SPACELIKE HYPERSURFACES IN RIEMANNIAN OR LORENTZIAN SPACE FORMS SATISFYING $L_k x = Ax + b$

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ABSTRACT. We study connected orientable spacelike hypersurfaces $x : M^n \to \mathbb{M}_q^{n+1}(c)$, isometrically immersed into the Riemannian or Lorentzian space forms of curvature $c = -1, 0, 1$, and index $q = 0, 1$, satisfying the condition $L_k x = Ax + b$, where L_k is the *linearized operator* of the $(k+1)$th mean curvature H_{k+1} of the hypersurface for a fixed integer $0 \le k < n$, A is a constant matrix and b is a constant vector. We show that the only hypersurfaces satisfying that condition are hypersurfaces with zero H_{k+1} and constant H_k (when $c \ne 0$), open pieces of totally umbilic hypersurfaces and open pieces of the standard Riemannian product of two totally umbilic hypersurfaces.

1. **Introduction**

In 1966, Takahashi [11] determined the n-dimensional submanifolds isometrically immersed into the Euclidean space $\mathbb{R}^{n+m}$ whose position vector field was an eigenvector of the Laplace operator Δ with the same eigenvalue. Many people generalized this result in different directions (see [2–5,8,9]). As is well-known, the Laplace operator of a hypersurface $M^n \subset \mathbb{R}^{n+1}$ arises naturally as the linearized operator of the first variation of the mean curvature for normal variations of M^n. As such, Δ is

the first one of a sequence of n operators, $L_0 = \Delta$, $L_1 = \square$, ... , L_{n-1}, where L_k is the linearized operator of the first variation of the $(k+1)$th mean curvature arising from normal variations of the hypersurface. The operator $\square$ was introduced in [7]. Based on this background, Alias et al. ([3–5]) considered hypersurfaces in space forms whose position vector fields satisfies the general condition $L_k x = Ax + b$, where A and b are as specified in the abstract.

   In [4], hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$, satisfying the condition $L_k x = Ax + b$, are characterized, where $A \in \mathbb{R}^{(n+1)\times(n+1)}$ and $b \in \mathbb{R}^{n+1}$. This work is generalized in [5] to the hypersurfaces in the Riemannian space forms $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$ satisfying $L_k x = Ax + b$, for a self-adjoint matrix A and a vector b. Here, we extend the result of [4] and [5] to spacelike hypersurfaces in the Riemannian or Lorentzian space forms $\mathbb{R}_q^{n+1}$, $\mathbb{S}_q^{n+1}$ and $\mathbb{H}_q^{n+1}$, with $q = 0, 1$, whose position vector x satisfies the condition $L_k x = Ax + b$, A and b are as specified in the abstract. Our main results are theorems 1.1 - 1.5 below. We should emphasize that in [5], A is self-adjoint, but here, we have managed to omit this restricting condition. One of main ingredients of the proof of our result is Lemma 3.3. Its proof is based on the moving frame method, and is completely different from that of Lemma 4.1 in [5]. Here are our new results.

Theorem 1.1. *Let $M^n(n > 2)$ be a connected orientable spacelike hypersurface isometrically immersed into the (pseudo-)Euclidean space by the map $x : M^n \to \mathbb{R}_q^{n+1}$, where $q = 0$ or 1. Then, x satisfies the condition $L_k x = Ax + b$, for an integer $0 \le k < n$, a matrix $A \in M(n+1, \mathbb{R})$ and a vector $b \in \mathbb{R}^{n+1}$, if and only if M^n is an open piece of one of the following hypersurfaces:*
(i) A k-minimal hypersurface;
(ii) $\mathbb{S}^n(c)$, if $q = 0$ ($\mathbb{H}^n(-c)$, if $q = 1$), where $c > 0$;
(iii) $\mathbb{S}^m(c) \times \mathbb{R}^{n-m}$, if $q = 0$ ($\mathbb{H}^m(-c) \times \mathbb{R}^{n-m}$, if $q = 1$), where $c > 0$ and $k < m < n$.

Theorem 1.2. *Let $M^n(n > 2)$ be a connected orientable spacelike hypersurface isometrically immersed into the (pseudo-)sphere by the map $x : M^n \to \mathbb{S}_q^{n+1} \subset \mathbb{R}_q^{n+2}$, where $q = 0, 1$, and assume that H_k (i.e. the k-th mean curvature of M^n) is constant on M. Then, x satisfies the condition $L_k x = Ax + b$, for an integer $0 \le k < n$, $A \in M(n+2, \mathbb{R})$ and a non-zero vector $b \in \mathbb{R}^{n+2}$, if and only if M^n is an open piece of one of the following hypersurfaces:*

(i) $\mathbb{S}^n(r)$, for some $0 < r < 1$, if $q = 0$ $(r > 1$, if $q = 1)$;
(ii) $\mathbb{R}^n$ or $\mathbb{H}^n(-r)$, for some $r > 0$, if $q = 1$.

Theorem 1.3. *Let $M^n (n > 2)$ be a connected orientable spacelike hypersurface isometrically immersed into the (pseudo-)hyperbolic space by the map $x : M^n \to \mathbb{H}_q^{n+1} \subset \mathbb{R}_{q+1}^{n+2}$, where $q = 0$ or 1, and assume that H_k is constant on M. Then, x satisfies the condition $L_k x = Ax + b$, for k, A, b as in Theorem 1.2, if and only if M^n is an open piece of one of the following hypersurfaces:*
(i) $\mathbb{H}^n(-r)$, for some $r > 1$, if $q = 0$ $(0 < r < 1$ if $q = 1)$;
(ii) $\mathbb{R}^n$, or $\mathbb{S}^n(r)$, for some $r > 0$ if $q = 0$.

Theorem 1.4. *Let $M^n (n > 2)$ be a connected orientable spacelike hypersurface isometrically immersed into the (pseudo-)sphere by the map $x : M^n \to \mathbb{S}_q^{n+1} \subset \mathbb{R}_q^{n+2}$, where $q = 0$ or 1. Then, x satisfies the condition $L_k x = Ax$, for k, A as in Theorem 1.2, if and only if M^n is an open piece of one of the following hypersurfaces:*
(i) A hypersurface with zero H_{k+1} and constant H_k;
(ii) $\mathbb{S}^m(\sqrt{1 - r^2}) \times \mathbb{S}^{n-m}(r)$, where $0 < r < 1$ and $0 < m < n$, if $q = 0$;
(iii) $\mathbb{H}^m(-\sqrt{r^2 - 1}) \times \mathbb{S}^{n-m}(r)$, where $r > 1$ and $0 < m < n$, if $q = 1$.

Theorem 1.5. *Let $M^n (n > 2)$ be a connected orientable spacelike hypersurface isometrically immersed into the (pseudo-)hyperbolic space by $x : M^n \to \mathbb{H}_q^{n+1} \subset \mathbb{R}_{q+1}^{n+2}$, where $q = 0$ or 1. Then, x satisfies the condition $L_k x = Ax$, for k, A as in Theorem 1.2, if and only if M^n is an open piece of one of the following hypersurfaces:*
(i) A hypersurface with zero H_{k+1} and constant H_k;
(ii) $\mathbb{H}^m(-\sqrt{1 + r^2}) \times \mathbb{S}^{n-m}(r)$, where $r > 0$ and $0 < m < n$, if $q = 0$;
(iii) $\mathbb{H}^m(-\sqrt{1 - r^2}) \times \mathbb{H}^{n-m}(-r)$, where $0 < r < 1$ and $0 < m < n$, if $q = 1$.

