, with the uniform Lipschitzian condition* $\lim_s k(s) \leq 1$. *Let X be a left invariant*

$\mathcal{S}-$stable subspace of $l^\infty(S)$ containing 1, and μ be a left invariant mean on X. Then, $T(\mu)$ is nonexpansive and $F(\mathcal{S}) \neq \emptyset$. Moreover, if E is smooth, then $F(\mathcal{S})$ is a sunny nonexpansive retract of C and the sunny nonexpansive retraction of C onto $F(\mathcal{S})$ is unique.

Lemma 2.5. ([7, 20]) *Let X be a real Banach space and let J be the duality mapping. Then, for any given $x, y \in X$ and $j(x+y) \in J(x+y)$, there holds the inequality,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle.$$

Lemma 2.6. ([18]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$, for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.7. ([21]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, \ n \geq 0,$$

where, $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section, we prove a strong convergence theorem for Lipschitzian semigroup in a smooth Banach space.

Theorem 3.1. *Let S be a left reversible semigroup and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as Lipschitzian mappings from a nonempty compact convex subset C of a smooth Banach space E into itself, with the uniform Lipschitzian condition $\lim_s k(s) \leq 1$, and f be a contraction of C into itself with coefficient $\alpha \in (0,1)$. Let X be a left invariant $\mathcal{S}$-stable subspace of $l^\infty(S)$ containing 1, $\{\mu_n\}$ be a strongly left regular sequence of means on X such that $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$ and $\{c_n\}$ be the sequence defined by (2.4). Suppose the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. The following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} \delta_n = 0$;

(iii) $\limsup_{n\to\infty} \frac{c_n}{\alpha_n} \leq 0$; *(note that, by Remark 2.1,* $\limsup_{n\to\infty} c_n \leq 0$*);*
(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
If for arbitrary given $x_1 \in C$*, the sequence* $\{x_n\}$ *is generated by* (1.5)*, then* $\{x_n\}$ *converges strongly to* $z \in F(\mathcal{S})$*, which is the unique solution of the variational inequality*

$$\langle (f - I)z, J(p - z)\rangle \leq 0, \forall p \in F(\mathcal{S}).$$

Equivalently, we have $z = Pfz$*, where* P *is the unique sunny nonexpansive retraction of* C *onto* $F(\mathcal{S})$*.*

Proof. First, we prove that $\{x_n\}$ is bounded. Let $p \in F(\mathcal{S})$. Then, by the nonexpansiveness of $T(\mu_n)$ and (2.5), we have

$$\begin{aligned}
\|y_n - p\| &= \|\delta_n x_n + (1-\delta_n)T(\mu_n)x_n - p\| \\
&\leq \delta_n\|x_n - p\| + (1-\delta_n)\|T(\mu_n)x_n - p\| \\
&\leq \delta_n\|x_n - p\| + (1-\delta_n)(\|x_n - p\| + c_n) \\
&= \delta_n\|x_n - p\| + (1-\delta_n)\|x_n - p\| + (1-\delta_n)c_n \\
&\leq \|x_n - p\| + c_n,
\end{aligned}$$

and so,

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - p\| \\
&= \|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) + \gamma_n(y_n - p)\| \\
&\leq \alpha_n\|f(x_n) - p\| + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\| \\
&\leq \alpha\alpha_n\|x_n - p\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| \\
&\quad +\gamma_n\|x_n - p\| + \gamma_n c_n \\
&= (1 - \alpha_n + \alpha\alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\| + \gamma_n c_n \\
&= (1 - \alpha_n(1-\alpha))\|x_n - p\| + \gamma_n c_n \\
&\quad +\alpha_n(1-\alpha)\frac{\|f(p) - p\|}{1-\alpha}.
\end{aligned}$$

By induction and (2.4), we get

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{\|f(p) - p\|}{1-\alpha}\}.$$

This implies that $\{x_n\}$ is bounded, and so are $\{f(x_n)\}$ and $\{y_n\}$. In fact, letting $M = \|p\| + \max\{\|x_1 - p\|, \frac{\|f(p)-p\|}{1-\alpha}\}$, for any $n \geq 1$, we have

$$\|T(\mu_n)x_n\| \leq \|T(\mu_n)x_n - p\| + \|p\| \leq \|x_n - p\| + \|p\| \leq M,$$

and then we also have $\|T(\mu_n)x_n\|$ is bounded.

