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AN INTRODUCTION TO HIGHER CLUSTER CATEGORIES

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ABSTRACT. In this survey, we give an overview over some aspects of the set of tilting objects in an $m-$cluster category, with focus on those properties which are valid for all $m \geq 1$. We focus on the following three combinatorial aspects: modeling the set of tilting objects using arcs in certain polygons, the generalized assicahedra of Fomin and Reading, and colored quiver mutation.

1. **Introduction**

Cluster categories were defined in [9] in order to use categorical methods to give a conceptual model for the combinatorics of cluster algebras, as defined by Fomin and Zelevinsky [16]. With contributions from many mathematicians, this theory and its generalisations have given new links between categorical representation theory and several branches of mathematics and mathematical physics. In addition, various problems concerning cluster algebras and related combinatorial problems have been solved. There are several recent survey papers on this topic, e.g., [25, 26, 1], discussing both categorical and combinatorial aspects of the theory.

In this survey, we discuss some combinatorial aspects of a generalisation of cluster categories, called m-cluster categories, or higher cluster

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categories. Such categories are not explicitly linked to cluster algebras. A survey on categorical aspects of higher cluster categories, and generalizations, is given in [27].

A cluster category is defined as an orbit category of the derived category of an hereditary finite dimensional algebra. Loosely speaking, it is obtained by identifying the AR-translation τ with the shift [1]. Keller [24] proved that a cluster category is triangulated, and that the canonical functor from the derived category to the cluster category is a triangle functor. The orbit category is a Calabi-Yau category of CY-dimension 2.

Keller's proof also showed that the orbit category obtained by identifying τ with the m-fold shift $[m]$ is triangulated. These categories has later been called m-cluster categories, and they are Calabi-Yau of dimension $m + 1$.

The main interest in 1-cluster categories, and some other triangulated categories of CY-dimension 2, is due to the combinatorial properties of the set of tilting objects (also called cluster tilting objects). The definition of tilting objects canonically extends to m-cluster categories.

In this survey, we give an overview over some combinatorial aspects of the set of tilting objects in an m-cluster category, with focus on those properties which are valid for all $m \geq 1$. In the two first sections, we give some more details and background and a precise definition. We also recall definitions and results on tilting theory in higher cluster categories. The results in these sections are mainly due to Wraalsen [37], Zhou [38] and Zhu [39]. Then, in the next three sections, we consider three different, but related, combinatorial aspects of the set of tilting objects $\mathcal{T}$ in m-cluster categories. First, we discuss work of Baur and Marsh, who modeled the combinatorics of $\mathcal{T}$ in the Dynkin case A or D using arcs in certain (unpunctured or punctured) polygons [7, 8]. Next, we discuss links to the Fomin-Reading generalised associahedra [15], due to Thomas [32] and Zhu [38]. Then, in Section 5, we explain colored quivers and mutation of such, as defined in a joint work with Thomas [12], and show how these can be used to describe combinatorial aspects of $\mathcal{T}$ for arbitrary finite quivers. We end, in Section 6, with some comments on other aspects of higher cluster categories and generalisations.

2. Background and definition

We give some background on derived categories, before we discuss the construction of the m-cluster categories. For more information on derived categories, see [18, 21]. For basic information on finite dimensional algebras and their representation theory, see the textbooks [4, 5].

2.1. The derived category.

Let H be a hereditary finite dimensional algebra over an algebraically closed field k. We assume H is basic, and hence H is isomorphic to a path algebra kQ of some finite quiver Q. Let $\operatorname{mod} H$ be the category of finite dimensional left H-modules, and let $D^b(H)$ be the (bounded) derived category. Let $[1]$ denote the shift functor on $D^b(H)$, and let $[-1]$ denote its inverse. The derived category is a Krull-Schmidt category, and its indecomposable objects are isomorphic to stalk complexes $M[i]$, where M is an indecomposable H-module, and i is some integer. For indecomposables $M[i]$ and $N[j]$, we have that the morphism spaces are given by

$$\operatorname{Hom}_{D^b(H)}(M[i], N[j]) = \begin{cases} \operatorname{Hom}_H(M,N) & \text{if } i = j \\ \operatorname{Ext}^1_H(M,N) & \text{if } j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

By results of Happel [18], the derived category $D^b(H)$ has Auslander-Reiten triangles. This implies that there is an autoequivalence τ on the derived category, with the property that for each indecomposable object M, there is a uniquely determined triangle

$$\tau M \to E \to M \to .$$

Furthermore, we have the Auslander-Reiten formula

$$\operatorname{Hom}_{D^b(H)}(M, N[1]) \simeq D \operatorname{Hom}_{D^b(H)}(N, \tau M),$$

where $D = \operatorname{Hom}(\cdot, k)$ is the ordinary duality.

We view objects in $\operatorname{mod} H$ as stalk complexes in degree 0. If M is a non-projective indecomposable module, then τM coincides with $\tau_H M$, where τ_H denotes the AR-translation in the module category. If P is an indecomposable projective, then $\tau P = I[-1]$, where $I = D \operatorname{Hom}_H(P, H)$ is indecomposable and injective.