2. **Preliminaries**

Here, we recall some basic preliminaries from [4,5] and [10]. By $\mathbb{R}_p^m$, we mean the vector space $\mathbb{R}^m$ with the scalar product,

$$< x, y >:= -\Sigma_{i=1}^p x_i y_i + \Sigma_{j>p} x_j y_j,$$

where $0 \le p < m$. Specially, $\mathbb{R}_0^m = \mathbb{R}^m$, and $\mathbb{R}_1^m$ is the Minkowski space. For $r > 0$ and $q = 0, 1$, $\mathbb{S}_q^{n+1}(r) = \{y \in \mathbb{R}_q^{n+2} | < y, y >= r^2\}$ denotes the (pseudo-)sphere or the de Sitter space of radius r and curvature

$1/r^2$, and $\mathbb{H}_q^{n+1}(-r) = \{y \in \mathbb{R}_{q+1}^{n+2} | <y,y> = -r^2\}$ denotes the (pseudo-)hyperbolic space or the anti-de Sitter space of radius r and curvature $-1/r^2$. The simply connected space form $\mathbb{M}_q^{n+1}(c)$ of curvature c and index q is $\mathbb{R}_q^{n+1}$, for $c = 0$, $\mathbb{S}_q^{n+1} = \mathbb{S}_q^{n+1}(1)$, for $c = 1$ (with induced metric from $\mathbb{R}_q^{n+2}$), and $\mathbb{H}_q^{n+1} = \mathbb{H}_q^{n+1}(-1)$, for $c = -1$ (with inherited metric from $\mathbb{R}_{q+1}^{n+2}$). When $q = 0$, we take a component of H_0^{n+1}. For an immersed hypersurface $x : M^n \to \mathbb{M}_q^{n+1}(c)$, the symbols ∇ and $\bar{\nabla}$ denote the Levi-Civita connections on M^n and $\mathbb{M}_q^{n+1}(c)$, respectively. Furthermore, ∇^0 denotes the Levi-Civita connection on $\mathbb{R}_q^{n+2}$ or $\mathbb{R}_{q+1}^{n+2}$. The Weingarten formula for a spacelike hypersurface $x : M^n \to \mathbb{M}_q^{n+1}(c)$ is $\bar{\nabla}_V W = \nabla_V W - \epsilon < SV, W > \mathbf{N}$, for $V, W \in \chi(M)$, where $\epsilon = 2q - 1$, $q \in \{0, 1\}$ and S is the shape operator of M associated to a unit normal vector field $\mathbf{N}$ on M with $< \mathbf{N}, \mathbf{N} > = -\epsilon$. Furthermore, in the case $|c| = 1$, $\mathbb{M}_q^{n+1}(c)$ is a hyperquadric, in $\mathbb{R}_q^{n+2}$ or $\mathbb{R}_{q+1}^{n+2}$, with the unit normal vector field x and the Gauss formula $\nabla_V^0 W = \bar{\nabla}_V W - c < V, W > x$. So, we have

$$(2.1) \ \nabla_V^0 W = \nabla_V W - \epsilon < SV, W > \mathbf{N} - c < V, W > x, \ \forall V, W \in \chi(M).$$

Since M is spacelike, S can be diagonalized. Denote its eigenvalues (the principal curvatures of M) by the functions $\kappa_1, ..., \kappa_n$ on M, define the elementary symmetric function as

$$s_j := \sum_{1 \le i_1 < ... < i_j \le n} \kappa_{i_1}...\kappa_{i_j},$$

and the jth mean curvature of M by $\binom{n}{j}H_j = (-\epsilon)^j s_j$, as (5.19) in [1]. The hypersurface M^n in $\mathbb{R}_p^{n+1}$ is called *j-minimal,* if its $(j+1)$th mean curvature H_{j+1} is identically zero.

In particular, $H_1 = -\epsilon(1/n)tr(S)$ and $\mathbf{H} = H_1\mathbf{N}$ are respectively the mean curvature and the mean curvature vector of M. In general, H_j is extrinsic (respectively, intrinsic), when j is an odd (respectively, even) number, since the sign of H_j depends on the chosen orientation only in the odd case.

For a spacelike hypersurface M in the space form $\mathbb{M}_q^{n+1}(c)$, we introduce, as (4) in [6], the Newton transformations $P_j : \chi(M) \to \chi(M)$, associated with the shape operator S of M, inductively by

$$P_0 = I, \ P_j = (-\epsilon)^j s_j I + \epsilon S \circ P_{j-1} \ , \ (j = 1, ..., n),$$

where I is the identity on $\chi(M)$. One can see that P_j has an explicit formula, $P_j = (-\epsilon)^j \Sigma_{l=0}^{j}(-1)^l s_{j-l} S^l = \sum_{l=0}^{j} \binom{n}{j-l} \epsilon^l H_{j-l} S^l$, where $H_0 = 1$ and $S^0 = I$. According to the characteristic polynomial of S, $Q_S(t) = det(tI - S) = \sum_{l=0}^{n} (-1)^{n-l} s_{n-l} t^l$, the Cayley-Hamilton theorem gives $P_n = 0$.

Using the explicit formula of P_j, one can see that it is self-adjoint and commutes with S. Therefore, $S(p)$ and $P_j(p)$ are simultanuously diagonalizable at each point $p \in M$. Let $e_1, ..., e_n$ be a local orthonormal tangent frame on M that diagonalizes S and P_j as $Se_i = \kappa_i e_i$ and $P_j e_i = \mu_{i,j} e_i$, for $i = 1, 2, ..., n$, where $\mu_{i,j} = (-\epsilon)^j \sum_{i_1 < ... < i_j, \ i_l \neq i} \kappa_{i_1} ... \kappa_{i_j}$, (for $j = 0, 1, ..., n-1$). Using this and the useful identity

$$(2.2) \qquad \epsilon \kappa_i \mu_{i,j} = \mu_{i,j+1} - (-\epsilon)^{j+1} s_{j+1} = \mu_{i,j+1} - \binom{n}{j+1} H_{j+1},$$

and the notation $c_j = (n-j)\binom{n}{j} = (j+1)\binom{n}{j+1}$, the following properties of P_k may be obtained easily:

$$(2.3) \qquad tr(P_j) = (-\epsilon)^j (n-j) s_j = c_j H_j,$$

$$(2.4) \qquad tr(S \circ P_j) = (-\epsilon)^j (j+1) s_{j+1} = -\epsilon c_j H_{j+1},$$

$$tr(S^2 \circ P_j) = \binom{n}{j+1}[nH_1 H_{j+1} - (n-j-1)H_{j+2}],$$

$$tr(P_j \circ \nabla_X S) = -\epsilon \binom{n}{j+1} < grad(H_{j+1}), X >, \ \forall X \in \chi(M).$$

By Corollary 34 in [10], page 115, ∇S is symmetric. Then, from the last equation, we obtain

$$(2.5) \qquad \sum_{i=1}^{n} (\nabla_{e_i} S)(P_{j-1} e_i) = -\epsilon \binom{n}{j} grad(H_j).$$

The *linearized operator* $L_j : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ of the $(j+1)$th mean curvature of M is defined by $L_j(f) := tr(P_j \circ \nabla^2 f)$, where $\nabla^2 f$ is given by $< \nabla^2 f(X), Y >= Hess(f)(X, Y)$.

For an spacelike hypersurface $x : M^n \to \mathbb{M}_q^{n+1}(c) \subset \mathbb{R}_t^{n+1+|c|}$, where $c \in \{-1, 0, 1\}$, $t := q + (1/2)(|c| - c)$ and $q \in \{0, 1\}$, with a (locally) unit normal vector field $\mathbf{N}$, from [1] and [4–6], we have $grad < x, \mathbf{a} >= \mathbf{a}^T$ and $grad < \mathbf{N}, \mathbf{a} >= -S\mathbf{a}^T$, for every $\mathbf{a} \in \mathbb{R}_t^{n+1+|c|}$, and also

$$\epsilon L_j \mathbf{N} = \binom{n}{j+1} grad(H_{j+1}) + \binom{n}{j+1}[nH_1 H_{j+1} - (n-j-1)H_{j+2}]\mathbf{N} - cc_j H_{j+1} x,$$

and

$$(2.6) \qquad L_j x = c_j H_{j+1} \mathbf{N} - cc_j H_j x, \ (for \ j = 0, ..., n-1).$$

