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ITERATIVE METHODS FOR FINDING NEAREST COMMON FIXED POINTS OF A COUNTABLE FAMILY OF QUASI-LIPSCHITZIAN MAPPINGS

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Abstract. We prove a strong convergence result for a sequence generated by Halpern's type iteration for approximating a common fixed point of a countable family of quasi-Lipschitzian mappings in a real Hilbert space. Consequently, we apply our results to the problem of finding a common fixed point of asymptotically nonexpansive mappings, an equilibrium problem, and a variational inequality problem for continuous monotone mappings.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be *Lipschitzian* if there exists a positive constant L such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \text{for all } x, y \in C.$$

In this case, T is also said to be L-Lipschitzian. We denote by $\mathrm{F}(T)$ the set of fixed points of T. A mapping T is said to be *quasi-Lipschitzian* if

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$\mathrm{F}(T) \neq \varnothing$ and there exists a positive constant L such that

$$\|Tx - y\| \leq L\|x - y\| \quad \text{for all } x \in C \text{ and } y \in \mathrm{F}(T).$$

In this case we also say that T is quasi-L-Lipschitzian.

Remark 1.1. *It follows directly from the definitions above that:*

(i) *If T is L-Lipschitzian with $\mathrm{F}(T) \neq \varnothing$, then T is quasi-L-Lipschitzian.*

(ii) *If T is quasi-L_1-Lipschitzian and $L_1 < L_2$, then T is quasi-L_2-Lipschitzian.*

(iii) *T is (quasi-) 1-Lipschitzian if and only if T is (quasi-) nonexpansive.*

Throughout the paper, we deal with quasi-L-Lipschitzian mappings where $L \geq 1$. There are many methods for approximating fixed points of mappings. In 1953, Mann [12] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \quad \text{for all } n \in \mathbb{N}, \tag{1.1}$$

where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Reich [24]. Recently, the present authors [16, 17, 18, 19, 20, 21] extended the iteration (1.1) to obtain weak and strong convergence theorems for a countable family of (quasi-) L_n-Lipschitzian mappings $\{T_n\}$ with some appropriate additional conditions by the following iteration:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_n x_n \quad \text{for all } n \in \mathbb{N},$$

where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. In an infinite-dimensional Hilbert space, strong convergence of Mann iteration is not generally guaranteed [5]. Some attempts to construct an iteration method so that strong convergence is guaranteed have recently been made [3, 8, 10, 13, 14, 15, 27, 28, 29, 30]. Halpern [8] introduced the following iterative scheme for approximating a fixed point of T

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n \quad \text{for all } n \in \mathbb{N}, \tag{1.2}$$

where $x, x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. This iteration process is called Halpern's type iteration. Strong convergence of this type iterative sequence was also studied by Wittmann [28]. In 1996, Bauschke [1] extended the iteration (1.2) to obtain strong convergence theorems for a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ by the following iteration:

$$x_{n+1} = \alpha_n x + (1-\alpha_n) T_{n(\bmod N)} x_n \quad \text{for all } n \in \mathbb{N}, \tag{1.3}$$

where $x, x_1 \in C$, $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\bmod N$ takes values in $\{1, 2, \ldots, N\}$. Recently, O'Hara et al. [22] extended the iteration (1.3) to obtain strong convergence theorems for a countable family of nonexpansive mappings.

In this paper, we establish strong convergence theorem for finding common fixed points of a countable family of quasi-L_n-Lipschitzian mappings in a real Hilbert space. As a consequence, several convergence theorems for quasi-nonexpansive mappings and asymptotically nonexpansive mappings are deduced. Finally, we apply our results to equilibrium problems and variational inequality problems for continuous monotone mappings.