2.2. An example of type A.

Let Q be the quiver

$$1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4 \, .$$

Consider the path algebra $H = kQ$, and let e_i be the idempotent in H corresponding to the vertex i. There are 10 indecomposable modules in $\operatorname{mod} H$. These are the 4 projectives $P_i = He_i$ and the 4 injectives $I_i = D(e_iH)$, in addition to the two modules $X \cong (P_1 \amalg P_3)/P_2$ and $Y = P_3/P_4$. The AR-quiver of the module category is given as follows, where the action of τ is indicated by the dotted arrows:

$$\begin{tikzcd}
 & P_1 \arrow[dr] & & I_4 \arrow[ll, dotted] \arrow[dr] & \\
P_2 \arrow[ur] \arrow[dr] & & X \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & I_3 \arrow[ll, dotted] \\
 & P_3 \arrow[ur] \arrow[dr] & & I_2 \arrow[ll, dotted] \arrow[ur] \arrow[dr] & \\
P_4 \arrow[ur] & & Y \arrow[ll, dotted] \arrow[ur] & & I_1 \arrow[ll, dotted]
\end{tikzcd}$$

In the derived category, the AR-translation τ is defined on all objects, and actually becomes an autoequivalence. A segment of the AR-quiver of the derived category looks as follows, where for an indecomposable M, we have that τM is the neighbour directly to left:

$$\begin{tikzcd}[column sep=tiny]
 & & I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & I_4 \arrow[dr] & & P_4[1] \arrow[dr] & & Y[1] \arrow[dr] & & I_1[1] \arrow[dr] & & P_1[2] \arrow[dr] & & P_4[2] & \\
\cdots & I_2[-1] \arrow[ur] \arrow[dr] & & P_2 \arrow[ur] \arrow[dr] & & X \arrow[ur] \arrow[dr] & & I_3 \arrow[ur] \arrow[dr] & & P_3[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & & P_2[2] \arrow[ur] \arrow[dr] & & X[2] \arrow[ur] \arrow[dr] & & \cdots \\
 & & I_3[-1] \arrow[ur] \arrow[dr] & & P_3 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & X[1] \arrow[ur] \arrow[dr] & & I_3[1] \arrow[ur] \arrow[dr] & & P_3[2] \arrow[ur] \arrow[dr] & & I_2[2] & \\
\cdots & I_4[-1] \arrow[ur] & & P_4 \arrow[ur] & & Y \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & I_4[1] \arrow[ur] & & P_4[2] \arrow[ur] & & Y[2] \arrow[ur] & & \cdots
\end{tikzcd}$$

2.3. **The m-cluster category.** Consider now the autoequivalence $G = \tau^{-1}[m]$ on $D^b(H)$, and define the m-cluster category to be the orbit category $\mathcal{C} = D^b(H)/G$.

The objects of $\mathcal{C}$ are the G-orbits of objects in $D^b(H)$; we use the same notation for an object in $D^b(H)$ and its orbit in $\mathcal{C}$. The morphism spaces in $\mathcal{C}$ are given by

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) = \amalg_i \operatorname{Hom}_{\mathcal{D}^b(H)}(X, G^i Y).$$

Keller [24] proved that $\mathcal{C}$ is triangulated, and that the canonical functor $D^b(H) \to \mathcal{C}$ is a triangle functor. It follows from [9] that $\mathcal{C}$ is a Krull-Schmidt category with almost split triangles and translation functor induced from $D^b(H)$, and it can be shown that the AR-formula

$$\operatorname{Hom}_{\mathcal{C}}(M, N[1]) \simeq D \operatorname{Hom}_{\mathcal{C}}(N, \tau M),$$

still holds in $\mathcal{C}$.

There is a canonical embedding of $\operatorname{mod} H$ into $D^b(H)$. Let $\operatorname{mod} H[0]$ denote the image under this embedding, and let $\operatorname{mod} H[i]$ be defined in the obvious way. We say that $\operatorname{mod} H[0] \vee \cdots \vee \operatorname{mod} H[m-1] \vee H[m]$ is a standard domain in $D^b(H)$. It is clear from the definition of $\mathcal{C}$, that any indecomposable object in $\mathcal{C}$ is up to isomorphism induced by an object in the standard domain.

Example 2.4. *We consider the path algebra of example given in 2.2. Now, the* 2*-cluster category is of finite type, consisting of 24 indecomposable objects: 2 copies of the 10 indecomposable objects in the module category and one additional copy of the 4 indecomposable projectives. The AR-quiver looks as follows, where one should note that objects on the left border and the right border are identified:*

$$\begin{tikzcd}
& P_1 \arrow[dr] & & I_4 \arrow[dr] & & P_4[1] \arrow[dr] & & Y[1] \arrow[dr] & & I_1[1] \arrow[dr] & & P_1[2] \arrow[dr] & & P_1 \\
P_2 \arrow[ur] \arrow[dr] & & X \arrow[ur] \arrow[dr] & & I_3 \arrow[ur] \arrow[dr] & & P_3[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & & P_2[2] \arrow[ur] \arrow[dr] & & P_2 \arrow[ur] \arrow[dr] & \\
& P_3 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & X[1] \arrow[ur] \arrow[dr] & & I_3[1] \arrow[ur] \arrow[dr] & & P_3[2] \arrow[ur] \arrow[dr] & & P_3 \\
P_4 \arrow[ur] & & Y \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & I_4[1] \arrow[ur] & & P_4[2] \arrow[ur] & & P_4 \arrow[ur] &
\end{tikzcd}$$

3. Tilting objects and exchange triangles

Tilting theory in module categories over finite dimensional algebras was initiated more than 30 years ago; see [3]. The original motivation was to compare module categories. Happel [18] introduced the use of derived categories in the theory, and showed that algebras related by tilting are derived equivalent.

In the setting of hereditary algebras, a tilting module in $\operatorname{mod} H$ is a module T with $\operatorname{Ext}^1_H(T,T) = 0$ and with n indecomposable non-isomorphic direct summands, where H has n isomorphism-classes of simples.

In the work of Riedtmann and Schofield [30], Unger [35], and several others, combinatorial properties on the set of tilting modules were studied. In particular, the simplicial complex defined by the set of direct summands in tilting modules was introduced; see [36] for more background on combinatorial aspects of tilting modules for finite dimensional algebras.