Now, let the isometric immersion $x : M^n \to \mathbb{M}_q^{n+1}(c) \subset \mathbb{R}_t^{n+1+|c|}$ satisfy $L_k x = Ax + b$, for an integer $0 < k < n$, a matrix $A \in M(n+1+|c|, \mathbb{R})$ and a vector $b \in \mathbb{R}^{n+1+|c|}$. Then, for $i = 1, .., n$, we have

$$(2.7) \qquad Ae_i = -c_k(\kappa_i H_{k+1} + cH_k)e_i + c_k < grad(H_{k+1}), e_i > \mathbf{N}$$

$$-cc_k < grad(H_k), e_i > x,$$

$$(2.8) \quad H_{k+1}A\mathbf{N} = +\epsilon\binom{n}{k+1}H_{k+1}grad(H_{k+1}) - 2(P_k \circ S)(grad(H_{k+1}))$$

$$-2cP_k grad(H_k) - cH_k b^T - c[\epsilon c_k H_{k+1}^2 + cH_k < b, x > +L_k H_k]x$$

$$+\epsilon\{\binom{n}{k+1}H_{k+1}[nH_1 H_{k+1}-(n-k-1)H_{k+2}]+cH_k < b, \mathbf{N} > +\epsilon L_k H_{k+1}\}\mathbf{N},$$

$$(2.9) \quad (< AX, N > - < AN, X >)SY-(< AY, N > - < AN, Y >)SX$$

$$= \epsilon c[< c_k grad(H_k) - b^T, X > Y - < c_k grad(H_k) - b^T, Y > X].$$

To get formulae $(2.7) - (2.9)$, as in [5], one can get

$$AX = -c_k[H_{k+1}SX - < grad(H_{k+1}), X > \mathbf{N} + cH_k X + c < \nabla H_k, X > x],$$

which gives (2.7). One may obtain (2.8), similar to (18) in [5]. Finally, we have easily $< AX, Y >=< X, AY >$, for every $X, Y \in \chi(M)$, which, by covariant derivation, gives

$$< A\nabla_Z^0 X, Y > + < AX, \nabla_Z^0 Y >=< \nabla_Z^0 X, AY > + < X, A\nabla_Z^0 Y > .$$

This, By using (2.1), (2.6) and the symmetry of S, gives (2.9).

3. Main results

In order to prove theorems $1.1 - 1.5$, we state the following auxiliary lemma.

Lemma 3.1. *Let* $x : M^n \to \mathbb{M}_q^{n+1}(c)$ *(where* $n \geq 3$*) be a connected hypersurface satisfying* $L_k x = Ax + b$*, for* A, b *as in Theorem* 1.2 *and integers* $0 < k < n$*,* $c \in \{-1, 0, 1\}$*,* $q \in \{0, 1\}$*. Let* $\{e_1, ..., e_n\}$ *be the local orthonormal tangent frame of principal directions on* M*. Define* $U := \{p \in M | H_{k+1}(p) \neq 0\}$*,* $\omega_{ij}(e_l) :=< \nabla_{e_l} e_i, e_j >$ *and*

$$\Omega_{i,j} := \epsilon\kappa_j \left(2\frac{\mu_{i,k+1}}{H_{k+1}} - (k+4)\binom{n}{k+1}\right) < grad(H_{k+1}), e_i >$$

$$+ c(2\frac{\kappa_j\mu_{i,k}}{H_{k+1}} - \epsilon c_k) < grad(H_k), e_i > + c(\epsilon + \frac{\kappa_j H_k}{H_{k+1}}) < b, e_i > .$$

Then, on U*, for every* $i, j, l \in \{1, ..., n\}$*, where* $i \neq l$*, we have*

$(i) \quad < \kappa_i \ grad(H_{k+1}) + c \ grad(H_k), e_l > = \ -H_{k+1} < grad(\kappa_i), e_l >;$

(ii) $\omega_{il}(e_l)\ (\kappa_l\ -\ \kappa_i)\ H_{k+1}\ =\ <\ \kappa_l\ grad(H_{k+1})\ +\ c\ grad(H_k)\ ,\ e_i\ >;$

(iii) $\omega_{ij}(e_l)(\kappa_i - \kappa_j)H_{k+1} = 0$, when i, j and l are mutually different;

(iv) $\Omega_{i,l} = 0$;

(v) $\mu_{i,j} < grad(\kappa_i), e_i >= -\epsilon\binom{n}{j+1} < grad(H_{j+1}), e_i >$

$\quad -\ \epsilon\left(j\binom{n}{j+1}\frac{H_{j+1}}{H_{k+1}}\ +\ \frac{\mu_{i,j+1}}{H_{k+1}}\right) < grad(H_{k+1}), e_i >$

$\quad +\ \frac{c}{H_{k+1}}(\ c_jH_j\ -\ \mu_{i,j}) < grad(H_k), e_i >;$

(vi) If $< b, e_i >= 0$, then

$\quad H_{k+1} < grad(\kappa_i), e_i >= -\kappa_i\left(\frac{2k\ -\ 1}{k\ +\ 1}\ +\ \frac{2\mu_{i,k+1}}{c_kH_{k+1}}\right) < grad(H_{k+1}), e_i >$

$\quad -2c\left(\frac{k}{k\ +\ 1}\ +\ \frac{\mu_{i,k+1}}{c_kH_{k+1}}\right) < grad(H_k), e_i > .$

Proof. $(i)-(iii)$: Using the local orthonormal tangent frame field $\{e_1, \cdots, e_n\}$ of principal directions on M, we compute both sides of the equation

$$< \nabla^0_{e_l}(Ae_i), e_j >=< A\nabla^0_{e_l}e_i, e_j >$$

on U, for $i, l, j \in \{1, ..., n\}$. Assume that $i \neq l$ and recall that $\nabla^0_{e_l}x = e_l$, $\nabla^0_{e_l}\mathbf{N} = -\kappa_l e_l$, $\nabla^0_{e_l}e_i = \nabla_{e_l}e_i - \epsilon\kappa_l\delta_{li}\mathbf{N} - c\delta_{li}x = \Sigma^n_{j=1}\omega_{ij}(e_l)e_j$ to easily have $\omega_{ij}(e_l) = -\omega_{ji}(e_l)$.

Using (2.7), by a computation, we obtain

$(3.1)\quad < \nabla^0_{e_l}(Ae_i), e_j >= -c_k\delta_{lj} < \kappa_l\ grad(H_{k+1}) + c\ grad(H_k), e_i >$

$\quad -c_k\delta_{ij} < grad(\kappa_iH_{k+1} + cH_k), e_l > -c_k(\kappa_iH_{k+1} + cH_k)\omega_{ij}(e_l).$

On the other hand, using (2.7), we get

$(3.2)\quad\quad\quad < A\nabla^0_{e_l}e_i, e_j >= -c_k\omega_{ij}(e_l)(\kappa_jH_{k+1} + cH_k).$

When $j = i$, we have $\omega_{ij} = 0$ and since $i \neq l$, we have $j \neq l$ and $\delta_{lj} = 0$. By comparing equations (3.1) and (3.2), we get (i) when $j = i$, (ii) when $j = l$ and (iii) when i, j, l are mutually different.

(iv): Equation (2.9) for $X = e_i$, $Y = e_l$, when $i \neq l$, gives

$$(< Ae_i, \mathbf{N} > - < A\mathbf{N}, e_i >)\kappa_l - \epsilon c < c_k grad(H_k) - b^T, e_i >= 0,$$

which by (2.2), (2.7), (2.8) and $< \mathbf{N}, \mathbf{N} >= -\epsilon$ gives (iv).

(v): By applying (2.5) and $P_je_m = \mu_{m,j}e_m$, we get

$$\sum^n_{m=1} \mu_{m,j}(\nabla_{e_m}S)e_m\ =\ -\epsilon\binom{n}{j+1}grad(H_{j+1}).$$

But, for every i, m we have

$< (\nabla_{e_m}S)e_m, e_i >=< \nabla_{e_m}(Se_m) - S\nabla_{e_m}e_m, e_i >$

$$=< grad(\kappa_m), e_m > \delta_i^m + (\kappa_m - \kappa_i)\omega_{mi}(e_m),$$

from which we obtain

$$\mu_{i,j} < grad(\kappa_i), e_i >= -\epsilon\binom{n}{j+1} < grad(H_{j+1}), e_i >$$

$$- \sum_{l=1}^{n} \mu_{l,j}(\kappa_l - \kappa_i)\omega_{li}(e_l).$$

Then, by (i) and (iii), we get

$$(3.3) \qquad \mu_{i,j} < grad(\kappa_i), e_i >= -\epsilon\binom{n}{j+1} < grad(H_{j+1}), e_i >$$

$$+ \sum_{l=1,\ l\neq i}^{n} \frac{\mu_{l,j}}{H_{k+1}} < \kappa_l\ grad(H_{k+1}) + c\ grad(H_k), e_i > .$$

On the other hand, by (2.2) and (2.4), we have the identity

$$\sum_{l=1,\ l\neq i}^{n} \kappa_l\mu_{l,j} = -\epsilon\left(j\binom{n}{j+1}H_{j+1} + \mu_{i,j+1}\right).$$

Using this identity and (2.3), the result can be obtained from (3.3).

