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COMPOSITE INTERPOLATION METHOD AND THE CORRESPONDING DIFFERENTIATION MATRIX

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Abstract. Properties of the hybrid of block-pulse functions and Lagrange polynomials based on the Legendre-Gauss-type points are investigated and utilized to define the composite interpolation operator as an extension of the well-known Legendre interpolation operator. The uniqueness and interpolating properties are discussed and the corresponding differentiation matrix is also introduced. The applicability and effectiveness of the method are illustrated by two numerical experiments.

1. **Introduction and preliminaries**

When solving partial differential equations via pseudospectral methods, the fundamental representation of a smooth function is in terms of its values at the discrete Gauss-type points [5, 6]. Derivatives of the function is approximated by analytic derivatives of the interpolating polynomials. For instance, Chebyshev and Legendre pseudospectral methods are the commonly used methods in this respect. The problem of finding an efficient algorithm for evaluating the entries of Chebyshev and Legendre pseudospectral differentiation matrices has been the subject of

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some papers ([2] and references therein). Here, we present an extension of the well-known Legendre interpolation method and its corresponding differentiation matrix. We also introduce the preliminaries of the Gauss pseudospectral method required for our subsequent development. The reader is referred to [3, 4] for details.

Let $x_0, x_1, \ldots, x_M$ be the Legendre-Gauss-type points, i.e., the zeros of the Legendre polynomial of degree $M+1$, and $w_0, w_1, \ldots, w_M$ be the corresponding quadrature weights. For a continuous function u, let $I_M(u)$ denote the Lagrange interpolating polynomial of degree M, interpolating the function u at the points $x_0, x_1, \ldots, x_M$, i.e.,

$$I_M(u)(x) = \sum_{m=0}^{M} u(x_m)\, L_m(x), \tag{1.1}$$

where, for $m = 0, 1, \ldots, M$, $L_m(x)$ denotes the Lagrange polynomial of degree M corresponding to the point x_m, defined by

$$L_m(x) = \prod_{i=0,\ i\neq m}^{M} \left(\frac{x - x_i}{x_m - x_i} \right).$$

For a fixed M, the Lagrange polynomials $L_0(x), L_1(x), \ldots, L_M(x)$ are mutually orthogonal with respect to the L^2-inner product on the interval $[-1, 1]$ and, in fact, form an orthogonal basis for $\mathbb{P}_M$ consisting of all polynomials of degree at most M on the interval $[-1, 1]$. In [1], page 74, it is showed that the interpolant $I_M(u)$ is the orthogonal projection of u upon $\mathbb{P}_M$ with respect to the discrete inner product

$$< u, v >_M = \sum_{m=0}^{M} u(x_m)\, v(x_m)\, w_m, \tag{1.2}$$

defined for every u and v, which are continuous functions on the interval $[-1, 1]$.

To compute the derivative of $I_M(u)$ at the point x_m, the Lagrange interpolation formula (1.1) is differentiated, yielding

$$\frac{d}{dx} I_M(u)(x_m) = \sum_{m'=0}^{M} d_{mm'}\, u(x_{m'}), \tag{1.3}$$

where $d_{mm'} = \frac{d}{dx} L_{m'}(x_m)$. Here, the matrix $D_M = [d_{mm'}]_{m,m'=0}^{M}$ is the classical $(M+1) \times (M+1)$ Gauss pseudospectral differentiation matrix [2].

Remark 1.1. *For the rest of our work, we transfer the interval* $[-1,1]$ *to the interval* $[0,1]$ *and confine our attention to the functions on the interval* $[0,1]$.

The remainder of our work is organized as follows: In Section 2, we introduce the composite interpolation formula by using the hybrid functions and discuss the uniqueness and interpolating properties and the approximation errors. In Section 3, we introduce the corresponding differentiation matrix. In Section 4, we report our numerical experiments and demonstrate the accuracy and efficiency of the method for approximating two functions and their derivatives.

2. **Composite interpolation**

In this section, we first introduce the hybrid of block-pulse functions and Lagrange polynomials based on the Legendre-Gauss-type points and define the composite interpolation operator using the hybrid functions and then discuss some approximation errors.