2. **Preliminaries**

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Then

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle$$

and

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\lambda \in [0,1]$. In particular,

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle \tag{2.2}$$

for all $x, y \in H$. We write $x_n \to x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges strongly (weakly, resp.) to x. It is also known that H satisfies:

- The *Opial's condition* [23], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

  $$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

  holds for every $y \in H$ with $y \ne x$.
- If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2 \quad \text{for all } y \in H. \tag{2.3}$$

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

Such a mapping P_C is called the *metric projection* of H onto C. We know that P_C is nonexpansive. Furthermore, for $x \in H$ and $z \in C$,

$$z = P_C x \quad \text{if and only if} \quad \langle x - z, z - y \rangle \geq 0 \quad \text{for all } y \in C.$$

Lemma 2.1 ([29], Lemma 2.1)**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \beta_n \quad \textit{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{\beta_n\}$ is a sequence of real numbers with $\limsup_{n\to\infty} \beta_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

To deal with a family of mappings, the following conditions are introduced: Let K be a subset of a Banach space, let $\{T_n\}$ be a family of mappings of K into itself with $\bigcap_{n=1}^{\infty} \mathrm{F}(T_n) \neq \varnothing$. $\{T_n\}$ is said to satisfy

(a) the ZLC-*condition* [31] if for each bounded sequence $\{z_n\}$ in K, there exists a family of nonexpansive mapping of K into itself $\mathfrak{T}$ such that $\|T_n z_n - T(T_n z_n)\| \to 0$ for all $T \in \mathfrak{T}$ and $\bigcap_{n=1}^{\infty} \mathrm{F}(T_n) = \mathrm{F}(\mathfrak{T}) \neq \varnothing$, where $\mathrm{F}(\mathfrak{T})$ is the set of all common fixed points of all mappings in $\mathfrak{T}$;

(b) the H-*condition* [9] if for each bounded sequence $\{z_n\}$ in K,

$$\lim_{n\to\infty} \|z_{n+1} - T_n z_n\| = 0 \quad \Rightarrow \quad \omega_w\{z_n\} \subset \bigcap_{n=1}^{\infty} \mathrm{F}(T_n),$$

where $\omega_w\{z_n\}$ denotes the set of all weak subsequential limits of $\{z_n\}$.

Recall that a mapping T is *demi-closed at* y, if $x_n \rightharpoonup x$ and $Tx_n \to y$, then $Tx = y$.

Lemma 2.2 ([7], Theorem 10.3)**.** *Let K be a nonempty closed convex subset of a reflexive Banach space which satisfies Opial's condition and let T be a nonexpansive mapping of K into itself. Then $I - T$ is demi-closed at zero.*

Lemma 2.3. *Let K be a nonempty closed subset of a reflexive Banach space which satisfies Opial's condition and let $\{T_n\}$ be a family of mappings of K into itself which satisfies the* ZLC-*condition. Then $\{T_n\}$ satisfies the* H-*condition.*

Proof. Let $\{z_n\}$ be a bounded sequence in K such that

$$\lim_{n\to\infty} \|z_{n+1} - T_n z_n\| = 0.$$

Since $\{T_n\}$ satisfies the ZLC-condition, there exists a family of nonexpansive mapping of K into itself $\mathfrak{T}$ such that $\|T_n z_n - T(T_n z_n)\| \to 0$ for all $T \in \mathfrak{T}$ and $\bigcap_{n=1}^{\infty} \mathrm{F}(T_n) = \mathrm{F}(\mathfrak{T}) \neq \varnothing$. Then

$$\begin{aligned}
&\|z_{n+1} - T z_{n+1}\| \\
&\leq \|z_{n+1} - T_n z_n\| + \|T_n z_n - T(T_n z_n)\| + \|T(T_n z_n) - T z_{n+1}\| \\
&\leq 2\|z_{n+1} - T_n z_n\| + \|T_n z_n - T(T_n z_n)\| \to 0,
\end{aligned}$$

for all $T \in \mathfrak{T}$. By Lemma 2.2, $I - T$ is demi-closed at zero. So, we get $\omega_w\{z_n\} \subset \mathrm{F}(T)$ for all $T \in \mathfrak{T}$. This implies that $\{T_n\}$ satisfies the H-condition. □

Lemma 2.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}$ be a family of quasi-L_n-Lipschitzian mappings of C into itself with $L_n \to 1$ and $\bigcap_{n=1}^{\infty} \mathrm{F}(T_n) \neq \varnothing$ (that is, for every $n \in \mathbb{N}$, $x \in C$ and $u \in \mathrm{F}$, $\|T_n x - u\| \leq L_n \|x - u\|$ holds). If $\{T_n\}$ satisfies the* H-*condition, then $\bigcap_{n=1}^{\infty} \mathrm{F}(T_n)$ is closed and convex.*