(vi): Similar to $(i) - (iii)$, by the identity $< A\nabla_{e_i}^0 e_i - \nabla_{e_i}^0(Ae_i), e_i >= 0$, (2.7), (2.8) and (2.2), we get

$$c_k H_{k+1} < grad(\kappa_i), e_i > + 2c\left(k\binom{n}{k+1} + \frac{\mu_{i,k+1}}{H_{k+1}}\right) < grad(H_k), e_i >$$

$$+ \kappa_i\left((2k-1)\binom{n}{k+1} + 2\frac{\mu_{i,k+1}}{H_{k+1}}\right) < grad(H_{k+1}), e_i >= 0,$$

which gives the result. $\square$

The next lemma is similar to Lemma 5 in [4].

Lemma 3.2. *Let* $x : M^n \to \mathbb{R}_q^{n+1}$, $q \in \{0, 1\}$, *be an orientable connected spacelike hypersurface in the Euclidean or Minkowski space satisfying* $L_k x = Ax + b$, k, A, b *as in Theorem* 1.1. *Then,* H_{k+1} *is constant on* M.

Proof. The case $k = 0$ is obtained by Proposition 3.2 in [3]. The proof of the case $k \geq 1$ is exactly similar to that of Lemma 5 in [4]. $\square$

The following key lemma, which is the essential ingredient of the proofs of theorems $1.2 - 1.5$, generalizes Lemma 4.1 of [5]. In the lemma, the range of x is $\mathbb{M}_q^{n+1}(c)$, for $q = 0, 1$ and A is an arbitrary matrix. Our proof is completely different from that of Lemma 4.1 of [5].

Lemma 3.3. *Let $n \geq 3$, $0 < k < n$ and $x : M^n \to \mathbb{M}_q^{n+1}(c) \subset \mathbb{R}_t^{n+2}$ be an orientable connected spacelike hypersurface satisfying $L_k x = Ax + b$, where $q \in \{0, 1\}$, $c \in \{-1, 1\}$, $t := q + (1/2)(|c| - c)$ and A, b as in Theorem 1.2. Then,*
(i) H_k is constant if and only if H_{k+1} is constant.
(ii) If $b = 0$, then H_k and H_{k+1} are constant.

Proof. (*i*) Assume that H_k is constant on M. It is enough to show that H_{k+1} is constant on the open subset $V = \{p \in M | H_{k+1}(p) \neq 0\}$. Using the local orthonormal tangent frame field $\{e_1, ..., e_n\}$ of principal directions on M, we show that

$$V_i = \{p \in V| < grad(H_{k+1}), e_i >_p \neq 0\} = \emptyset, \ (for \ i = 1, 2, ..., n).$$

For each i, we take $J_i := \{1, ..., n\} - \{i\}$ and write V_i as

$$V_i = W_{i,1} \dot{\cup} W_{i,2} \dot{\cup} W_{i,3},$$

where $\dot{\cup}$ is the disjoint union and

$$(3.4) \qquad\qquad W_{i,1} = \{p \in V_i | \exists l \in J_i, \kappa_l(p) = \kappa_i(p)\},$$

$$(3.5) \ \ W_{i,2} = \{p \in V_i | \forall j \in J_i, (\kappa_j(p) \neq \kappa_i(p) \ \wedge \ \exists l \in J_i, \kappa_j(p) \neq \kappa_l(p))\},$$

$$(3.6) \qquad\qquad W_{i,3} = V_i - W_{i,1} \dot{\cup} W_{i,2}.$$

For any i, $1 \leq i \leq n$, in three steps we prove that $W_{i,1} = W_{i,2} = W_{i,3} = \emptyset$.

   Step 1 ($W_{i,1} = \emptyset$): If $W_{i,1} \neq \emptyset$, at each point $p \in W_{i,1}$, there exists an $l \in J_i$ such that $\kappa_l(p) = \kappa_i(p)$, and by Lemma 3.1 (*ii*), $\kappa_l(p) = \kappa_i(p) = 0$. Then, by Lemma 3.1 (*iv*), $< b, e_i >_p = 0$. Since $H_{k+1} \neq 0$, one may choose $j \in J_i - \{l\}$ such that $\kappa_j(p) \neq 0$. By Lemma 3.1 (*iv*), we get

$$\kappa_j(p) \left((k+4)\binom{n}{k+1} \ - \ 2\frac{\mu_{i,k+1}(p)}{H_{k+1}(p)} \right) < \nabla H_{k+1}, e_i >_p = 0.$$

So, on $W_{i,1}$, $\mu_{i,k+1} = (2 + \frac{k}{2})\binom{n}{k+1} H_{k+1}$. On the other hand, as $\kappa_i = 0$, by (2.2), $\mu_{i,k+1} = \binom{n}{k+1} H_{k+1}$. Hence, we get $H_{k+1} = 0$, which is a contradiction. Hence, $W_{i,1} = \emptyset$.
   Step 2 ($W_{i,2} = \emptyset$): If $W_{i,2} \neq \emptyset$, at each point $p \in W_{i,2}$, for every $j \in J_i$, there exists $l \in J_i$ such that $\kappa_l(p)$, $\kappa_j(p)$ and $\kappa_i(p)$ are mutually distinct. By Lemma 3.1 (*iv*), $\Omega_{i,j} = \Omega_{i,l} = 0$, which gives $< b, e_i >_p = 0$,

and $\mu_{i,k+1}(p) = (2+\frac{k}{2})\binom{n}{k+1}H_{k+1}(p)$. Then, by Lemma 3.1 (v), for $j = k$, we get

(3.7) $\mu_{i,k}\theta_i = -3\epsilon(1+\frac{k}{2})\binom{n}{k+1} \neq 0$ $(where,\ \theta_i = \frac{< grad(\kappa_i), e_i >}{< grad(H_{k+1}), e_i >})$.

These equations and (2.2) give that

(3.8) $$\theta_i = -3\frac{\kappa_i}{H_{k+1}}.$$

On the other hand, by Lemma 3.1(v), for $j = k - 1$, we get

$$\mu_{i,k-1}\theta_i = -\frac{\epsilon}{H_{k+1}}((k-1)\binom{n}{k}H_k + \mu_{i,k}),$$

from which, by (2.2), (3.7) and (3.8), we get

(3.9) $$H_k\kappa_i = \frac{(n-k)\epsilon}{k+1}H_{k+1},$$

which, by covariant derivation on the open set $W_{i,2}$, gives

(3.10) $$< grad(H_{k+1}), e_i > = \frac{k+1}{(n-k)\epsilon}H_k < grad(\kappa_i), e_i > .$$

Then, $\theta_i = \frac{(n-k)\epsilon}{(k+1)H_k}$, which by (3.8) and (3.9), we get $H_{k+1} = 0$. This is a contradiction, and hence $W_{i,2} = \emptyset$. So, $V_i = W_{i,3}$.