Let $\{\omega_n\}$ be a sequence in C. By Saeidi ([13], Theorem 3.1, STEP 1, p. 7), we can show that

$$\lim_{n\to\infty} \|T(\mu_{n+1})\omega_n - T(\mu_n)\omega_n\| = 0. \tag{3.1}$$

Next, we show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, and by Lemma 2.3, we observe that

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|\delta_{n+1}x_{n+1} + (1-\delta_{n+1})T(\mu_{n+1})x_{n+1} \\
&\quad - (\delta_n x_n + (1-\delta_n)T(\mu_n)x_n)\| \\
&= \|\delta_{n+1}x_{n+1} - \delta_{n+1}x_n + \delta_{n+1}x_n \\
&\quad + (1-\delta_{n+1})T(\mu_{n+1})x_{n+1} - (1-\delta_{n+1})T(\mu_n)x_n \\
&\quad + (1-\delta_{n+1})T(\mu_n)x_n - \delta_n x_n - (1-\delta_n)T(\mu_n)x_n\| \\
&= \|\delta_{n+1}(x_{n+1} - x_n) + (\delta_{n+1} - \delta_n)x_n \\
&\quad + (1-\delta_{n+1})(T(\mu_{n+1})x_{n+1} - T(\mu_n)x_n) \\
&\quad + (\delta_n - \delta_{n+1})T(\mu_n)x_n\| \\
&\le \delta_{n+1}\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|(\|x_n\| + \|T(\mu_n)x_n\|) \\
&\quad + \|T(\mu_{n+1})x_{n+1} - T(\mu_n)x_n\| \\
&\le \delta_{n+1}\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|(\|x_n\| + \|T(\mu_n)x_n\|) \\
&\quad + \|T(\mu_{n+1})x_{n+1} - T(\mu_n)x_{n+1}\| \\
&\quad + \|T(\mu_n)x_{n+1} - T(\mu_n)x_n\| \\
&\le \delta_{n+1}\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|(\|x_n\| + \|T(\mu_n)x_n\|) \\
&\quad + \|T(\mu_{n+1})x_{n+1} - T(\mu_n)x_{n+1}\| + \|x_{n+1} - x_n\| + c_n.
\end{aligned}$$

Setting $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$, we see that $z_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$. Then, we compute

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \|\frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}\| \\
&= \|\frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n y_n}{1-\beta_n}\| \\
&= \|\frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_{n+1}f(x_n)}{1-\beta_{n+1}} \\
&\quad + \frac{\alpha_{n+1}f(x_n)}{1-\beta_{n+1}} - \frac{\gamma_{n+1}y_n}{1-\beta_{n+1}} + \frac{\gamma_{n+1}y_n}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n y_n}{1-\beta_n}\|
\end{aligned}$$

$$
\begin{aligned}
&= \|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1})-f(x_n)) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(y_{n+1}-y_n) \\
&\quad +(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n})f(x_n) + (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n})y_n\| \\
&\leq \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|y_{n+1}-y_n\| \\
&\quad +|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|\|f(x_n)\| \\
&\quad +|\frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}} - \frac{1-\beta_n-\alpha_n}{1-\beta_n}|\|y_n\| \\
&= \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|y_{n+1}-y_n\| \\
&\quad +|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\| + \|y_n\|) \\
&\leq \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\| + \|y_n\|) \\
&\quad +\|y_{n+1}-y_n\| \\
&\leq \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\| + \|y_n\|) \\
&\quad +\delta_{n+1}\|x_{n+1}-x_n\| + |\delta_{n+1}-\delta_n|(\|x_n\| + \|T(\mu_n)x_n\|) \\
&\quad +\|T(\mu_{n+1})x_{n+1} - T(\mu_n)x_{n+1}\| + \|x_{n+1}-x_n\| + c_n.
\end{aligned}
$$