Proof. It follows directly from [19, Lemma 2.8]. □

3. **Strong Convergence Theorems**

In this section, using the Halpern's type iteration we obtain a strong convergence theorem for a countable family of quasi-Lipschitzian mappings.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}$ be a family of quasi-L_n-Lipschitzian mappings of C into itself with $L_n \geq 1$ and $\mathrm{F} := \bigcap_{n=1}^{\infty} \mathrm{F}(T_n) \neq \varnothing$ (that is, for every $n \in \mathbb{N}$, $x \in C$ and $u \in \mathrm{F}$, $\|T_n x - u\| \leq L_n \|x - u\|$ holds). Assume that $\{\alpha_n\}$ is a sequence in $(0, 1]$ which satisfies the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(L) $\lim_{n\to\infty} \frac{L_n - 1}{\alpha_n} = 0$.

Let $\{x_n\}$ be a sequence in C defined as follows: $x, x_1 \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T_n x_n \quad \textit{for all } n \in \mathbb{N}. \tag{3.1}$$

If $\{T_n\}$ *satisfies the* H*-condition, then the sequence* $\{x_n\}$ *converges strongly to* $P_{\mathrm{F}}x$*, where* P_{F} *is the projection of* H *onto* F*.*

Proof. Let $u \in \mathrm{F}$. Since $\lim_{n\to\infty} \frac{L_n - 1}{\alpha_n} = 0$, there exists $N \in \mathbb{N}$ such that $\frac{L_n-1}{\alpha_n} < \frac{1}{2}$ for all $n \geq N$. Choose a constant $M > 0$ so that

$$\|x_N - u\| \leq M \quad \text{ and } \|x - u\| \leq \frac{M}{2}.$$

We proceed by induction to show that $\|x_n - u\| \leq M$ for all $n \geq N$. Assume that $\|x_k - u\| \leq M$ for some $k \geq N$. From the iteration process (3.1), we estimate as follows:

$$\begin{aligned}
\|x_{k+1} - u\| &\leq \alpha_k \|x - u\| + (1 - \alpha_k)\|T_k x_k - u\| \\
&\leq \alpha_k \|x - u\| + (1 - \alpha_k) L_k \|x_k - u\| \\
&= \alpha_k \|x - u\| + (1 - \alpha_k)(L_k - 1)\|x_k - u\| + (1 - \alpha_k)\|x_k - u\| \\
&\leq \alpha_k \frac{M}{2} + (1 - \alpha_k)\alpha_k \frac{M}{2} + (1 - \alpha_k) M \\
&\leq \alpha_k \frac{M}{2} + \alpha_k \frac{M}{2} + (1 - \alpha_k) M = M.
\end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded and hence so is $\{T_n x_n\}$. So, from $\alpha_n \to 0$, we get

$$\|x_{n+1} - T_n x_n\| = \alpha_n \|x - T_n x_n\| \to 0.$$

Since $\{T_n\}$ satisfies the H-condition, $\omega_w\{x_n\} \subset \mathrm{F}$. We next show

$$\limsup_{n\to\infty} \langle x - z, x_n - z\rangle \leq 0, \tag{3.2}$$

where $z := P_{\mathrm{F}}x$. To this end, we choose a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\limsup_{n\to\infty} \langle x - z, x_n - z\rangle = \lim_{i\to\infty} \langle x - z, x_{n_i} - z\rangle \quad \text{and} \quad x_{n_i} \rightharpoonup w \in \mathrm{F}.$$

So, we get

$$\lim_{i\to\infty} \langle x - z, x_{n_i} - z\rangle = \langle x - z, w - z\rangle \leq 0.$$