In this section, we will define tilting objects in higher cluster categories. Using the natural embedding of a module category into a cluster category, it is easy to see that the tilting modules will be mapped to the tilting objects. In fact, for 1-cluster categories, all tilting objects are of this form (up to derived equivalence). In the case of higher cluster categories, there are more tilting objects, as we will observe in later examples.

3.1. **Tilting theory in m-cluster categories.** An object M in an m-cluster category is called rigid if $\mathrm{Ext}^i_{\mathcal{C}}(M,M)=0$, for $i=1,\dots,m$. A finite collection of rigid objects $\{X_i\}$ is said to be Ext-compatible if the direct sum $\amalg X_i$ is rigid. M is called maximal rigid if the indecomposable direct summands in M form a maximal Ext-compatible collection. A tilting object M in $\mathcal{C}$ is a rigid object with the additional property that if an object X satisfies $\mathrm{Ext}^i(M,X)=0$, for $i=1,\dots,m$. Then, this implies that X is in $\operatorname{add} M$.

Zhu [38] (see also [37]) showed that tilting and maximal rigid objects coincide. This was shown in [9] for the case $m=1$. Recall that an object X is called *basic* if any indecomposable object occurs at most once in a direct sum decomposition of X.

Theorem 3.1. [38] *The followings are equivalent for a basic rigid object T in an m-cluster category.*

(a) *T is maximal rigid.*
(b) *T is tilting.*
(c) *T has n isomorphism classes of indecomposable direct summands.*

Note that it follows from this that every (basic) rigid object is a direct summand in a tilting object.

Complements 3.2. *Let $T=\amalg_{i=1}^n T_i$ be a tilting object in an m-cluster category, and fix an indecomposable direct summand T_k.*

We call $B_k=T/T_k$ an *almost complete tilting object*, and indecomposable objects X such that $B_k \amalg X$ is tilting, are called complements to B_k. Indeed, T_k is a complement. Let $T_k \xrightarrow{f} B'_k$ be a minimal left $\operatorname{add} B_k$-approximation of T_k. This means:

- B'_k is in $\operatorname{add} B_k$.
- Any map from T_k to an object in $\operatorname{add} B_k$, factors through the map f.
- If $gf=f$, for some endomorphism $g\colon B'_k\to B'_k$, then g is an automorphism.

Let

$$T_k \to B'_k \to T^*_k \to \tag{3.1}$$

be the induced triangle in $\mathcal{C}$. Then, one can show that T_k^* is also a complement to B_k with $T_k^* \not\simeq T_k$. The triangle (3.1) is called an *exchange triangle*. One can of course iterate this procedure to produce new complements and exchange triangles. However, one can show that after m iterations, such that totally $m+1$ complements are constructed, no new complements will occur. Also, one can show that B_k has no further complements than those constructed in this way. More precisely, we have the following theorem.

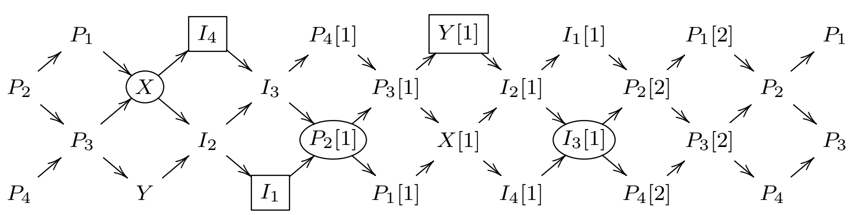
Theorem 3.3. [37, 39] *The almost complete tilting object B_k in the m-cluster category $\mathcal{C}$ has exactly $m+1$ complements $T_k^{(c)}$, for $c = 0, 1, \ldots, m$, occurring in m exchange triangles*

$$T_k^{(c)} \xrightarrow{f_k^{(c)}} B_k^{(c)} \xrightarrow{g_k^{(c+1)}} T_k^{(c+1)} \xrightarrow{h_k^{(c+1)}} . \tag{3.2}$$

The fact that we get $m+1$ complements in this way was proved in [23], while the fact that there are no further complements was proved independently in [39] and [37].

It is pointed out in [39] that exchange is transitive on the set of tilting objects; i.e., any tilting object can be reached from any other tilting object by a finite sequence of exchanges. This was proved in [9] for $m = 1$, using ideas in [19].

Example 3.3 *We revisit our example given in 2.2. The boxed objects are the direct summands of an almost complete tilting object $B = I_4 \amalg I_1 \amalg Y[1]$, and the encircled object are the three complements of B:*



The three exchange triangles are:

$$X \to I_1 \amalg I_4 \to P_2[1] \to$$

$$P_2[1] \to Y[1] \to I_3[1] \to$$

$$I_3[1] \to 0 \to I_3[2](= X) \to .$$

4. **A graphical description**

Independent of the ideas in [9], Caldero et al. [13] defined a family of categories, using diagonals in regular n-gons as objects. They also showed that their categories are equivalent to the cluster categories of Dynkin type A. Later, Schiffler [31] used a similar approach to describe the cluster categories of type D. He considered punctured n-gons instead.

Generalising this, Baur and Marsh gave a graphical interpretation of m-cluster categories in type A [8] and in type D [7]; see [6] for a survey. Here, we will give a brief discussion of their ideas in type A, including an example.