Therefore, we observe that

$$
\begin{aligned}
\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| &\leq (\frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}} + \delta_{n+1})\|x_{n+1}-x_n\| \\
&\quad +|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\| + \|y_n\|) \\
&\quad +|\delta_{n+1}-\delta_n|(\|x_n\| + \|T(\mu_n)x_n\|) \\
&\quad +\|T(\mu_{n+1})x_{n+1} - T(\mu_n)x_{n+1}\| + c_n.
\end{aligned}
$$

It follow from (i), (ii), (iv), (3.1) and Lemma 2.3, that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \leq 0.$$

Applying Lemma 2.6, we obtain $\lim_{n\to\infty}\|z_n - x_n\| = 0$, and also

$$\|x_{n+1}-x_n\| = (1-\beta_n)\|z_n - x_n\| \to 0,$$

as $n \to \infty$. Therefore, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.2}$$

Next, we show that the set of all limit points of $\{x_n\}$ is a subset of $F(\mathcal{S})$. Let p be a limit point of $\{x_n\}$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ converging strongly to p. Note that

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
= \; & \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n(\delta_n x_n + (1-\delta_n)T(\mu_n)x_n) - x_n\| \\
= \; & \|\alpha_n f(x_n) - (1-\beta_n)x_n + \gamma_n(\delta_n x_n + (1-\delta_n)T(\mu_n)x_n)\| \\
= \; & \|\alpha_n f(x_n) - (1-\beta_n)x_n + \gamma_n\delta_n x_n + \gamma_n T(\mu_n)x_n - \gamma_n\delta_n T(\mu_n)x_n\| \\
= \; & \|\alpha_n f(x_n) - (1-\beta_n)x_n + \gamma_n\delta_n x_n + (1-\alpha_n-\beta_n)T(\mu_n)x_n - \gamma_n\delta_n T(\mu_n)x_n\| \\
= \; & \|\alpha_n(f(x_n) - T(\mu_n)x_n) + (1-\beta_n)(T(\mu_n)x_n - x_n) + \gamma_n\delta_n(x_n - T(\mu_n)x_n)\| \\
= \; & \|\alpha_n(f(x_n) - T(\mu_n)x_n) + (-1+\beta_n+\gamma_n\delta_n)(x_n - T(\mu_n)x_n)\| \\
\leq \; & \alpha_n\|f(x_n) - T(\mu_n)x_n\| + (-1+\beta_n+\gamma_n\delta_n)\|x_n - T(\mu_n)x_n\|.
\end{aligned}$$

So,

$$\|x_n - T(\mu_n)x_n\| \leq \frac{1}{1-\beta_n-\gamma_n\delta_n}(\alpha_n\|f(x_n) - T(\mu_n)x_n\| - \|x_{n+1} - x_n\|).$$

Hence, by (i), (ii), (iv) and (3.2), we have

$$\lim_{n\to\infty} \|x_n - T(\mu_n)x_n\| = 0. \tag{3.3}$$

From this and Lemma 2.3, we obtain:

$$\begin{aligned}
\limsup_{k\to\infty} \|p - T(\mu_{n_k})p\| \; & \leq \; \limsup_{k\to\infty}(\|p - x_{n_k}\| + \|x_{n_k} - T(\mu_{n_k})x_{n_k}\| \\
& \qquad + \|T(\mu_{n_k})x_{n_k} - T(\mu_{n_k})p\|) \\
& \leq \; \limsup_{k\to\infty}(2\|p - x_{n_k}\| + \|x_{n_k} - T(\mu_{n_k})x_{n_k}\| + c_{n_k}) \\
& \leq \; 0.
\end{aligned}$$

Therefore, applying Corollary 2.2, we get $p \in F(\mathcal{S})$.