   Step 3 $(W_{i,3} = \emptyset)$: If $W_{i,3} \neq \emptyset$, on which, we have for all $l, j \in J_i$, $\kappa_l = \kappa_j$ and $\kappa_j \neq \kappa_i$, then, for each m,

(3.11) $$\mu_{i,m} = (-\epsilon)^m \binom{n-1}{m}\kappa_j^m.$$

Lemma 3.1 (v), for $j = k$, jointly with (2.2) and (3.11), give

(3.12) $$H_{k+1}\theta_i = \frac{(k+2)(n-k-1)}{k+1}\kappa_j + (k+1)\kappa_i.$$

Similarly, from Lemma 3.1 (v), for $j = k-1$, (2.2) and (3.11), we obtain $H_{k+1}\theta_i = (n-k)\kappa_j + (k-1)\kappa_i$, which, comparing with (3.12), gives

(3.13) $$\kappa_i = (1 - \frac{n}{2k+2})\kappa_j.$$

Using (2.2), (3.11) and (3.13), we get $H_k = \frac{1}{2}(-\epsilon\kappa_j)^k, H_{k+1} = \frac{1}{2}(-\epsilon\kappa_j)^{k+1}$. These equations imply that κ_j and H_{k+1} are constant on the open set $V_i = W_{i,3}$. This is a contradiction, and hence $V_i = W_{i,3} = \emptyset$.

   Therefore, H_{k+1} is constant on V, and by continuity of H_{k+1} on M, it is constant on M.

Conversely , assume that H_{k+1} is constant on M. In order to prove that $grad(H_k) = 0$ on $V = \{p \in M | H_k(p) \neq 0\}$, we show that

$$V_i = \{p \in V | < grad(H_k), e_i >_p \neq 0\} = \emptyset \ (for, i = 1, 2, ..., n).$$

If $H_{k+1} = 0$, then by Lemma 3.1 (ii), $grad(H_k) = 0$ on V, and the proof is complete. Assume that $H_{k+1} \neq 0$. For any i, $1 \leq i \leq n$, we take $V_i = W_{i,1} \dot{\cup} W_{i,2} \dot{\cup} W_{i,3}$, where the $W_{i,j}$ are as in $(3.4) - (3.6)$. The claim is that $W_{i,1} = W_{i,2} = W_{i,3} = \emptyset$. For any i, $1 \leq i \leq n$, we prove the claim in the following steps.

Step 1 $(W_{i,1} = \emptyset)$: If $W_{i,1} \neq \emptyset$, at each point $p \in W_{i,1}$ there exists an $l \in J_i$ such that $\kappa_l(p) = \kappa_i(p)$. So, by Lemma 3.1 (ii), we obtain $< grad(H_k), e_i >= 0$ on $W_{i,1}$, which is a contradiction. Hence, $W_{i,1} = \emptyset$.

Step 2 $(W_{i,2} = \emptyset)$: If $W_{i,2} \neq \emptyset$, at each $p \in W_{i,2}$, by definition, for every $j \in J_i$, there exists an $l \in J_i$ such that $\kappa_l(p)$, $\kappa_j(p)$ and $\kappa_i(p)$ are mutually distinct. By Lemma 3.1 (iv), $\Omega_{i,j} = \Omega_{i,l} = 0$, which gives $c_k < grad(H_k), e_i >_p = < b, e_i >_p$, and

$$(3.14) \qquad\qquad \mu_{i,k}(p) = -\frac{1}{2}c_k H_k(p).$$

Now, By Lemma 3.1 (v), for $j = k$, we have

$$(3.15) \ \ \mu_{i,k}(\psi_i + \frac{c}{H_{k+1}}) = cc_k \frac{H_k}{H_{k+1}}, \ \ (where \ \psi_i = \frac{< grad(\kappa_i), e_i >}{< grad(H_k), e_i >}).$$

From these equations, we get $\psi_i = -3\frac{c}{H_{k+1}} \neq 0$ on $W_{i,2}$. Also, by Lemma 3.1 (v), when $j = k - 1$, we have

$$(3.16) \qquad\qquad \mu_{i,k-1}(\psi_i + \frac{c}{H_{k+1}}) = \binom{n}{k}(-\epsilon + ck\frac{H_{k-1}}{H_{k+1}}),$$

which, using (2.2) and (3.15), we get

$$(3.17) \qquad H_k(\psi_i - (n - k - 1)\frac{c}{H_{k+1}}) = \kappa_i(1 - c\epsilon k\frac{H_{k-1}}{H_{k+1}}).$$

Similarly, one may obtain that for every $j \in J_i$,

$$c_k < grad(H_k), e_j >= < b, e_j >,$$

and $(\mu_{j,k} + \frac{1}{2}c_k H_k(p)) < grad(H_k), e_j >= 0$.

If $< b, e_j >_p \neq 0$, for some $j \in J_i$ and $p \in W_{i,2}$, then $\psi_i(p) = \psi_j(p) = -3\frac{c}{H_{k+1}}$. Also, with the assumption $< b, e_j >_p \neq 0$, as (3.17), one can

obtain

(3.18) $\qquad H_k(\psi_j - (n-k-1)\dfrac{c}{H_{k+1}}) = \kappa_j(1 - c\epsilon k\dfrac{H_{k-1}}{H_{k+1}}).$

Then, by (3.17) and (3.18), we get $(\kappa_i - \kappa_j)(1 - c\epsilon k\frac{H_{k-1}}{H_{k+1}}) = 0$, and hence $H_{k+1} = c\epsilon k H_{k-1}$. As a result, by (3.17), $H_k(p) = 0$, which is a contradiction. So, $c_k < grad(H_k), e_j >=< b, e_j >= 0$ on $W_{i,2}$, for all $j \in J_i$.

By (2.7), $Ae_j = -c_k(\kappa_j H_{k+1} + cH_k)e_j$ and $\kappa_j H_{k+1} + cH_k$ is constant on the open set $W_{i,2}$, for all $j \in J_i$, and so is $d_{lj} = \kappa_l - \kappa_j$, for every pair $j, l \in J_i$. Then, for a fixed $j \in J_i$ and for every $l \in J_i$ we have $\kappa_l = d_{lj} + \kappa_j$. So, by (3.14), H_k has a polynomial expression in terms of κ_j, and consequently $\kappa_j H_{k+1} + cH_k$ is a polynomial in κ_j and constant on $W_{i,2}$. Therefore, κ_j and H_k are constant on $W_{i,2}$. This is a contradiction, and hence $W_{i,2} = \emptyset$.

Step 3 $(W_{i,3} = \emptyset)$: Assume that $V_i = W_{i,3} \neq \emptyset$. On $W_{i,3}$, by definition, we have for all $l, j \in J_i$, $\kappa_l = \kappa_j \neq \kappa_i$, and, $\mu_{l,m} = \mu_{j,m}$, for $m = 0, 1, ..., n-1$. We also get the formulas (3.11), (3.15) and (3.16). From (2.2), (3.15) and (3.11), we obtain

(3.19) $\qquad H_{k+1}\psi_i = c(n-k-1) + ck\dfrac{\kappa_i}{\kappa_j}.$

Similarly, from (2.2), (3.16) and ((3.11), the following formula holds:

(3.20) $H_{k+1}\psi_i = \dfrac{-\epsilon}{k}[(n-k-1)\kappa_j^2 + (k+1)\kappa_i\kappa_j] + c(n-k) + c(k-1)\dfrac{\kappa_i}{\kappa_j}.$

Comparing (3.19) and (3.20), we get $\kappa_i = \frac{ck\kappa_j - \epsilon(n-k-1)\kappa_j^3}{ck + \epsilon(k+1)\kappa_j^2}$. By (2.2), one may express H_k and H_{k+1} in terms of κ_j. As H_{k+1} is a constant, κ_j satisfies a polynomial equation. So, κ_j and H_k are constant on the open set $V_i = W_{i,3}$. This is a contradiction. Hence, $V_i = W_{i,3} = \emptyset$.

Therefore, H_k is constant on V, and by continuity, it is constant on M.

(ii)- Assume that $b = 0$. By (i), it is enough to prove that H_{k+1} is constant on M. We take V and V_i as in (i), and show that $V_i = \emptyset$, for $i = 1, 2, ..., n$. Taking $V_i = W_{i,1} \dot\cup W_{i,2} \dot\cup W_{i,3}$, where the $W_{i,j}$ $(1 \le j \le 3)$ are as in $(3.4)-(3.6)$, we claim that $W_{i,1} = W_{i,2} = W_{i,3} = \emptyset$, and prove it in the following steps. *Step 1* $(W_{i,1} = \emptyset)$: If $W_{i,1} \neq \emptyset$, then at each point $p \in W_{i,1}$, there exists an $l \in J_i$ such that $\kappa_l(p) = \kappa_i(p)$. By Lemma 3.1 (ii), we get $\kappa_l(p) < grad(H_{k+1}), e_i >_p = -c < grad(H_k), e_i >_p$, which by Lemma 3.1 (iv) and (2.2), we have $< grad(H_k), e_i >_p = 0$.