4.1. **A category from polygons.** We discuss here the results of Baur and Marsh [8] for Dynkin type A. We want to construct a certain category of diagonals of an $(nm+2)$-gon $\mathcal{P} = \mathcal{P}_{nm+2}$, where m and n are positive integers, and $n > 1$. This category will be equivalent to the m-cluster category of a Dynkin quiver of type A_{n-1}. The indecomposable objects in the m-cluster category $\mathcal{C}$ of type A_{n-1} will correspond to m-diagonals in $\mathcal{P}$. Here, an m-diagonal is a diagonal with the property that it divides $\mathcal{P}$ into an $(mi + 2)$-gon (for some positive integer i), and its complement, which is then an $(m(n - i) + 2)$-gon.

The actual reconstruction of the cluster category from this data is made in three steps:

- construct a quiver Γ which is isomorphic, as a *stable translation quiver*, to the AR-quiver of the cluster category,
- take the *mesh category* of Γ, and
- take the additive category generated by the mesh category.

We shall first explain these notions, and then see how Γ is constructed. The AR-quiver Δ of a cluster category is an example of a stable translation quiver. The AR-translation gives a bijective map $\tau\colon \Delta_0 \to \Delta_0$ with the following property: given any two vertices x, y, the number of arrows $x \to y$ equals the number of arrows $\tau y \to x$.

A (locally finite) quiver Γ without loops, such that a translation-function τ_Γ with the same property as τ above exists, is called a *stable translation quiver*.

Given a stable translation quiver Γ with translation function τ_Γ, one can define a *mesh category* $M_{(\Gamma,\tau_\Gamma)}$. The objects in this category are the vertices of Γ, and these are then the indecomposable objects in the additive category generated by $M_{(\Gamma,\tau)}$. Here we only consider quivers

Γ without multiple arrows. In this case, the maps in $M_{(\Gamma,\tau)}$ are all linear combination of paths modulo a certain ideal I generated by the *mesh relations*. For every vertex v, there is one mesh relation, which is constructed as follows. Let $\{b_i : v_i \to v\}$ be all arrows ending in v, and let $a_i : \tau v \to v_i$ be the arrow corresponding to b_i. Then, the sum $\sum b_i a_i$ is the mesh relation for v.

We now describe how to get a stable translation quiver Γ from the $(mn+2)$-gon. Label the vertices of the polygon $1, \ldots, mn+2$ (in a clockwise oriented cycle), and let (i,j) denote an m-diagonal between the vertices i and j. We now construct a finite quiver Γ, by letting the vertices correspond to the m-diagonals. We denote by $(i,j) = (j,i)$ the vertex corresponding to the diagonal between i and j. We draw an arrow $(i,j) \to (i,j+m)$, if $(i,j+m)$ is an m-diagonal, and an arrow $(i,j) \to (i+m,j)$, if $(i+m,j)$ is an m-diagonal. In addition, we define a translation τ_Γ by mapping (i,j) to $(i-m,j-m)$.

Theorem 4.1. [8] *The m-cluster category of type A_{n-1} is equivalent to the additive category of the mesh category $M_{(\Gamma,\tau_\Gamma)}$, where Γ is the constructed from the $(mn+2)$-gon as above.*

Example 4.2. *Let $m=2$ and $n=5$, and consider the 12-gon. It gives rise to the following stable translation quiver, which is easily seen to be isomorphic to the AR-quiver of the m-cluster category of type A_4 from example given in 2.3:*

$$\begin{tikzcd}[column sep=tiny]
& (3,6) \arrow[dr] & & (5,8) \arrow[dr] & & (7,10) \arrow[dr] & & (9,12) \arrow[dr] & & (11,2) \arrow[dr] & & (1,4) \arrow[dr] & & (3,6) \\
(1,6) \arrow[ur] \arrow[dr] & & (3,8) \arrow[ur] \arrow[dr] & & (5,10) \arrow[ur] \arrow[dr] & & (7,12) \arrow[ur] \arrow[dr] & & (9,2) \arrow[ur] \arrow[dr] & & (11,4) \arrow[ur] \arrow[dr] & & (1,6) \arrow[ur] \arrow[dr] & \\
& (1,8) \arrow[ur] \arrow[dr] & & (3,10) \arrow[ur] \arrow[dr] & & (5,12) \arrow[ur] \arrow[dr] & & (7,2) \arrow[ur] \arrow[dr] & & (9,4) \arrow[ur] \arrow[dr] & & (11,6) \arrow[ur] \arrow[dr] & & (1,8) \\
(11,8) \arrow[ur] & & (1,10) \arrow[ur] & & (3,12) \arrow[ur] & & (5,2) \arrow[ur] & & (7,4) \arrow[ur] & & (9,6) \arrow[ur] & & (11,8) \arrow[ur] &
\end{tikzcd}$$

4.2. **Interpretation of tilting objects and exchange.** The construction described above also has an additional important feature. The correspondence between indecomposable objects in the m-cluster category of type A_{n-1} and the category of diagonals of $\mathcal{P}_{mn+2}$ is defined such that two indecomposable objects X and Y in $\mathcal{C}$ are Ext-compatible if and only if the m-diagonals corresponding to X and Y do not cross. The maximal sets of non-crossing m-diagonals in $\mathcal{P}$ are called $(m+2)$-angulations. They always have $n-1$ elements and correspond to the tilting objects in $\mathcal{C}_{A_{n-1}}$. If we remove an m-diagonal in an $(m+2)$-angulation, we can replace it with m different m-diagonals, to obtain m different $(m+2)$-angulations. This corresponds to replacing one indecomposable summand T_k in a tilting object T with one of the m complements of T/T_k different from T_k.