2.1. **Hybrid of block-pulse functions and Lagrange polynomials.**

The hybrid of block-pulse functions and Lagrange polynomials ϕ_{nm}, $n = 1, 2, \ldots, N$, $m = 0, 1, \ldots, M$, are defined on the interval $[0, 1)$ as

$$\phi_{nm}(x) = \begin{cases} L_m\left(2Nx - 2n + 1\right) & \text{if } \frac{n-1}{N} \leq x < \frac{n}{N}, \\ 0 & \text{otherwise,} \end{cases}$$

where n and m are the orders of the block-pulse function and the Lagrange polynomial, respectively.

For every integers $N \geq 1$ and $M \geq 0$, we use the following notation.

Notation. Let $\mathbb{P}_M^N$ denote the set of all functions whose restriction to each interval $(\frac{n-1}{N}, \frac{n}{N})$, $n = 1, 2, \ldots, N$, is a polynomial of degree at most M.

It is easily verified that $\mathbb{P}_M^N$ is a linear subspace of $L^2(0, 1)$, the space of all square Lebesgue-integrable functions with the L^2-norm.

Proposition 2.1. *(a) For a fixed integer $N \geq 1$, $\bigcup_{M=0}^{\infty} \mathbb{P}_M^N$ is dense in $L^2(0,1)$, (b) for fixed integers $N \geq 1$ and $M \geq 0$, $\{\phi_{nm} : n = 1,2,\ldots,N, m = 0,1,\ldots,M\}$ is an orthogonal basis for $\mathbb{P}_M^N$.*

Proof. Suppose $N \geq 1$ is a fixed integer. Since the interval $[0,1]$ is compact, by the classical Weierstrass approximation theorem, the set of all polynomials $\mathbb{P} = \bigcup_{M=0}^{\infty} \mathbb{P}_M$ is dense in $C[0,1]$, the set of all continuous functions, with respect to the L^2-norm; on the other hand, $C[0,1]$ is dense in $L^2(0,1)$ (see [7], Theorem 3.14). Since $\mathbb{P} \subseteq \bigcup_{M=0}^{\infty} \mathbb{P}_M^N$, therefore $\bigcup_{M=0}^{\infty} \mathbb{P}_M^N$ is dense in $L^2(0,1)$ and (a) holds. If δ denotes the Kronecker delta function, for every $n, n' = 1,2,\ldots,N$ and $m, m' = 0,1,\ldots,M$, then we have

$$\begin{aligned}\int_0^1 \phi_{nm}(x)\,\phi_{n'm'}(x)\,dx &= \delta_{nn'}\int_0^1 \phi_{nm}(x)\,\phi_{nm'}(x)\,dx \\ &= \frac{1}{2N}\,\delta_{nn'}\int_{-1}^1 L_m(x)\,L_{m'}(x)\,dx \\ &= \frac{w_m}{2N}\,\delta_{nn'}\,\delta_{mm'},\end{aligned}$$

where for the first equality, we use the property of disjointness of the intervals and for the second and third equalities, we use the change of variable rule and the Gaussian integration formula, respectively. Therefore, the hybrid functions ϕ_{nm}, $n = 1,2,\ldots,N$, $m = 0,1,\ldots,M$, are mutually orthogonal. Since the dimension of the linear space $\mathbb{P}_M^N$, which is $N(M+1)$, equals the cardinal number of the hybrid functions, (b) holds.

In other words, Proposition 2.1 states that for a fixed integer $N \geq 1$, $\bigcup_{M=0}^{\infty}\{\phi_{nm} : n = 1,2,\ldots,N, m = 0,1,\ldots,M\}$ is a complete basis for $L^2(0,1)$; for the definition of a complete basis in Hilbert spaces, see [7], Theorem 4.18 (b). □

2.2. **Composite interpolation formula.** In this section, for every integer $N \geq 1$, we let $PC(N)$ denote the set of all functions, whose restriction to each interval $(\frac{n-1}{N}, \frac{n}{N})$, $n = 1,2,\ldots,N$, has a continuous extension to the interval $[\frac{n-1}{N}, \frac{n}{N}]$. It is easily seen that $PC(N)$ is a linear subspace of $L^2(0,1)$ and also contains $\mathbb{P}_M^N$, for every integer $M \geq 0$.

Hereafter, $N \geq 1$ and $M \geq 0$ are fixed integers and also u and v are two arbitrary functions in $PC(N)$.

For $n = 1, 2, \ldots, N$ and $m = 0, 1, \ldots, M$, we set

$$x_{nm} = \frac{1}{2N}(x_m + 2n - 1),$$

i.e., the corresponding point of x_m in the interval $[\frac{n-1}{N}, \frac{n}{N}]$ by means of the natural linear transformation.