Now (3.2) is proved. Finally we prove that $x_n \to z$. From (2.2), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n(x - z) + (1 - \alpha_n)(T_n x_n - z)\|^2 \\
&\le (1 - \alpha_n)^2 \|T_n x_n - z\|^2 + 2\alpha_n \langle x - z, x_{n+1} - z\rangle \\
&\le (1 - \alpha_n)^2 L_n^2 \|x_n - z\|^2 + 2\alpha_n \langle x - z, x_{n+1} - z\rangle \\
&= (1 - \alpha_n)^2 \|x_n - z\|^2 + \alpha_n (1 - \alpha_n)^2 \frac{L_n^2 - 1}{\alpha_n} \|x_n - z\|^2 \\
&\quad + 2\alpha_n \langle x - z, x_{n+1} - z\rangle \\
&\le (1 - \alpha_n) \|x_n - z\|^2 + \alpha_n \frac{L_n - 1}{\alpha_n} (L_n + 1) \|x_n - z\|^2 \\
&\quad + 2\alpha_n \langle x - z, x_{n+1} - z\rangle.
\end{aligned}$$

Setting

$$\beta_n = \frac{L_n - 1}{\alpha_n} (L_n + 1) \|x_n - z\|^2 + 2\langle x - z, x_{n+1} - z\rangle,$$

we get

$$\|x_{n+1} - z\|^2 \le (1 - \alpha_n) \|x_n - z\|^2 + \alpha_n \beta_n.$$

Since $\lim_{n\to\infty} \frac{L_n - 1}{\alpha_n} = 0$ by (3.2), $\limsup_{n\to\infty} \beta_n \le 0$. By Lemma 2.1, we conclude that $x_n \to z$. This completes the proof. □

Remark 3.2. *For a given family of quasi-L_n-Lipschitzian mappings, we can always find a sequence $\{\alpha_n\}$ in $(0, 1]$ such that the conditions* (C1), (C2) *and* (L) *are satisfied. In fact, if $L_n \to 1$, we can set* $\alpha_n = \max\left\{\frac{1}{n}, \frac{L_n - 1 + \sqrt{L_n - 1}}{L_n + \sqrt{L_n - 1}}\right\}$.

Setting $L_n \equiv 1$ in Theorem 3.1, we have the following.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}$ be a family of quasi-nonexpansive mappings of C into itself with* $\mathrm{F} := \bigcap_{n=1}^{\infty} \mathrm{F}(T_n) \ne \varnothing$ *(that is, for every $n \in \mathbb{N}$, $x \in C$ and $u \in \mathrm{F}$, $\|T_n x - u\| \le \|x - u\|$ holds). If $\{T_n\}$ satisfies the* H*-condition, then the sequence $\{x_n\}$ defined by* (3.1)*, where $\{\alpha_n\}$ is a sequence in* $(0, 1]$ *satisfying* (C1) *and* (C2)*, converges strongly to $P_{\mathrm{F}} x$.*

Remark 3.4. *Corollary 3.3 extends and improves Theorem 2.1 of* [31] *in the following ways:*

(i) *Since every nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive, Corollary 3.3 is applicable for a wider class of mappings.*

(ii) *The* ZLC-*condition is weakened and replaced by the* H-*condition (see Lemma 2.3).*

4. **Applications**

In this section, we show that the H-condition studied in the previous section is satisfies by various classes of mappings.

4.1. **Convergence theorems for asymptotically nonexpansive mappings.** Let C be a subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers such that $k_n \in [1, \infty)$, $k_n \to 1$, and

$$\|T^n x - T^n y\| \le k_n \|x - y\| \quad \text{for all } x, y \in C \text{ and } n \in \mathbb{N}.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [6] as a natural generalization of the class of nonexpansive mappings. They proved that if C is nonempty bounded closed and convex, and T is an asymptotically nonexpansive self-mapping of C, then T has a fixed point.

Lemma 4.1 ([11], Lemma 2.2)**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S and T be two commutative asymptotically nonexpansive mappings of C into itself with asymptotical coefficients $\{s_n\}$ and $\{t_n\}$, respectively. For any $x \in C$ and $n \in \mathbb{N}$, put $R_n x = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} \sum_{i+j=k} S^i T^j x$. Then for each $r > 0$, there holds*

$$\lim_{i\to\infty} \limsup_{n\to\infty} \sup_{x \in C \cap B_r} \|R_n x - S^i(R_n x)\| = 0$$

and

$$\lim_{j\to\infty} \limsup_{n\to\infty} \sup_{x \in C \cap B_r} \|R_n x - T^j(R_n x)\| = 0,$$

where $B_r = \{x \in H : \|x\| \le r\}$.