Example 4.4 *The tilting object $B \amalg X$ of example given in 3.3, corresponds to a 4-angulation of a 12-gon as in Figure 1.*

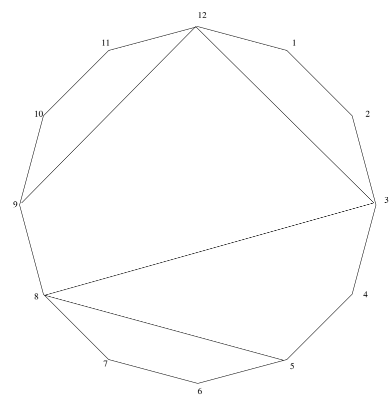


FIGURE 1. The 4-angulation corresponding to $B \amalg X$

If we remove the 2-diagonal corresponding to X in this 12-angulation, we can replace it with $m = 2$ different 2-diagonals, and obtain the two 4-angulations of Figure 2. These correspond to the tilting objects $B \amalg P_2[1]$ and $B \amalg I_3[1]$.

5. The simplicial complex of m-clusters

An *(abstract) simplicial complex* is a nonempty family Δ of finite subsets of a fixed universal set, with the property that if X is in Δ, then also every subset $Y \subset X$ is in Δ.

An m-cluster category $\mathcal{C} = \mathcal{C}_H$ gives in a canonical way rise to a simplicial complex $\Delta(\mathcal{C})$: take the set of isomorphism classes of indecomposables in $\mathcal{C}$ as the universal set, and let $\Delta(\mathcal{C})$ consist of the subsets X with the property that the elements in X are Ext-compatible.

Consider now the case, where $\mathcal{C} = \mathcal{C}_H$ is the m-cluster category of $H = kQ$, and Q is a *Dynkin quiver*. Corresponding to the underlying graph of Q, there is a finite root system Φ.

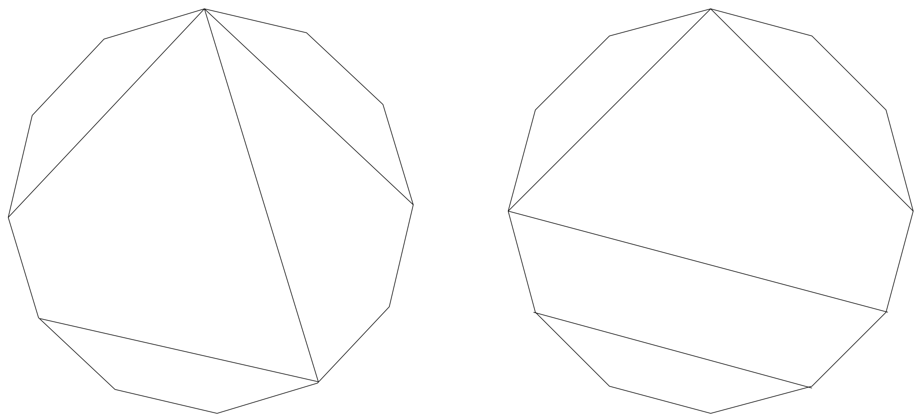


Figure 2. The 4-angulations corresponding to $B \amalg P_2[1]$ (left) and $B \amalg I_3[1]$ (right)

Starting with a finite root system and a positive integer m, Fomin and Reading [15] have defined another simplicial complex, the *m-cluster complex*, and one of the original motivations of studying tilting theory in m-cluster categories, was to compare their simplicial complex to $\Delta(\mathcal{C})$. This was done independently by Thomas [32] and Zhu [38]. Zhu also dealt with non-simply laced Dynkin graphs and their corresponding root systems. The m-cluster complexes naturally generalizes the 1-cluster complexes, which play a crucial role in the study of cluster algebras [16].

For a finite root system Φ, Fomin and Reading considered the set $\Phi^m_{\geq -1}$ of *coloured almost positive roots*. This set consists of m copies of the positive roots, and one set of copies of the negative simple roots. This is the universal set for the m-cluster complex. Then, they defined a notion of compatibility of elements in this set. This is combinatorially defined, and we leave out the details here, but refer instead to [15, Section 2]. The m-cluster complex consists of all sets of compatible elements in $\Phi^m_{\geq -1}$.

Fomin and Reading show that m-cluster complexes satisfy some nice conditions.

Theorem 5.1. [15] *Consider a root system with n simple positive roots, or equivalently a Dynkin graph with n vertices.*

(a) *All facets (inclusion-maximal sets) in the m-cluster complex have cardinality n.*

(b) *Each set in the m-cluster complex of cardinality $n-1$ is a subset of exactly $n+1$ facets.*

For a given Dynkin quiver Q, it is well known that the set of indecomposable $H = kQ$-modules is in bijection with the set of positive roots of the corresponding root system. Hence, it is clear that the indecomposable objects $\operatorname{ind}\mathcal{C}_H$ in the cluster category $\mathcal{C}_H$ are in bijection with the set $\Phi^m_{\geq -1}$ of colored almost positive roots.

Now, assume Q has alternating orientation, i.e., each vertex is either a sink or a source. In this case, Thomas [32] and Zhu [38] defined a bijection W between these two sets in such a way that Ext-compatible objects in the cluster category were mapped to compatible elements in $\Phi^m_{\geq -1}$. Hence, they obtained the following result.

Theorem 5.2. *Using the bijection W to identify the set of indecomposable objects in the cluster category $\mathcal{C}_H$ with the set $\Phi^m_{\geq -1}$, the m-cluster complex coincides with $\Delta(\mathcal{C})$.*

Let M_α be the indecomposable $H = kQ$-module corresponding to the positive root α. The bijection map W basically extends, in a canonical way, this correspondence to a correspondence between the indecomposables in $\mathcal{C}$ of the form $M[i]$, for $0 \leq i \leq m-1$, and the m copies of the positive roots. The indecomposables $P[m] = I[-1]$ are identified with the negative simple roots.