Analogous to (1.2), we define the bilinear form $< .,. >_{N,M}$ on $PC(N)$ by

$$< u, v >_{N,M} = \frac{1}{2N} \sum_{n=1}^{N} \sum_{m=0}^{M} u(x_{nm})\, v(x_{nm})\, w_m.$$

Then, the following proposition holds.

Proposition 2.2. *We have*
(a) $< .,. >_{N,M}$ *is symmetric, i.e.,* $< u, v >_{N,M} = < v, u >_{N,M}$.
(b) $< .,. >_{N,M}$ *is positive semi-definite, i.e.,* $< u, u > \geq 0$.
(c) For u *and* v *in* $\mathbb{P}_M^N$,

$$\int_0^1 u(x)\, v(x)\, dx = < u, v >_{N,M} .$$

Proof. From the definition, it is readily verified that (a) and (b) hold. For (c), using the change of variable rule and the Gaussian integration formula, we have

$$\begin{aligned} \int_0^1 u(x)\, v(x)\, dx &= \sum_{n=1}^{N} \int_{\frac{n-1}{N}}^{\frac{n}{N}} u(x)\, v(x)\, dx \\ &= \frac{1}{2N} \sum_{n=1}^{N} \sum_{m=0}^{M} u(x_{nm})\, v(x_{nm})\, w_m \\ &= < u, v >_{N,M} . \end{aligned}$$

Note that the $(M+1)$-point Gaussian integration formula is exact for the polynomials up to degree $2M+1$ (see [1], page 70).

According to Proposition 2.2, the symmetric positive semi-definite bilinear form $< .,. >_{N,M}$ coincides with the L^2-inner product on $\mathbb{P}_M^N$. Furthermore, the hybrid functions ϕ_{nm}, $n = 1, 2, \ldots, N$, $m = 0, 1, \ldots, M$, are mutually orthogonal also with respect to $< .,. >_{N,M}$, by Proposition 2.1, (b). $\square$

Definition 2.3. *We define I_M^N to be the operator which maps u to the orthogonal projection of u on $\mathbb{P}_M^N$ with respect to $< .,. >_{N,M}$, i.e., for every p in $\mathbb{P}_M^N$, $< I_M^N(u), p >_{N,M} = < u, p >_{N,M}$.*

The following theorem states the uniqueness and interpolating properties of the operator I_M^N.

Theorem 2.4. *The operator I_M^N is uniquely determined on $PC(N)$ by the following properties:*
(a) $I_M^N(u) \in \mathbb{P}_M^N$.
(b) $I_M^N(u)(x_{nm}) = u(x_{nm}), \quad n = 1, 2, \ldots, N,\ m = 0, 1, \ldots, M$.
Furthermore, analogous to (1.1),

$$I_M^N(u)(x) = \sum_{n=1}^{N} \sum_{m=0}^{M} u(x_{nm})\, \phi_{nm}(x). \tag{2.1}$$

Proof. By the definition, (a) holds. According to (a) and Proposition 2.1 (b),

$$I_M^N(u)(x) = \sum_{n=1}^{N} \sum_{m=0}^{M} c_{nm}\, \phi_{nm}(x).$$

For every $n = 1, 2, \ldots, N$ and $m = 0, 1, \ldots, M$, we have

$$0 = < I_M^N(u) - u, \phi_{nm} >_{N,M} = \frac{w_m}{2N}\big(c_{nm} - u(x_{nm})\big),$$

and thus $c_{nm} = u(x_{nm})$. It is easily verified that the properties (a) and (b) uniquely determine the operator I_M^N. $\square$

Theorem 2.4 suggests that we call the well-defined operator I_M^N to be "the composite interpolation operator".

2.3. **Approximation errors.** In this section, we give some estimates for the composite interpolation error $u - I_M^N(u)$ in terms of the Sobolev norms.

We recall that the Sobolev norm of integer order $s \geq 0$ in the interval (a, b) is given by

$$||u||_{H^s(a,b)} = \left(\sum_{k=0}^{s} \int_a^b |u^{(k)}(x)|^2 \, dx \right)^{1/2} = \left(\sum_{k=0}^{s} ||u^{(k)}||^2_{L^2(a,b)} \right)^{1/2},$$

where $u^{(k)}$ denotes the (distributional) derivative of u of order k.