From Lemma 4.1, we have the following result.

Lemma 4.2. *Let C, S, T, R_n be the same as Lemma 4.1. Assume that* $\mathrm{F}(S) \cap \mathrm{F}(T) \neq \varnothing$. *Then $\{R_n\}$ is a family of L_n-Lipschitzian mappings of C into itself and satisfies the* H-*condition, where*

$$L_n = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} \sum_{i+j=k} s_i t_j.$$

Proof. It is easy to see that $\{R_n\}$ is a family of L_n-Lipschitzian mappings of C into itself. Moreover, by Lemma 4.1, we have $\bigcap_{n=1}^{\infty} \mathrm{F}(R_n) = \mathrm{F}(S) \cap \mathrm{F}(T) \neq \varnothing$. Next, we prove that $\{R_n\}$ satisfies the H-condition. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} \|z_{n+1} - R_n z_n\| = 0$ and $z \in \omega_w\{z_n\}$. Then, there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $z_{n_k+1} \rightharpoonup z$ and $R_{n_k} z_{n_k} \rightharpoonup z$. Since $\{z_n\}$ is bounded, let $r > 0$ be such that $\{z_n\} \subset C \cap B_r$. Then

$$\begin{aligned}
\|z_{n_k+1} - S^i z\| &\le \|z_{n_k+1} - R_{n_k} z_{n_k}\| + \|R_{n_k} z_{n_k} - S^i(R_{n_k} z_{n_k})\| \\
&\quad + \|S^i(R_{n_k} z_{n_k}) - S^i z_{n_k+1}\| + \|S^i z_{n_k+1} - S^i z\| \\
&\le (1 + s_i)\|z_{n_k+1} - R_{n_k} z_{n_k}\| + \sup_{x \in C \cap B_r} \|R_{n_k} x - S^i(R_{n_k} x)\| \\
&\quad + s_i \|z_{n_k+1} - z\|.
\end{aligned}$$

It follows from $\lim_{i\to\infty} s_i = 1$ and Lemma 4.1 that

$$\limsup_{i\to\infty} \limsup_{k\to\infty} \|z_{n_k+1} - S^i z\| \le \limsup_{k\to\infty} \|z_{n_k+1} - z\|. \tag{4.1}$$

From (2.3) and (2.1), we have

$$\begin{aligned}
&\limsup_{k\to\infty} \|z_{n_k+1} - z\|^2 + \left\|\frac{S^i z - z}{2}\right\|^2 \\
&= \limsup_{k\to\infty} \left\|z_{n_k+1} - \frac{S^i z + z}{2}\right\|^2 \\
&= \limsup_{k\to\infty} \left(\frac{1}{2}\|z_{n_k+1} - S^i z\|^2 + \frac{1}{2}\|z_{n_k+1} - z\|^2 - \frac{1}{4}\|S^i z - z\|^2\right) \\
&\le \frac{1}{2}\limsup_{k\to\infty} \|z_{n_k+1} - S^i z\|^2 + \frac{1}{2}\limsup_{k\to\infty} \|z_{n_k+1} - z\|^2 - \frac{1}{4}\|S^i z - z\|^2
\end{aligned}$$

and hence

$$\|S^i z - z\|^2 \le \limsup_{k\to\infty} \|z_{n_k+1} - S^i z\|^2 - \limsup_{k\to\infty} \|z_{n_k+1} - z\|^2.$$

This together with (4.1) gives

$$\lim_{i\to\infty} \|S^i z - z\| = 0.$$

Since S is uniformly continuous, $Sz = z$ and then $z \in \mathrm{F}(S)$. Similarly, we can get $z \in \mathrm{F}(T)$. Hence $z \in \mathrm{F}(S) \cap \mathrm{F}(T) = \bigcap_{n=1}^{\infty} \mathrm{F}(R_n)$. This implies that $\{R_n\}$ satisfies the H-condition. This completes the proof. □

Applying Theorem 3.1 and Lemma 4.2, we have the following result.