Using this, the authors in [32, 38] gave a conceptual, and type-free proof of Theorem 5.1, by combining Theorem 5.2 with the results in Section 3. Here, we should note that the results needed concerning the number of direct summands for tilting objects and the number of complements were proved in [32, 38] for the Dynkin case.

6. Mutation of colored quivers

We will now discuss another combinatorial approach to m-cluster categories, motivated by the fact that tilting and exchange in 1-cluster categories give a categorical model for the Fomin-Zelevinsky quiver mutation. We will first recall the notion of quiver mutation.

6.1. **The Fomin-Zelevinsky quiver mutation.** Let $Q = (q_{ij})$ be a finite quiver with vertices $1, \ldots, n$, with q_{ij} arrows from i to j, and with no loops or oriented 2-cycles (parallel underlying edges with opposite directions). For a fixed vertex v, we get a new quiver $\mu_v(Q)$, also without loops or oriented two-cycles. This operation, called *quiver mutation in*

v, can be described in various ways. Having the generalisation to $m > 1$ in mind, we choose the following formulation.

- For each pair of arrows $i \to v \to j$ in Q, add an arrow $i \to j$.
- If, between some pairs of vertices, there appear parallel underlying edges with opposite directions (oriented 2-cycles), remove the same number of arrows in each direction, until there are no oriented 2-cycles.
- Reverse all arrows starting in or ending in v.

It is straightforward to check that this operation satisfies $\mu_v(\mu_v(Q)) = Q$. It is also straightforward to verify that the quiver $\mu_v(Q) = (\widetilde{q_{ij}})$ is determined by the following formula, which is a reformulation of the FZ-mutation formula:

$$\widetilde{q_{ij}} = \begin{cases} q_{ji} & \text{if } v = i \text{ or } v = j \\ \max\{0, q_{ij} - q_{ji} + q_{iv}q_{vj} - q_{jv}q_{vi}\} & \text{if } i \neq v \neq j. \end{cases} \tag{6.1}$$

For a tilting object T in a cluster category $\mathcal{C}$, we can consider the endomorphism-algebra $\mathrm{End}_{\mathcal{C}}(T)$. This is again a finite dimensional basic k-algebra, and therefore is isomorphic to a factor algebra of a path algebra of a finite quiver Q_T (the Gabriel quiver of T).

Consider now a 1-cluster category. Let $T = B \amalg M$ and $T' = B \amalg M^*$ be two tilting objects, and let Q_T and Q_{T^*} be their respective Gabriel-quivers. The main result of [10] is that

$$Q_{T^*} = \mu_v(Q_T), \tag{6.2}$$

where v corresponds to the indecomposable object M. This can be considered a categorification of the FZ-quiver mutation.

It is natural to ask for a generalization of the above to the case $m > 1$. We give an example to show that there can be no direct generalization in terms of the Gabriel quiver of T.

Example 6.2. *Consider the* 3*-cluster category of type* A_2*. Let* P_1 *be the simple projective,* P_2 *be the indecomposable projective of length* 2*, and* I_2 *be the simple injective. Then, the AR-quiver of the* 3*-cluster category has 11 vertices:*

$$\begin{tikzcd}[column sep=small]
 & P_2 \arrow[dr] & & P_1[1] \arrow[dr] & & I_2[1] \arrow[dr] & & P_2[2] \arrow[dr] & & P_1[3] \arrow[dr] & & P_1 \arrow[dr] & \\
P_1 \arrow[ur] & & I_2 \arrow[ur] & & P_2[1] \arrow[ur] & & P_1[2] \arrow[ur] & & I_2[2] \arrow[ur] & & P_2[3] \arrow[ur] & & P_2
\end{tikzcd}$$

Consider the almost complete tilting object $P_2[2]$, and the four completions,

$$T_a = P_2[2] \amalg P_1, \qquad T_b = P_2[2] \amalg P_1[1]$$
$$T_c = P_2[2] \amalg P_1[2] \quad \text{and} \qquad T_d = P_2[2] \amalg I_2[2].$$

The following picture describes the Gabriel quivers of the endomorphism rings of these tilting objects, with the direction of exchange indicated by the broken arrows:

$$\begin{tikzcd}
T_a\colon & \cdot & \cdot & {} \arrow[r, dotted] & {} & \cdot & \cdot & \colon T_b \arrow[d, dotted] \\
T_d\colon \arrow[u, dotted] & \cdot & \cdot \arrow[l] & {} & {} \arrow[l, dotted] & \cdot \arrow[r] & \cdot & \colon T_c
\end{tikzcd}$$

From this, it is clear that more information than the Gabriel quiver of a tilting object T is needed, in order to generalize the formula (6.2).

6.2. **Colored quivers and mutation.** It turns out that instead of the Gabriel quivers, we can now deal with colored quivers.

An m-colored multi-quiver Q consists of vertices $1, \dots, n$ and colored arrows $i \overset{(c)}{\to} j$, where c is in $\{0, 1, \dots, m\}$. We let $q_{ij}^{(c)}$ denote the number of arrows from i to j of color (c).

Colored quiver mutation was introduced in [12]. Given a vertex v in an m-colored quiver Q, define a new colored quiver $\mu_v(Q)$ by modifying Q as follows:

- For each pair of arrows

$$i \xrightarrow{(c)} v \xrightarrow{(0)} j$$

with c in $\{0, 1, \dots m\}$, add two arrows: one arrow of color (c) from i to j and one arrow of color $(m - c)$ from j to i.
- If, for some pairs of vertices, there appear parallel arrows with different colors from i to j, remove the same number of arrows of each color.
- Change the color of all arrows ending at v, by adding one. Change the color of all arrows starting at v, by subtracting one.