Next, we show that $\limsup_{n\to\infty}\langle (f-I)z, J(x_n - z)\rangle \leq 0$, where, $z = Pfz$. We know from Corollary 2.4 and the proof of Corollary 2.2 [13], that there exists a unique sunny nonexpansive retraction P of C onto $F(\mathcal{S})$. The Banach Contraction Mapping Principle guarantees that Pf has a unique fixed point z, which by (2.3) is the unique solution of

$$\langle (f-I)z, J(p-z)\rangle \leq 0, \quad \forall p \in F(\mathcal{S}). \tag{3.4}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k\to\infty}\langle (f-I)z, J(x_{n_k} - z)\rangle = \limsup_{n\to\infty}\langle (f-I)z, J(x_n - z)\rangle. \tag{3.5}$$

Without loss of generality, we can assume that $\{x_{n_k}\}$ converges to some $p \in C$ such that $p \in F(\mathcal{S})$. Smoothness of E and a combination of (3.4) and (3.5) give

$$\limsup_{n\to\infty}\langle (f-I)z, J(x_n-z)\rangle = \langle (f-I)z, J(p-z)\rangle \le 0, \tag{3.6}$$

as required.

Finally, we show that the sequence $\{x_n\}$ converges strongly to $z = Pfz$. Now, we have

$$\begin{aligned}
\|y_n - z\| &= \|\delta_n x_n + (1-\delta_n)T(\mu_n)x_n - z\| \\
&= \|(1-\delta_n)(T(\mu_n)x_n - z) + \delta_n(x_n - z)\| \\
&\le (1-\delta_n)\|T(\mu_n)x_n - z\| + \delta_n\|x_n - z\| \\
&\le (1-\delta_n)\|x_n - z\| + c_n + \delta_n\|x_n - z\| \\
&= \|x_n - z\| + c_n. \qquad (3.7)
\end{aligned}$$

By using Lemma 2.5, (3.7) and (2.2), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - z\|^2 \\
&= \|(\gamma_n(y_n - z) + \beta_n(x_n - z)) + \alpha_n(f(x_n) - z)\|^2 \\
&\le \|\gamma_n(y_n - z) + \beta_n(x_n - z)\|^2 \\
&\quad + 2\alpha_n\langle f(x_n) - z, J(x_{n+1} - z)\rangle \\
&= \|(1-\beta_n)\frac{\gamma_n}{1-\beta_n}(y_n - z) + \beta_n(\frac{1-\beta_n}{1-\beta_n})(x_n - z)\|^2 \\
&\quad + 2\alpha_n\langle f(x_n) - f(z), J(x_{n+1} - z)\rangle \\
&\quad + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&\le (1-\beta_n)\|\frac{\gamma_n}{1-\beta_n}(y_n - z)\|^2 + \beta_n\|x_n - z\|^2 \\
&\quad + 2\alpha\alpha_n\|x_n - z\|\|x_{n+1} - z\| \\
&\quad + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&\le \frac{\gamma_n^2}{1-\beta_n}\|y_n - z\|^2 + \beta_n\|x_n - z\|^2 \\
&\quad + \alpha\alpha_n(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\quad + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&\le \frac{\gamma_n^2}{1-\beta_n}\|x_n - z\|^2 + \frac{\gamma_n^2 c_n}{1-\beta_n} + \beta_n\|x_n - z\|^2 \\
&\quad + \alpha\alpha_n\|x_n - z\|^2 + \alpha\alpha_n\|x_{n+1} - z\|^2 \\
&\quad + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= (\frac{\gamma_n^2}{1-\beta_n} + \beta_n + \alpha\alpha_n)\|x_n - z\|^2 + \frac{\gamma_n^2 c_n}{1-\beta_n} \\
&\quad +\alpha\alpha_n\|x_{n+1} - z\|^2 + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&= (\frac{((1-\beta_n) - \alpha_n)^2}{1-\beta_n} + \beta_n + \alpha\alpha_n)\|x_n - z\|^2 + \frac{\gamma_n^2 c_n}{1-\beta_n} \\
&\quad + \alpha\alpha_n\|x_{n+1} - z\|^2 + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&= (\frac{(1-\beta_n)^2 - 2(1-\beta_n)\alpha_n + \alpha_n^2}{1-\beta_n} + \beta_n + \alpha\alpha_n)\|x_n - z\|^2 \\
&\quad + \frac{\gamma_n^2 c_n}{1-\beta_n} + \alpha\alpha_n\|x_{n+1} - z\|^2 + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&= (1 - \beta_n - 2\alpha_n + \frac{\alpha_n^2}{1-\beta_n} + \beta_n + \alpha\alpha_n)\|x_n - z\|^2 + \frac{\gamma_n^2 c_n}{1-\beta_n} \\
&\quad + \alpha\alpha_n\|x_{n+1} - z\|^2 + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&= ((1 - \alpha\alpha_n) + (2\alpha\alpha_n - 2\alpha_n) + \frac{\alpha_n^2}{1-\beta_n})\|x_n - z\|^2 + \frac{\gamma_n^2 c_n}{1-\beta_n} \\
&\quad + \alpha\alpha_n\|x_{n+1} - z\|^2 + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\leq (1 - \frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n} \\
&\quad + \frac{\alpha_n^2}{(1-\alpha\alpha_n)(1-\beta_n)})\|x_n - z\|^2 + \frac{\gamma_n^2 c_n}{(1-\alpha\alpha_n)(1-\beta_n)} \\
&\quad + \frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&\leq (1 - \frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n})\|x_n - z\|^2 + \frac{\alpha_n}{1-\alpha\alpha_n}(\frac{\alpha_n}{1-\beta_n}\|x_n - z\|^2 \\
&\quad + \frac{\gamma_n^2 c_n}{\alpha_n(1-\beta_n)} + 2\langle f(z) - z, J(x_{n+1} - z)\rangle) \\
&:= (1 - \sigma_n)\|x_n - z\|^2 + \rho_n,
\end{aligned}
$$