Theorem 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S and T be two commutative asymptotically nonexpansive mappings of C into itself with asymptotical coefficients $\{s_n\}$ and $\{t_n\}$, respectively. Assume that* $\mathrm{F} := \mathrm{F}(S) \cap \mathrm{F}(T) \neq \varnothing$. *Let $L_n = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} \sum_{i+j=k} s_i t_j$ and let $\{x_n\}$ be a sequence in C defined as follows: $x, x_1 \in C$ and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} \sum_{i+j=k} S^i T^j x_n \tag{4.2}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0,1]$ *satisfying* (C1), (C2) *and* (L). *Then the sequence $\{x_n\}$ converges strongly to $P_{\mathrm{F}} x$.*

Setting $L_n \equiv 1$ in Theorem 4.3, we have the following.

Corollary 4.4 ([25], Theorem 1)**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S and T be two commutative nonexpansive mappings of C into itself with* $\mathrm{F} := \mathrm{F}(S) \cap \mathrm{F}(T) \neq \varnothing$. *Then the sequence $\{x_n\}$ defined by* (4.2), *where $\{\alpha_n\}$ is a sequence in* $(0,1]$ *satisfying* (C1) *and* (C2), *converges strongly to $P_{\mathrm{F}} x$.*

4.2. **Some applications for the equilibrium problem.** Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $f : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$f(x,y) \geq 0 \quad \text{for all } y \in C. \tag{4.3}$$

Numerous problems in physics, optimization, and economics reduce to find a solution of (4.3). The set of solutions of (4.3) is denoted by $\mathrm{EP}(f)$. In 2005, Combettes and Hirstoaga [4] introduced an iterative scheme for finding the best approximation to the initial data when $\mathrm{EP}(f)$ is not empty.

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions which are generally assumed (see [2]):

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for any $x, y \in C$;

(A3) f is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} f(tz + (1-t)x, y) \leq f(x,y);$$

(A4) $f(x,\cdot)$ is convex and lower semicontinuous for each $x \in C$.

By [2, Corollary 1] and [4, Lemma 2.12], we have the following lemma.

Lemma 4.5. *Let C be a nonempty closed convex subset of a real Hilbert space H, let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4) *and let $r > 0$ and $x \in H$. Then there exists a unique $x^* \in C$ such that*

$$f(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - x\rangle \geq 0 \quad \text{for all } y \in C.$$

Let T_r be a mapping of H into C defined by $T_r(x) = x^*$ for all $x \in H$. Then, the following hold:

(i) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \|x - y\|^2 - \|(T_r x - x) - (T_r y - y)\|^2;$$

(ii) $\mathrm{F}(T_r) = \mathrm{EP}(f)$;

(iii) $\mathrm{EP}(f)$ *is closed and convex.*

We present a convergence theorem for an equilibrium problem with a new control parameter which is complementary to Song and Zheng's result [26, Corollary 5.3].

Lemma 4.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4) *and* $\mathrm{EP}(f) \neq \varnothing$. *If $\{r_n\}$ is a sequence in $(0, \infty)$ satisfying* $\lim_{n\to\infty} r_n = \infty$, *then $\{T_{r_n}\}$ is a family of firmly nonexpansive mappings of H into C with $\bigcap_{n=1}^{\infty} \mathrm{F}(T_{r_n}) = \mathrm{EP}(f)$ and satisfies the* H*-condition.*

Proof. We note that $\bigcap_{n=1}^{\infty} \mathrm{F}(T_{r_n}) = \mathrm{EP}(f) \neq \varnothing$. Let $\{z_n\}$ be a bounded sequence in H such that $\lim_{n\to\infty} \|z_{n+1} - T_{r_n} z_n\| = 0$ and $z \in \omega_w\{z_n\}$. For each $n \in \mathbb{N}$, let $y_n = T_{r_n} z_n$. Then $z_{n_i+1} \rightharpoonup z$ and $y_{n_i} \rightharpoonup z$ for some subsequence $\{n_i\}$ of $\{n\}$. We note that $\{z_n - y_n\}$ is bounded. Since $\lim_{n\to\infty} r_n = \infty$, we have

$$\frac{z_n - y_n}{r_n} \to 0. \tag{4.4}$$

Notice that

$$f(y_n, y) + \frac{1}{r_n}\langle y - y_n, y_n - z_n\rangle \geq 0 \quad \text{for all } y \in C.$$