Then, $\kappa_i(p) = \kappa_l(p) = 0$. So, applying Lemma 3.1 (iv), for every $j \in J_i$, we have

$$\kappa_j((k+4)\binom{n}{k+1}) - 2\frac{\mu_{i,k+1}}{H_{k+1}}) < \nabla H_{k+1}, e_i >= 0,$$

and hence $\mu_{i,k+1} = (2 + \frac{k}{2})\binom{n}{k+1}H_{k+1}$ on $W_{i,1}$. Also, since $\kappa_i = 0$, by (2.2), we get $\mu_{i,k+1} = \binom{n}{k+1}H_{k+1}$. As a result, we get a contradiction as $(1 + \frac{k}{2})\binom{n}{k+1}H_{k+1} = 0$. So, $W_{i,1} = \emptyset$.

   Step 2 $(W_{i,2} = \emptyset)$: If $W_{i,2} \neq \emptyset$, at each point $p \in W_{i,2}$, for every $j \in J_i$, there exists an $l \in J_i$, such that $\kappa_l(p)$, $\kappa_j(p)$ and $\kappa_i(p)$ are mutually distinct. Then, by Lemma 3.1 (iv), we have $\Omega_{i,j} = \Omega_{i,l} = 0$, which gives $< grad(H_k), e_i >_p = 0$. Now, exactly as in (3.7)-(3.10), we obtain a contradiction. Hence, $W_{i,2} = \emptyset$.

   Step 3 $(W_{i,3} = \emptyset)$: If $W_{i,3} \neq \emptyset$, on which we have for all $l, j \in J_i$, $\kappa_l = \kappa_j$ and $\kappa_j \neq \kappa_i$. So, for each m, we have (3.11). Taking $\varphi_i := \frac{<grad(H_k), e_i>}{<grad(H_{k+1}), e_i>}$ and $\vartheta_i := \frac{<grad(\kappa_i), e_i>}{<grad(H_{k+1}), e_i>}$, by Lemma 3.1 (iv) and (3.11), we get, for every $l \in J_i$,

(3.21) $\kappa_l[(\frac{n}{k+1} - 1)(k+2)\kappa_l + (k+4)\kappa_i] = -c((n-k+1)\kappa_l + (k+1)\kappa_i)\varphi_i.$

From Lemma 3.1 (v), for $j = k$, we get for every $l \in J_i$,

(3.22) $H_{k+1}\vartheta_i = (\frac{n}{k+1} - 1)(k+2)\kappa_l + (k+1)\kappa_i + c(n-k-1+k\frac{\kappa_i}{\kappa_l})\varphi_i,$

and for $j = k - 1$, we get
$H_{k+1}\vartheta_i = (n-k)\kappa_l + (k-1)\kappa_i$

(3.23) $\quad - [\epsilon\frac{n-k-1}{k}\kappa_l^2 + \epsilon\frac{k+1}{k}\kappa_i\kappa_l - c(n-k) - c(k-1)\frac{\kappa_i}{\kappa_l}]\varphi_i,$

which, comparing with (3.22), gives
(3.24)
$$(2 - \frac{n}{k+1})\kappa_l - 2\kappa_i = [c(\frac{\kappa_i}{\kappa_l} - 1) + \epsilon\frac{(n-k-1)\kappa_l + (k+1)\kappa_i}{k}\kappa_l]\varphi_i.$$

Now, by (3.21), $n - k + 1 + (k+1)\frac{\kappa_i}{\kappa_l} \neq 0$. So, we have

(3.25) $\qquad\qquad \varphi_i = \frac{(k+2)(\frac{n}{k+1} - 1)\kappa_l + (k+4)\kappa_i}{-c(n-k+1+(k+1)\frac{\kappa_i}{\kappa_l})}.$

Using (3.25), from (3.24), we get

$\frac{\epsilon}{k}(n-k-1)^2(k+2)\kappa_l^3 - c((n-k)^2+n-3k-4)\kappa_l = c(2nk+n-2k^2+4k+6)\kappa_i$
(3.26)

$+c(k+1)(k-2)\frac{\kappa_i^2}{\kappa_l} - \frac{2\epsilon}{k}(n-k-1)(k+1)(k+3)\kappa_i\kappa_l^2 - \frac{\epsilon}{k}(k+1)^2(k+4)\kappa_i^2\kappa_l,$

from which, by covariant derivation in direction of e_i, we obtain

$[\frac{3\epsilon}{k}(n-k-1)^2(k+2)\kappa_l^2 - c((n-k)^2+n-3k-4) + c(k+1)(k-2)\frac{\kappa_i^2}{\kappa_l^2}$
$+ \frac{4\epsilon}{k}(n-k-1)(k+1)(k+3)\kappa_i\kappa_l + \frac{\epsilon}{k}(k+1)^2(k+4)\kappa_i^2] < grad(\kappa_l), e_i >=$
$\{c(2nk+n-2k^2+4k+6) + 2c(k+1)(k-2)\frac{\kappa_i}{\kappa_l} - \frac{2\epsilon}{k}(k+1)^2(k+4)\kappa_i\kappa_l$

(3.27) $- \frac{2\epsilon}{k}(n-k-1)(k+1)(k+3)\kappa_l^2\} < grad(\kappa_i), e_i > .$

Now, by Lemma 3.1 (i) we have $\kappa_l + c\varphi_i = -\frac{<grad(\kappa_l),e_i>}{<grad(\kappa_i),e_i>}H_{k+1}\vartheta_i$, which using (3.22) and (3.25) and by multiplying by $ck(k+1)\frac{1}{\kappa_l}$, we get

$\epsilon(n-k-1)^2(k+2)(4k^2+7k+3n+3)\kappa_l^2 \ - \epsilon(k+1)^2(8nk^2+15nk-38n-12k^3-45k^2$
$-3k+30)\kappa_i^2 \ - ck(n^3+4n^2k^2+5n^2k-8nk^3-7nk^2+14nk+2n^2+13n+4k^4+k^3-22k^2$
$-31k-12) \ + ck(4n^2k^2+n^2k-16nk^3-10nk^2+22nk-2n^2+16n+12k^4+3k^3-52k^2-65k-22)\frac{\kappa_i}{\kappa_l} \ + ck(k+1)(8nk^2-5nk-7n-12k^3+9k^2+33k+12)\frac{\kappa_i^2}{\kappa_l^2}$
$-\epsilon(k+1)(n-k-1)(4nk^2+nk-24n-12k^3-45k^2-29k+4)\kappa_i\kappa_l$

(3.28) $+ ck(k+1)^2(k-2)(4k+1)\frac{\kappa_i^3}{\kappa_l^3} \ -\epsilon(k+1)^3(k+4)(4k-5)\frac{\kappa_i^3}{\kappa_l} = 0.$

Taking $\bar{x} := \kappa_l^2$ and $\bar{y} := \frac{\kappa_i}{\kappa_l}$, from (3.26) by dividing by κ_l, we get

$\bar{x} = \epsilon ck\frac{(k+1)(k-2)\bar{y}^2+(2nk+n-2k^2+4k+6)\bar{y}+(n-k)^2+n-3k-4}{(k+1)^2(k+4)\bar{y}^2+2(k+1)(n-k-1)(k+3)\bar{y}+(n-k-1)^2(k+2)}.$

Clearly, one can see that the denominator of the last fraction does not have any real root for integers $n > k > 0$. So, from (3.28) we get a polynomial of degree five in terms of $\bar{y}$, as $c_0 + c_1\bar{y} + ... + c_5\bar{y}^5 = 0$, where,

$c_0 = 2ck(k+2)(n-k-1)^2(3k+11n+6k^2+11nk-2n^2+2n^2k+3k^3-n^3),$

$c_1 = ck(n-k-1)(-96-228k+260n-78k^2+550nk-80n^2-30n^3k+24n^2k^2-6nk^3-68n^2k+302nk^2+150k^4+174k^3-20n^3+12n^2k^3-$

$18nk^4 + 30k^5 - 8n^3k^2)$

$c_2 = ck(k+1)(-336 - 888k + 676n + 1274nk - 288n^2 - 50n^3k + 78n^2k^2 - 222nk^3 - 240n^2k + 448nk^2 + 300k^4 + 204k^3 - 20n^3 + 30n^2k^3 - 72nk^4 + 60k^5 - 10n^3k^2)$,

$c_3 = ck(k+1)^2(432 + 744k - 460n + 132k^2 - 386nk + 44n^2 - 38n^2k + 48nk^3 + 122nk^2 - 60k^4 - 240k^3 - 10n^2k^2)$,

$c_4 = 6ck(k+1)^3(k+4)(k-2)(5k+5-2n)$

$c_5 = -6ck(k+1)^4(k-2)(k+4)$.