Here, for our subsequent development, we state some results obtained in [1] as the following theorem.

Theorem 2.5. *([1], Section 5.4.3, page 289) Suppose $u \in H^s(-1, 1)$ with $s \geq 1$. Then*

$$||u - \sum_{m=0}^{M} u(x_m) \, L_m||_{L^2(-1,1)} \leq C \, M^{-s} \, |u|_{H^{s;M}(-1,1)}, \tag{2.2}$$

and for $1 \leq r \leq s$,

$$||u - \sum_{m=0}^{M} u(x_m) \, L_m||_{H^r(-1,1)} \leq C \, M^{2r-\frac{1}{2}-s} \, |u|_{H^{s;M}(-1,1)}. \tag{2.3}$$

Here,

$$|u|_{H^{s;M}(-1,1)} = \Big(\sum_{k=\min(s,M+1)}^{s} ||u^{(k)}||^2_{L^2(-1,1)} \Big)^{1/2},$$

and also C denotes a positive constant that depends upon the type of the norm, but which is independent of the function u and the integer M.

We use the following notation.

Notation. For $s \geq 0$, we let $PH^s(N)$ denote the set of all functions whose restriction to each interval $(\frac{n-1}{N}, \frac{n}{N})$, $n = 1, 2, \ldots, N$, is in $H^s(\frac{n-1}{N}, \frac{n}{N})$.

According to the above notation, it is readily verified that $PH^s(N)$ is a linear subspace of $L^2(0, 1)$ and also, in the case $s \geq 1$, is contained in $PC(N)$.

For convenience, we introduce the following semi-norm defined for $u \in PH^s(N)$, $0 \le r \le s$, $M \ge 0$ and $N \ge 1$, as:

$$|u|_{H^{r;s;M;N}(0,1)} = \Big(\sum_{k=\min(s,M+1)}^{s} N^{2r-2k} \sum_{n=1}^{N} ||u_n^{(k)}||^2_{L^2(\frac{n-1}{N},\frac{n}{N})} \Big)^{1/2},$$

where u_n, $n = 1, 2, \ldots, N$, denote the restriction of u to the interval $(\frac{n-1}{N}, \frac{n}{N})$, $n = 1, 2, \ldots, N$, respectively. Note that whenever $M \ge s-1$, we have

$$|u|_{H^{r;s;M;N}(0,1)} = N^{r-s} \Big(\sum_{n=1}^{N} ||u_n^{(s)}||^2_{L^2(\frac{n-1}{N},\frac{n}{N})} \Big)^{1/2}. \tag{2.4}$$

Remark 2.6. *In the case $N = 1$, $|\,.\,|_{H^{r;s;M;N}}$ coincides with $|\,.\,|_{H^{s;M}}$, as introduced and used in* [1].

To state our main results, the following lemma will be required.

Lemma 2.7. *For a fixed n, $n = 1, 2, \ldots, N$, suppose $u_n : (\frac{n-1}{N}, \frac{n}{N}) \to \mathbb{R}$ be a function in $H^s(\frac{n-1}{N}, \frac{n}{N})$. Consider the function $\mathcal{F}_n u_n : (-1, 1) \to \mathbb{R}$ such that $(\mathcal{F}_n u_n)(x) = u_n(\frac{1}{2N}(x + 2n - 1))$, for all $x \in (-1, 1)$. Then, for $0 \le l \le s$, we have*

$$||(\mathcal{F}_n u_n)^{(l)}||_{L^2(-1,1)} = C\, N^{\frac{1}{2}-l}\, ||u_n^{(l)}||_{L^2(\frac{n-1}{N},\frac{n}{N})},$$

where C is independent of the integer N.

Proof. For $0 \le l \le s$, we have

$$\begin{aligned} ||(\mathcal{F}_n u_n)^{(l)}||^2_{L^2(-1,1)} &= \int_{-1}^{1} |(\mathcal{F}_n u_n)^{(l)}(x)|^2\, dx \\ &= \int_{-1}^{1} |u_n^{(l)}(\frac{1}{2N}(x + 2n - 1))|^2\, dx \\ &= \int_{\frac{n-1}{N}}^{\frac{n}{N}} (2N)^{-2l}\, |u_n^{(l)}(t)|^2\, (2N)\, dt \\ &= C\, N^{1-2l}\, ||u_n^{(l)}||^2_{L^2(\frac{n-1}{N},\frac{n}{N})}, \end{aligned}$$

where for the third equality, we used the change of variable rule by setting $t = \frac{1}{2N}(x + 2n - 1)$ and for the forth equality, we set $2^{1-2l} = C$.