where, $\sigma_n := \frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n}$ and $\rho_n := \frac{\alpha_n}{1-\alpha\alpha_n}(\frac{\alpha_n}{1-\beta_n}\|x_n - z\|^2 + \frac{\gamma_n^2 c_n}{\alpha_n(1-\beta_n)} + 2\langle f(z) - z, J(x_{n+1} - z)\rangle)$. Now, from (i), (iii), (iv), (3.6) and Lemma 2.7, we get $\|x_n - z\| \to 0$, as $n \to \infty$. This completes the proof. □

Corollary 3.2. *Let S be a left reversible semigroup and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as Lipschitzian mappings from a nonempty compact convex subset C of a smooth Banach space E into itself, with the*

uniform Lipschitzian condition $\lim_s k(s) \leq 1$*. Let* X *be a left invariant* $\mathcal{S}$*-stable subspace of* $l^\infty(S)$ *containing 1,* $\{\mu_n\}$ *be a strongly left regular sequence of means on* X *such that* $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$ *and* $\{c_n\}$ *be the sequence defined by* (2.4)*. Suppose the sequences* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\delta_n\}$ *in* $(0,1)$ *satisfy* $\alpha_n + \beta_n + \gamma_n = 1$*,* $n \geq 1$*. The following conditions are satisfied:*
(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} \delta_n = 0$*;*
(iii) $\limsup_{n\to\infty} \frac{c_n}{\alpha_n} \leq 0$*; (by Remark 2.1,* $\limsup_{n\to\infty} c_n \leq 0$*);*
(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*.*
If arbitrary given $x_1 \in C$*, the sequence* $\{x_n\}$ *is generated by*

$$\begin{cases} y_n = \delta_n x_n + (1-\delta_n) T(\mu_n) x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n, \forall n \geq 1, \end{cases} \tag{3.8}$$

then $\{x_n\}$ *converges strongly to* $z \in F(\mathcal{S})$*, which is the unique solution of the variational inequality,*

$$\langle (f - I)z, J(p - z) \rangle \leq 0, \forall p \in F(\mathcal{S}).$$

Equivalently, we have $z = Pfz$*, where* P *is the unique sunny nonexpansive retraction of* C *onto* $F(\mathcal{S})$*.*