If the last polynomial equation has no real root, then we have a contradiction. Otherwise it has at most five distinct real roots, which implies that there exists an open subset of $W_{i,3} = V_i$ on which, $\bar{y}$, $\bar{x}$, κ_l, κ_i and H_{k+1} are constant, which is a contradiction. Therefore, we get $V_i = W_{i,3} = \emptyset$. So, H_{k+1} is constant on V, and hence on M. $\qquad\square$

4. **Proof of the theorems**

Proof. **(1.1)**. For the case $q = 0$, see Theorem 1 in [4]. Now, we consider the case $q = 1$. By examples 5.1 and 5.2, the hypersurfaces of type $(i) - (iii)$ do satisfy the formula $L_k x = Ax + b$. Conversely, By Lemma 3.2, H_{k+1} is constant on M. If $H_{k+1} = 0$, then M is k-minimal. Assume that $H_{k+1} \neq 0$. As the proof of Theorem 1 in [4], one may show that S satisfies the equation $c_k H_{k+1} S^2 - \beta S = 0$, where $\beta = \frac{(n-k-1)H_{k+2}}{(k+1)H_{k+1}} - \frac{nH_1}{k+1}$ is constant on M, i.e., M is isoparametric. So, for each i, we have $\kappa_i^2 + (\beta/c_k H_{k+1})\kappa_i = 0$. Since M is connected, we may assume that there exists an $m \geq 1$ such that $\kappa_i = -\beta/(c_k H_{k+1})$, for $i = 1, ..., m$, and $\kappa_i = 0$, for $i = m+1, ..., n$. If $m = n$, then $M = \mathbb{H}^n(-c)$, which gives (ii). If $m < n$, as in [12], TM decomposes as $TM = T_1 \oplus T_2$, where $T_1 := span\{e_1, ..., e_m\}$ and $T_2 := span\{e_{m+1}, ..., e_n\}$. By Lemma 3.1 (iii), if $\kappa_i \neq \kappa_j$, then $\omega_{ij} = 0$. So, for every $1 \leq i \leq m$ and $m+1 \leq j \leq n$, we have $\omega_{ij} = 0$, and T_1 and T_2 are integrable. Therefore, M decomposes as $M = M_1 \times M_2$, where M_1 and M_2 are the integral manifolds of T_1 and T_2, respectively. Hence, M_1 is an open piece of $\mathbb{H}^m(-c)$ and M_2 is an open piece of $\mathbb{R}^{n-m}$. Therefore, M is an open piece of $\mathbb{H}^m(-c) \times \mathbb{R}^{n-m}$. $\qquad\square$

Proof. **(1.2 and 1.3)**. By examples 5.2 - 5.5, and examples 3.2 and 3.3 in [5], each of the hypersurfaces mentioned in Theorems 1.2 and 1.3 satisfies the formula $L_k x = Ax + b$. Conversely, since H_k is constant, by assumption, so is H_{k+1} by Lemma 3.3 (i). If $H_{k+1} = 0$, by Example 5.1, we get $b = 0$, which is a contradiction. So, $H_{k+1} \neq 0$. In this case, similar

to the proof of Theorem 1.7 in [5], we obtain that the shape operator S satisfies the equation $c_k H_{k+1} S^2 + (\alpha + 2cc_k H_k) S + cc_k H_{k+1} I = 0$, where α is constant. So, M is an isoparametric hypersurface in $\mathbb{S}_q^{n+1}$ or $\mathbb{H}_q^{n+1}$. Then, by theorems 1 and 2 in [12], M is an open subset of one of the hypersurfaces mentioned in theorems 1.2 and 1.3. $\qquad\square$

Proof **(1.4 and 1.5)**. By examples 5.1, 5.6 and 5.7, and examples 3.1 and 3.4 in [5], each of the hypersurfaces mentioned in theorems 1.4 and 1.5 satisfies the formula $L_k x = Ax$. Conversely, by Lemma 3.3 (ii), H_k and H_{k+1} are constant on M. If $H_{k+1} = 0$, then there is nothing to prove. Assume that $H_{k+1} \neq 0$. Similar to the proof of Theorem 1.2 in [5], the shape operator S satisfies the equation $c_k H_{k+1} S^2 + (\alpha + 2cc_k H_k) S + cc_k H_{k+1} I = 0$, where α is constant, and hence M is an isoparametric hypersurface in $\mathbb{S}_q^{n+1}$ or $\mathbb{H}_q^{n+1}$, and by theorems 1 and 2 in [12], M is an open subset of one of the hypersurfaces mentioned in the theorems. $\qquad\square$

5. **Examples**

Following [4, 5] and [12], we give the following examples.

Example 5.1. Consider an spacelike connected orientable hypersurface $x : M^n \to \mathbb{M}_q^{n+1}(c)$, where $q \in \{0, 1\}$ and $c \in \{-1, 0, 1\}$. Assume that H_k is constant (when $c \neq 0$), and $H_{k+1} \equiv 0$. By (2.6), it satisfies $L_k x = Ax + b$, with $A = -cc_k H_k I_{k+2} \in \mathbb{R}^{(n+2)\times(n+2)}$ and $b = 0 \in \mathbb{R}^{n+2}$.

Example 5.2. Let M be $\mathbb{H}^m(-r) \times \mathbb{R}^{n-m} \subset \mathbb{R}_1^{n+1}$, with $r > 0$ and $1 \leq m \leq n$. In fact, $M = \{y \in \mathbb{R}_1^{n+1}| - y_1^2 + ... + y_{m+1}^2 = -r^2\}$. With the Gauss map $N(y) = \frac{-1}{r}(-y_1, y_2, ..., y_{m+1}, 0, ..., 0)$ on M, we get its principal curvatures $\kappa_1 = ... = \kappa_m = \frac{1}{r}$, $\kappa_{m+1} = ... = \kappa_n = 0$. Then, $\binom{n}{k+1} H_{k+1} = (-1)^{k+1} \binom{m}{k+1}(\frac{1}{r})^{k+1}$, for $k < m$, and $H_{k+1} = 0$, otherwise. By (2.6), $L_k x = Ax$, with $A = (-1)^k \binom{m}{k+1}(k+1)(\frac{1}{r})^{k+2} diag[-1, I_m, 0]$. When $m = n$, $M = \mathbb{H}^n(-r)$ is totally umbilic in $\mathbb{R}_1^{n+1}$.

Example 5.3. Take a unit vector $\mathbf{a} \in \mathbb{R}_1^{n+2}$ and $\sigma =< \mathbf{a}, \mathbf{a} >$. For each $r > \sqrt{|\sigma|}$, $M_r := \{y \in \mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2}| < v, \mathbf{a} >= \sqrt{r^2 + \sigma}\}$ is a totally umbilic hypersurface in $\mathbb{S}_1^{n+1}$. Similar to Example 3.2 in [5], the Gauss map is $\mathbf{N}(x) = \frac{1}{r}(\mathbf{a} - \sqrt{r^2 + \sigma}x)$, and so for all i, $\kappa_i = \frac{1}{r}\sqrt{r^2 + \sigma}$, and for each k, $H_k = (-1)^k[\frac{1}{r}\sqrt{r^2 + \sigma}]^k$. By (2.6), M_r satisfies $L_k x = Ax + b$, with $A = c_k(-1)^k \sigma \frac{\sqrt{r^2+\sigma}^k}{r^{k+2}} I_{n+2}$ and $b = c_k(-1)^{k+1} \frac{\sqrt{r^2+\sigma}^{k+1}}{r^{k+2}}\mathbf{a}$. When $\sigma = -1$ and $r \geq 1$, $M_r = \mathbb{S}^n(r)$. When $\sigma = 1$ and $r > 0$, $M_r = \mathbb{H}^n(-r)$.