For estimating the composite interpolation error $u - I_M^N(u)$, we have the following result. □

Theorem 2.8. *Suppose $u \in PH^s(N)$ with $s \geq 1$. Then*

$$\|u - I_M^N(u)\|_{L^2(0,1)} \leq C\, M^{-s}\, |u|_{H^{0;s;M;N}(0,1)}, \tag{2.5}$$

and for $1 \leq r \leq s$,

$$\|u - I_M^N(u)\|_{H^r(0,1)} \leq C\, M^{2r-\frac{1}{2}-s}\, |u|_{H^{r;s;M;N}(0,1)}, \tag{2.6}$$

where,

$$\|u - I_M^N(u)\|_{H^r(0,1)} = \left(\sum_{n=1}^{N} \|u_n - \sum_{m=0}^{M} u(x_{nm})\phi_{nm}\|^2_{H^r(\frac{n-1}{N},\frac{n}{N})}\right)^{1/2}.$$

As in Theorem 2.5, C denotes a positive constant that depends upon the type of the norm, but which is independent of the function u and the integers N and M.

Proof. For $r \geq 0$, we have

$$\begin{aligned}
&\|u - I_M^N(u)\|^2_{H^r(0,1)}\\
&= \sum_{n=1}^{N} \|u_n - \sum_{m=0}^{M} u(x_{nm})\phi_{nm}\|^2_{H^r(\frac{n-1}{N},\frac{n}{N})}\\
&= C \sum_{n=1}^{N} \sum_{p=0}^{r} N^{2p-1}\, \|(\mathcal{F}_n u_n)^{(p)} - (\sum_{m=0}^{M} (\mathcal{F}_n u_n)(x_m)\, L_m)^{(p)}\|^2_{L^2(-1,1)},
\end{aligned}$$

where for the second equality, we used Lemma 2.7. From (2.5), by setting $r = 0$, we get

$$\begin{aligned}
&\|u - I_M^N(u)\|^2_{L^2(0,1)}\\
&= C \sum_{n=1}^{N} N^{-1}\, \|\mathcal{F}_n u_n - \sum_{m=0}^{M} (\mathcal{F}_n u_n)(x_m)\, L_m\|^2_{L^2(-1,1)}\\
&\leq C\, M^{-2s}\, N^{-1} \sum_{n=1}^{N} \sum_{k=\min(s,M+1)}^{s} \|(\mathcal{F}_n u_n)^{(k)}\|^2_{L^2(-1,1)}\\
&= C\, M^{-2s}\, N^{-1} \sum_{k=\min(s,M+1)}^{s} N^{1-2k} \sum_{n=1}^{N} \|u_n^{(k)}\|^2_{L^2(\frac{n-1}{N},\frac{n}{N})}
\end{aligned}$$

$$= C\, M^{-2s} \sum_{k=\min(s,M+1)}^{s} N^{-2k} \sum_{n=1}^{N} ||u_n^{(k)}||^2_{L^2(\frac{n-1}{N},\frac{n}{N})}.$$

Here, we used (2.2) for the second step. Therefore (2.5) is established.

Also, for $1 \leq r \leq s$, we have

$$\begin{aligned}
&||u - I_M^N(u)||^2_{H^r(0,1)} \\
&= C \sum_{n=1}^{N} \sum_{p=0}^{r} N^{2p-1}\, ||(\mathcal{F}_n u_n)^{(p)} - (\sum_{m=0}^{M} (\mathcal{F}_n u_n)(x_m)\, L_m)^{(p)}||^2_{L^2(-1,1)} \\
&\leq C\, M^{4r-1-2s}\, N^{2r-1} \sum_{n=1}^{N} \sum_{k=\min(s,M+1)}^{s} ||(\mathcal{F}_n u_n)^{(k)}||^2_{L^2(-1,1)} \\
&= C\, M^{4r-1-2s}\, N^{2r-1} \sum_{k=\min(s,M+1)}^{s} N^{1-2k} \sum_{n=1}^{N} ||u_n^{(k)}||^2_{L^2(\frac{n-1}{N},\frac{n}{N})} \\
&= C\, M^{4r-1-2s} \sum_{k=\min(s,M+1)}^{s} N^{2r-2k} \sum_{n=1}^{N} ||u_n^{(k)}||^2_{L^2(\frac{n-1}{N},\frac{n}{N})},
\end{aligned}$$

where for the second step, we used (2.3). This yields (2.6).