Alternatively one can describe colored mutation via a formula which is a generalized version of the formula (6.1). If $Q = (q_{ij})$ is an m-colored quiver, then $Q' = \mu_v Q = (\widetilde{q_{ij}})$ is given by [1]

$$\widetilde{q}^{(c)}_{ij} = \begin{cases} q_{ij}^{(c+1)} & \text{if } v = i \\ q_{ij}^{(c-1)} & \text{if } v = j \\ \max\{0, q_{ij}^{(c)} - \sum_{t\neq c} q_{ij}^{(t)} + (q_{iv}^{(c)} - q_{iv}^{(c-1)})q_{vj}^{(0)} & \text{otherwise.} \\ \qquad\qquad +q_{iv}^{(m)}(q_{vj}^{(c)} - q_{vj}^{(c+1)})\} & \end{cases}$$

6.3. **The colored quiver of a tilting object.** Let $m \geq 1$ be an integer, and $\mathcal{C}$ be an m-cluster category. We want to assign to each tilting object $T = \amalg_{i=1}^n T_i$ in $\mathcal{C}$ a colored quiver $Q_T = (q_{ij}^{(c)})$ with n vertices corresponding to the indecomposable direct summands in T. To determine the colored arrows, we use the exchange triangles (3.2): we let $q_{ij}^{(c)}$ be the multiplicity of T_j as a direct summand in $B_i^{(c)}$. Note that the 0-colored arrows are indeed the arrows of the Gabriel quiver of T.

Not all colored quivers can be obtained as Q_T for a tilting object T. By definition, there are no loops (of any color) in Q_T, that is, $q_{ii}^{(c)} = 0$, for i and all c. Also, one can prove that Q_T is locally monochromatic: for fixed vertices i and j there are only arrows of one color from i to j. One can also prove that $q_{ij}^{(c)} = q_{ji}^{(m-c)}$, that is, for each arrow of color c, there is an arrow in the opposite direction with color $m - c$. There are also more known restrictions; see [12, Prop. 5.1].

It is an interesting open problem to find a set of properties that characterizes the colored quivers of type Q_T among all colored quivers.

One can now generalize the result in [10] to colored quivers of tilting objects in higher cluster categories.

Theorem 6.1. *Let $T = \amalg_{i=1}^n T_i$ and $T' = T/T_j \amalg T_j^{(1)}$ be tilting objects in an m-cluster category $\mathcal{C}$ such that there is an exchange triangle*

$$T_j \to B_j^{(0)} \to T_j^{(1)} \to . \tag{6.3}$$

Then, $Q_{T'} = \mu_j(Q_T)$.

Example 6.2. *Revisiting example given in 6.2, we now consider instead the colored quivers, and their mutations. Note that we always mutate in*

[1] Note that in [12], there is an unfortunate typo in the formula: the two first cases are mixed up.

the leftmost vertex:

$$\begin{array}{ccc} T_a\colon \quad \cdot \underset{(1)}{\overset{(2)}{\rightleftarrows}} \cdot & \dashrightarrow & \cdot \underset{(2)}{\overset{(1)}{\rightleftarrows}} \cdot \quad : T_b \\ \uparrow & & \downarrow \\ T_d\colon \quad \cdot \underset{(0)}{\overset{(3)}{\rightleftarrows}} \cdot & \dashleftarrow & \cdot \underset{(3)}{\overset{(0)}{\rightleftarrows}} \cdot \quad : T_c \end{array}$$

Example 6.3. *We consider again the case $m = 2$, with the quiver Q of type A_4 as in example given in 2.2.*

The colored quivers of the three tilting objects

$$T = I_1 \amalg I_4 \amalg Y[1] \amalg X, \quad T' = I_1 \amalg I_4 \amalg Y[1] \amalg P_2[1] \text{ and}$$
$$T'' = I_1 \amalg I_4 \amalg Y[1] \amalg I_3[1]$$

are given in Figure 3. Note that $Q_{T'}$ is given by colored mutation of Q_T at the vertex corresponding to X, that $Q_{T''}$ is given by colored mutation of $Q_{T'}$ at the vertex corresponding to $P_2[1]$, and that Q_T is given by colored mutation of $Q_{T''}$ at the vertex corresponding to $I_3[1]$.

6.4. **Finiteness of the mutation class.** *Let Q be an acyclic quiver. We can view this as an m-colored quiver, by regarding each arrow α in Q as an arrow of color (0), and then adding an arrow of color (m) in the opposite direction to α.*

Torkildsen [33] has proved the following, generalizing a similar statement of [11] for $m = 1$.

Theorem 6.4. [33] *The colored mutation class of a connected acyclic quiver Q is finite if and only if Q is either of Dynkin or extended Dynkin type, or has at most two vertices.*

In Dynkin type A, Torkildsen [34] has also found a formula for the number of elements in the mutation class, using a connection to the classical cell-growth problem [20]. Fomin and Reading [15] have shown that number of m-clusters (in the Dynkin case) is given by the Fuss-Catalan numbers.

6.5. **m-cluster tilted algebras.** colored quiver mutation gives some information on the m-cluster-tilted algebras, i.e., algebras of the form $\operatorname{End}_{\mathcal{C}}(T)$, for T a cluster-tilting object in an m-cluster category.

Using that any tilting object can be reached from any other tilting object by a sequence of exchanges [39], one obtains the following as a consequence of Theorem 6.1.