Example 5.4. In this example, we follow [12], page 132. Take $\varepsilon = (-1, 1, 0, ..., 0) \in \mathbb{R}_1^{n+2}$. Define the function $g : \mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2} \to \mathbb{R}$ by $g(x) = -x_1 + x_2$, and take $M_t := g^{-1}(e^{-t})$, for each $t \in \mathbb{R}$. In fact, $M_t = \{(f(y) + \sinh t, f(y) + \cosh t, y) \in \mathbb{S}_1^{n+1} | y \in \mathbb{R}^n\}$, where $f(y) = \frac{-e^t}{2} \Sigma_{i=1}^n y_i^2$. With respect to the Gauss map $\mathbf{N}(x) = e^t \varepsilon - x$ on M_t, one may obtain $\kappa_1 = ... = \kappa_n = 1$, and so $H_k = (-1)^k$. Therefore, by (2.6), M_t satisfies $L_k x = Ax + b$, for $k = 0, 1, ..., n-1$, with $A = 0$ and $b = c_k(-1)^{k+1} e^t \varepsilon$.

Example 5.5. Take a timelike unit vector $\mathbf{a} \in \mathbb{R}_2^{n+2}$ and $f : \mathbb{H}_1^{n+1} \subset \mathbb{R}_2^{n+2} \to \mathbb{R}$, given by $f(v) = <v, \mathbf{a}>$. For $0 < r \le 1$, $M_r := f^{-1}(-\sqrt{1 - r^2})$ $= \mathbb{H}^n(-r)$ is a totally umbilic hypersurface in $\mathbb{H}_1^{n+1}$. Similar to Example 3.3 in [5], with the Gauss map $\mathbf{N}(x) = \frac{1}{r}(\mathbf{a} - \sqrt{1 - r^2}x)$, one may see that for all i, $\kappa_i = \frac{\sqrt{1-r^2}}{r}$, and for each k, $H_k = (-1)^k(\frac{\sqrt{1-r^2}}{r})^k$. By (2.6), M_r satisfies $L_k x = Ax + b$, with $A = c_k(-1)^k \sigma \frac{\sqrt{1-r^2}^k}{r^{k+2}} I_{n+2}$ and $b = c_k(-1)^{k+1} \frac{\sqrt{1-r^2}^{k+1}}{r^{k+2}} \mathbf{a}$.

Example 5.6. Let M be the standard product

$$H^m(-\sqrt{r^2 - 1}) \times S^{n-m}(r) \subset \mathbb{S}_1^{n+1},$$

where $r > 1$ and $0 < m < n$. Similar to Example 3.4 in [5], M is a connected component of $M_r := \{y \in \mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+1} | y_{m+2}^2 + ... + y_{n+2}^2 = r^2\}$. Then, the Gauss map of M is $N(y) = \frac{-r}{\sqrt{r^2-1}}(y_1, ..., y_{m+1}, (1 - \frac{1}{r^2})y_{m+2}, ..., (1 - \frac{1}{r^2})y_{n+2})$, and its principal curvatures are $\kappa_1 = ... = \kappa_m = \frac{r}{\sqrt{r^2-1}}$, $\kappa_{m+1} = ... = \kappa_n = \frac{\sqrt{r^2-1}}{r}$. So, H_k and H_{k+1} are constant, and by (2.6), $L_k x = (\lambda x_1, ..., \lambda x_{m+1}, \mu x_{m+2}, ..., \mu x_{n+2}) = Ax + b$, where $\lambda = \frac{-r}{\sqrt{r^2-1}} c_k H_{k+1} - c_k H_k$, $\mu = \frac{-\sqrt{r^2-1}}{r} c_k H_{k+1} - c_k H_k$, $A = diag[\lambda, ..., \lambda, \mu, ..., \mu]$ and $b = 0$.

Example 5.7. Let M be the standard product

$$\mathbb{H}^m(-\sqrt{1 - r^2}) \times \mathbb{H}^{n-m}(-r) \subset \mathbb{H}_1^{n+1},$$

where $0 < r < 1$ and $0 < m < n$. M is a connected component of $M_r := \{y \in \mathbb{H}_1^{n+1} \subset \mathbb{R}_2^{n+1} | -y_2^2 + y_{m+3}^2 + ... + y_{n+2}^2 = -r^2\}$, with the Gauss map $N(y) = \frac{-r}{\sqrt{1-r^2}}(y_1, (1 - \frac{1}{r^2})y_2, y_3, ..., y_{m+2}, (1 - \frac{1}{r^2})y_{m+3}, ..., (1 - \frac{1}{r^2})y_{n+2})$ and the principal curvatures $\kappa_1 = ... = \kappa_m = \frac{r}{\sqrt{1-r^2}}$, $\kappa_{m+1} = ... = \kappa_n = -\frac{\sqrt{1-r^2}}{r}$. Then, H_k and H_{k+1} are constant and $L_k x =$

$(\lambda x_1, \mu x_2, \lambda x_3, ..., \lambda x_{m+2}, \mu x_{m+3}, ..., \mu x_{n+2}) = Ax + b$, where,
$\lambda = \frac{-r}{\sqrt{1-r^2}} c_k H_{k+1} + c_k H_k$, $\mu = \frac{\sqrt{1-r^2}}{r} c_k H_{k+1} + c_k H_k$, $b = 0$ and
$A = diag[\lambda, \mu, \lambda, ..., \lambda, \mu, ..., \mu]$.

Acknowledgments

References

[1] J. A. Aledo and L. J. Alias, On the curvatures of bounded complete spacelike hypersurfaces in the Lorentz-Minkowski space, *Manuscripta Math.* **101** (2000), no. 3, 401–413 .

[2] L. J. Alias, A. Ferrández and P. Lucas, Submanifolds in pseudo-Euclidean spaces satisfying the condition $\Delta x = Ax + B$, *Geom. Dedicata* **42** (1992), no. 3, 345–354.

[3] L. J. Alias, A. Ferrández and P. Lucas, Hypersurfaces in space forms satisfying the condition $\Delta x = Ax + B$, *Trans. Amer. Math. Soc.* **347** (1995), no. 5, 1793–1801.

[4] L. J. Alias and N. Gürbüz, An extension of Takahashi theorem for the linearized operators of the higher order mean curvatures, *Geom. Dedicata* **121** (2006) 113–127.

[5] L. J. Alias and S. M. B. Kashani, Hypersurfaces in space forms satisfying the condition $L_k x = Ax + b$, *Taiwanese J. Math* **14** (2010), no. 5, 1957–1977.

[6] L. J. Alias and J. M. Malacarne, Spacelike hypersurfaces with constant higher order mean curvature in Minkowski space-time, *J. Geom. Phys.* **41** (2002), no. 4, 359–375.

[7] S. Y. Cheng and S. T. Yau, Hypersurfaces with constant scalar curvature, *Math. Ann.* **225** (1977), no. 3, 195–204.

[8] O. J. Garay, An extension of Takahashi's theorem, *Geom. Dedicata* **34** (1990), no. 2, 105–112.

[9] T. Hasanis and T. Vlachos, Hypersurfaces of $\mathbb{E}^{n+1}$ satisfying $\Delta x = Ax + B$, *J. Austral. Math. Soc. Ser. A* **53** (1992), no. 3, 377–384.

[10] B. O'Neill, Semi-Riemannian Geometry with Applicatins to Relativity, Acad. Press Inc., 1983.

[11] T. Takahashi, Minimal immersions of Riemannian manifolds, *J. Math. Soc. Japan* **18** (1966) 380–385.

[12] L. Zhen-Qi and X. Xian-Hua, Space-like isoparametric hupersurfaces in Lorentzian space forms, *Front. Math. China* **1** (2006), no. 1, 130–137.

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