Note that by setting $M \geq s-1$ in (2.5) and (2.6) and using (2.4), we get

$$||u - I_M^N(u)||_{L^2(0,1)} \leq C\, M^{-s}\, N^{-s} \left(\sum_{n=1}^{N} ||u_n^{(s)}||^2_{L^2(\frac{n-1}{N},\frac{n}{N})} \right)^{1/2}, \tag{2.7}$$

and for $r \geq 1$,

$$\begin{aligned}
&||u - I_M^N(u)||_{H^r(0,1)} \\
&\qquad \leq C\, M^{2r-\frac{1}{2}-s}\, N^{r-s} \left(\sum_{n=1}^{N} ||u_n^{(s)}||^2_{L^2(\frac{n-1}{N},\frac{n}{N})} \right)^{1/2}.
\end{aligned} \tag{2.8}$$

□

Remark 2.9. *In the case that u is infinitely smooth, we can get*

$$\left(\sum_{n=1}^{N} ||u_n^{(s)}||^2_{L^2(\frac{n-1}{N},\frac{n}{N})} \right)^{1/2} = ||u^{(s)}||_{L^2(0,1)}, \qquad s \geq 0.$$

Therefore, in this case, (2.7) and (2.8) show that the rate of convergence of $I_M^N(u)$ to u is faster than $\frac{1}{N}$ to the power $M+1-r$ and any power $\frac{1}{M}$.

The error of approximating L^2-inner product by the bilinear form $< .,. >_{N,M}$ can be estimated as follows.

Corollary 2.10. *For $u \in PH^s(N)$ with $s \geq 1$ and $p \in \mathbb{P}_M^N$, by setting $M \geq s-1$, we have*

$$\left| \int_0^1 u(x)\, p(x)\, dx - < u, p >_{N,M} \right| \leq C\, M^{-s}\, N^{-s} \left(\sum_{n=1}^{N} ||u_n^{(s)}||^2_{L^2(\frac{n-1}{N}, \frac{n}{N})} \right)^{1/2} ||p||_{L^2(0,1)}, \tag{2.9}$$

where C is as in Theorem 2.8.

Proof. Using Proposition 2.2 (c) and the Cauchy-Schwartz inequality, we get

$$\begin{aligned} &\left| \int_0^1 u(x)\, p(x)\, dx - < u, p >_{N,M} \right| \\ &\qquad = \left| \int_0^1 u(x)\, p(x)\, dx - \int_0^1 I_M^N(u)(x)\, p(x)\, dx \right| \\ &\qquad \leq \int_0^1 \left| \Big(u(x) - I_M^N(u)(x) \Big)\, p(x) \right| dx \\ &\qquad \leq ||u - I_M^N(u)||_{L^2(0,1)}\, ||p||_{L^2(0,1)}. \end{aligned}$$

Therefore, (2.9) follows, using (2.7). $\square$

3. The differentiation matrix

In order to approximate the derivative of $I_M^N(u)$ by the composite interpolation operator, we need its values at the grid points x_{nm}, $n = 1, 2, \ldots, N$, $m = 0, 1, \ldots, M$. In this section, we compute the derivative of $I_M^N(u)$ at these grid points.

By differentiating the composite interpolation formula (2.1), for every $n = 1, 2, \ldots, N$ and $m = 0, 1, \ldots, M$, we get

$$\frac{d}{dx} I_M^N(u)(x_{nm}) = \sum_{m'=0}^{M} \tilde{d}_{mm'}\, u(x_{nm'}), \tag{3.1}$$

where $\tilde{d}_{mm'} = \frac{d}{dx}\phi_{nm'}(x_{nm}) = 2N \frac{d}{dx} L_{m'}(x_m) = 2N d_{mm'}$.