Proof. Taking $f(x) = u$ for all $x \in C$ in (1.5), we get (3.8), and we can conclude the desired conclusion easily. This completes the proof. □

Corollary 3.3. [13, Theorem 3.1] *Let* S *be a left reversible semigroup and* $\mathcal{S} = \{T(s) : s \in S\}$ *be a representation of* S *as Lipschitzian mappings from a nonempty compact convex subset* C *of a smooth Banach space* E *into itself, with the uniform Lipschitzian condition* $\lim_s k(s) \leq 1$ *and* f *be a contraction of* C *into itself with coefficient* $\alpha \in (0,1)$*. Let* X *be a left invariant* $\mathcal{S}$*-stable subspace of* $l^\infty(S)$ *containing 1,* $\{\mu_n\}$ *be a strongly left regular sequence of means on* X *such that* $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$ *and* $\{c_n\}$ *be the sequence defined by*

$$c_n = \sup_{x,y \in C} (\|T(\mu_n)x - T(\mu_n)y\| - \|x - y\|), \forall n.$$

Suppose the sequences $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *in* $(0,1)$ *satisfy* $\alpha_n + \beta_n + \gamma_n = 1$*,* $n \geq 1$*. The following conditions are satisfied:*
(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{c_n}{\alpha_n} \leq 0$*; (by Remark 2.1,* $\limsup_{n\to\infty} c_n \leq 0$*);*
(iii) $\liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*.*
If arbitrary given $x_1 \in C$*, the sequence* $\{x_n\}$ *is generated by* (1.4)*, then*

$\{x_n\}$ converges strongly to $z \in F(\mathcal{S})$, which is the unique solution of the variational inequality,

$$\langle (f-I)z, J(p-z)\rangle \leq 0, \forall p \in F(\mathcal{S}).$$

Equivalently, we have $z = Pfz$, where P is the unique sunny nonexpansive retraction of C onto $F(\mathcal{S})$.

Proof. Taking $\delta_n = 0$ for all $n \in \mathbb{N}$ in (1.5), we get (1.4), and we can conclude the desired conclusion easily. This completes the proof. □

4. **Application**

Corollary 4.1. *Let C be a nonempty compact convex subset of a smooth Banach space E and let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a strongly continuous semigroup of Lipschitzian mappings from C into itself, with the uniform Lipschitzian condition $\lim_s k(s) \leq 1$ and $\{t_n\}$ be an increasing sequence in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = 1$. Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$. Suppose the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. The following conditions are satisfied:*
(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} \delta_n = 0$;
(iii) $\limsup_{n\to\infty} \frac{c_n}{\alpha_n} \leq 0$,
where, $c_n = \sup_{x,y\in C}\{\|\frac{1}{t_n}\int_0^{t_n} T(s)xds - \frac{1}{t_n}\int_0^{t_n} T(s)yds\| - \|x-y\|\}$;
(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
If for arbitrary given $x_1 \in C$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} y_n = \delta_n x_n + (1-\delta_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \forall n \geq 1, \end{cases} \tag{4.1}$$

then $\{x_n\}$ converges strongly to $z \in F(\mathcal{S})$, which is the unique solution of the variational inequality,

$$\langle (f-I)z, J(p-z)\rangle \leq 0, \forall p \in F(\mathcal{S}).$$

Equivalently, we have $z = Pfz$, where P is the unique sunny nonexpansive retraction of C onto $F(\mathcal{S})$.

Proof. For $n \geq 1$, define $\mu_n(f) = \frac{1}{t_n}\int_0^{t_n} f(t)dt$ for each $f \in C(\mathbb{R}^+)$, where, $C(\mathbb{R}^+)$ is the space of all real valued bounded continuous functions on $\mathbb{R}^+$ with the supremum norm. Then, $\{\mu_n\}$ is a strongly regular sequence of means and $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$ [1]. Furthermore, for each $x \in C$, we have $T(\mu_n)x = \frac{1}{t_n}\int_0^{t_n} T(s)xds$. Therefore, we apply Theorem 3.1 to conclude the result. □

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