So, from (A2), we have

$$\left\langle y - y_n, \frac{y_n - z_n}{r_n} \right\rangle \geq f(y, y_n) \quad \text{for all } y \in C.$$

In particular,

$$\left\langle y - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{r_{n_i}} \right\rangle \geq f(y, y_{n_i}) \quad \text{for all } y \in C.$$

This together with (4.4), (A4) and $y_{n_i} \rightharpoonup z$ gives

$$0 \geq f(y,z) \quad \text{for all } y \in C.$$

Then, for $t \in (0,1]$ and $y \in C$,

$$\begin{aligned} 0 &= f(ty+(1-t)z, ty+(1-t)z) \\ &\leq tf(ty+(1-t)z,y) + (1-t)f(ty+(1-t)z,z) \\ &\leq tf(ty+(1-t)z,y) \end{aligned}$$

hence

$$f(ty+(1-t)z,y) \geq 0.$$

Letting $t \to 0^+$ and using (A3), we get

$$f(z,y) \geq 0 \quad \text{for all } y \in C$$

and hence $z \in \mathrm{EP}(f) = \bigcap_{n=1}^{\infty} \mathrm{F}(T_{r_n})$. This implies that $\{T_{r_n}\}$ satisfies the H-condition. This completes the proof. □

Using Corollary 3.3, we have the following theorem.

Theorem 4.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4) *and* $\mathrm{EP}(f) \neq \varnothing$*. Let $\{x_n\}$ be a sequence in C defined as follows: $x, x_1 \in H$ and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)T_{r_n}x_n \quad \textit{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1]$ *satisfying* (C1) *and* (C2)*, and $\{r_n\}$ is a sequence in* $(0,\infty)$ *with* $\lim_{n\to\infty} r_n = \infty$*. Then $\{x_n\}$ converges strongly to $P_{\mathrm{EP}(f)}x$.*

4.3. **Some applications for the variational inequality problem.** Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $A : C \to H$ be a mapping. The classical variational inequality problem is to find $x \in C$ such that

$$\langle Ax, y-x\rangle \geq 0 \quad \text{for all } y \in C.$$

The set of solutions of classical variational inequality problem is denoted by $\mathrm{VIP}(C,A)$.

The following lemma were also given in Nilsrakoo and Saejung [16].

Lemma 4.8 ([16], Lemmas 19, 20)**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a continuous monotone mapping of C into H, that is,*

$$\langle Ax-Ay, x-y\rangle \geq 0 \qquad \textit{for all } x,y \in C.$$

Define $f : C \times C \to \mathbb{R}$ *as follows*

$$f(x, y) = \langle Ax, y - x\rangle \quad \text{for all } x, y \in C.$$

Then, the following hold:

(i) f *satisfies* (A1)-(A4) *and* $\mathrm{VIP}(C, A) = \mathrm{EP}(f)$*;*
(ii) *for* $x \in H$, $u \in C$ *and* $r > 0$,

$$f(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0 \quad \text{for all } y \in C \quad \Leftrightarrow \quad u = P_C(x - rAu).$$

Using Theorem 4.7 and Lemma 4.8, we have the following theorem.

Theorem 4.9. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* A *be a continuous monotone mapping of* C *into* H *such that* $\mathrm{VIP}(C, A) \neq \varnothing$*. Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x, x_1 \in C$ *and*

$$\begin{cases} u_n = P_C(x_n - r_n A u_n) \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) u_n \quad \text{for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ *is a sequence in* $(0, 1]$ *satisfying* (C1) *and* (C2)*, and* $\{r_n\}$ *is a sequence in* $(0, \infty)$ *with* $\lim_{n\to\infty} r_n = \infty$*. Then* $\{x_n\}$ *converges strongly to* $P_{\mathrm{VIP}(C,A)}x$*.*

5. **Conclusion**

We propose the Halpern's type iteration to obtain a strong convergence theorem for a common fixed point of a countable family of certain quasi-Lipschitzian mappings in a real Hilbert space. We assume that the family of mappings satisfies the H-condition introduced by Hirstoaga in [9]. This is not restrictive because there are many examples satisfying the H-condition. Applications for equilibrium problems and variational inequality problems are also discussed.

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