If we let

$$\mathbf{u} = \Big(u(x_{10}), \ldots, u(x_{1M}), \ldots, u(x_{N0}), \ldots, u(x_{NM})\Big)^T$$

and

$$\frac{d}{dx}\mathbf{u} = \Big(\frac{d}{dx} I_M^N(u)(x_{10}), \ldots, \frac{d}{dx} I_M^N(u)(x_{1M}), \ldots, \frac{d}{dx} I_M^N(u)(x_{N0}), \ldots, \frac{d}{dx} I_M^N(u)(x_{NM})\Big)^T,$$

then, analogous to (1.3), we can restate (3.1) as

$$\frac{d}{dx}\mathbf{u} = D_M^N\, \mathbf{u},$$

where D_M^N is the $N(M+1) \times N(M+1)$ block diagonal matrix

$$D_M^N = 2N \begin{bmatrix} D_M & 0 & \cdots & 0 \\ 0 & D_M & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_M \end{bmatrix}. \tag{3.2}$$

Remark 3.1. *According to Remark 1.1, for the case $N = 1$, it is easily seen that the differentiation matrix (3.2) coincides with the transformed Gauss pseudospectral differentiation matrix (1.3).*

4. **Numerical experiments**

In this section, in order to assess the accuracy and efficiency of the composite interpolation method, we apply it to the following functions:

(i) $u(x) = e^x$, $-1 \leq x \leq 1$, and

TABLE 1. Numerical results for the function (i)

	$E_0(N,M)$ $N=1$	$N=2$	$N=3$	$N=4$	$E_1(N,M)$ $N=1$	$N=2$	$N=3$	$N=4$
$M=1$	1.64e-1	4.74e-2	2.17e-2	1.23e-2	1.33e-0	7.44e-1	5.08e-1	3.84e-1
$M=2$	2.70e-2	3.98e-3	1.22e-3	5.21e-4	3.60e-1	1.03e-1	4.76e-2	2.70e-2
$M=3$	3.35e-3	2.49e-4	5.12e-5	1.64e-5	6.48e-2	9.52e-3	2.92e-3	1.24e-3
$M=4$	3.33e-4	1.25e-5	1.71e-6	4.12e-7	8.69e-3	6.45e-4	1.32e-4	4.23e-5
$M=5$	2.77e-5	5.22e-7	4.77e-8	8.61e-9	9.25e-4	3.46e-5	4.73e-6	1.13e-6

TABLE 2. Numerical results for the function (ii)

	$E_0(N,M)$ $N=1$	$N=2$	$N=3$	$N=4$	$E_1(N,M)$ $N=1$	$N=2$	$N=3$	$N=4$
$M=1$	1.49e-1	7.45e-2	2.87e-2	1.86e-2	1.43e-0	1.15e-0	6.86e-1	5.77e-1
$M=2$	1.12e-1	0	7.22e-3	0	1.23e-0	0	2.38e-1	0
$M=3$	2.46e-2	0	1.57e-3	0	4.94e-1	0	9.51e-2	0
$M=4$	1.95e-2	0	1.25e-3	0	3.18e-1	0	6.12e-2	0
$M=5$	9.21e-3	0	5.91e-4	0	2.85e-1	0	5.49e-2	0

(ii) $u(x) = \begin{cases} 1-x^2 & \text{for} \quad -1 \le x \le 0, \\ 1+x^2 & \text{for} \quad 0 \le x \le 1. \end{cases}$

For this purpose, we first transfer the two functions onto the interval $[0,1]$ and then apply the method for approximating them and their derivatives. In order to report our numerical results, it is convenient to define

$$E_0(N,M) = ||u - I_M^N(u)||_{L^2(0,1)},$$

and

$$E_1(N,M) = ||\frac{du}{dx} - I_M^N\big(\frac{d}{dx} I_M^N(u)\big)||_{L^2(0,1)}.$$

Table 1 and Table 2 show the numerical results of $E_0(N,M)$ and $E_1(N,M)$ with different values of N and M for the functions introduced as (i) and (ii), respectively. As seen from the results of Table 1, for the function (i), which is infinitely smooth, the rate of convergence is faster than any power of both $\frac{1}{N}$ and $\frac{1}{M}$, as noted in Remark 2.9. Also, for the function (ii), Table 2 shows that the approximation errors decays as fast as the global smoothness of the underlying function in (ii) permits.

5. Conclusion

Table 1 shows that for a fixed number of Lagrange polynomials, by increasing the number of block-pulse functions, the accuracy increases. Also, Table 2 suggests that choosing a suitable number of block-pulse functions yields a very good accuracy. Therefore, it seems that the composite interpolation method is an efficient method for approximating both smooth and nonsmooth functions and their derivatives. Furthermore, the corresponding differentiation matrix is a block diagonal matrix, making the method very attractive for solving differential equations.

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