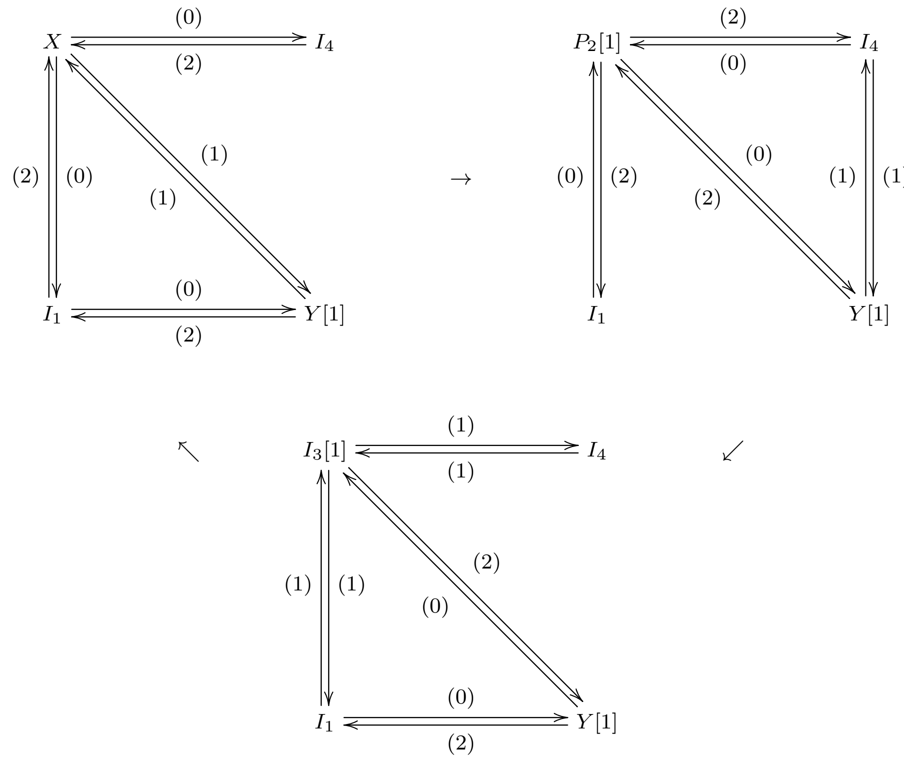


Figure 3. colored mutation at the upper left vertex

Theorem 6.5. [12] *Let $\mathcal{C} = \mathcal{C}_{kQ}$ for an acyclic quiver Q. Then the Gabriel quivers of all m-cluster tilted algebras are obtained by iterated colored mutation of Q.*

7. Other aspects and generalizations

In this survey, the main focus is on the combinatorial aspects of higher cluster categories. In this concluding section, we give some links to other aspects and generalizations, leaving out all the details.

7.1. **Calabi-Yau triangulated categories.** Consider a triangulated category $\mathcal{C}$ with split idempotents and with suspension functor Σ. Assume in addition that all Hom-spaces of $\mathcal{C}$ are finite dimensional over the algebraically closed field k, and that $\mathcal{C}$ admits a Serre functor ν, i.e.

there is a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X, \nu Y) \simeq D\,\mathrm{Hom}_{\mathcal{C}}(Y, X).$$

If, in addition, there is an isomorphism $\Sigma^{m+1} \simeq \nu$, then $\mathcal{C}$ is said to be Calabi-Yau of CY-dimension $m+1$ (for short, $m+1$-Calabi-Yau). Note that the m-cluster category satisfies all these properties with $\nu = \tau[1]$.

Rigid objects and tilting objects may now be defined exactly as in the case of m-cluster categories. In fact, one does not need to restrict to objects. In [28], a *(cluster) tilting subcategory* in an $m+1$-Calabi-Yau category is defined as a k-linear functorially finite subcategory $\mathcal{T}$ of $\mathcal{C}$, satisfying

- $\mathrm{Ext}^i(T, T') = 0$, for all T, T' in $\mathcal{T}$ and all $0 < i < m$, and
- if $X \in \mathcal{C}$ satisfies $\mathrm{Ext}^i(T, X) = 0$, for all T in $\mathcal{T}$ and all $0 < i < m$, then X belongs to $\mathcal{T}$.

Note that the additive closure $\mathrm{add}\,T$ of a tilting object T in an m-cluster category clearly satisfies this. Keller and Reiten [29] showed that one can characterize m-cluster categories as exactly those $m+1$-Calabi-Yau categories with an object T such that

- $\mathrm{add}\,T$ is a cluster tilting subcategory
- $\mathrm{Hom}(T, \Sigma^i T) = 0$, for $i = -m, \ldots, -1$, and
- $\mathrm{End}(T)$ is a hereditary algebra.

7.2. **Generalized higher cluster categories.** Amiot gave in [2] a more general definition of cluster categories in the case $m = 1$. Starting with a finite dimensional algebra A of global dimension at most 2, she constructs a certain triangulated category $\mathcal{C}_A$, which is equivalent to the ordinary cluster category in case A is hereditary. This category $\mathcal{C}_A$ is in general not Hom-finite. But, if A satisfies certain additional conditions, then $\mathcal{C}_A$ is Hom-finite, and in this case $\mathcal{C}_A$ is 2-Calabi-Yau and A is a tilting object in $\mathcal{C}_A$.

In a very recent paper, Lingyan Guo [17] generalized this construction to $m > 1$. More precisely, for finite dimensional algebra A of finite global dimension m, assume that the functor $\mathrm{Tor}_m^A(-, DA)$ is nilpotent. In this setting, she constructed a Hom-finite triangulated category $\mathcal{C}_A^{(m-1)}$, which was m-Calabi-Yau, and such that A was an $m-1$-cluster tilting object in $\mathcal{C}_A^{(m-1)}$.

In addition, both in [2] and [17], generalised (higher) cluster categories were also considered in the setting of quivers with (super-)potentials; see [14].

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