Bulletin of the Iranian Mathematical Society Vol. 39 No. 1 (2013), pp 175-185.

THE QUASI-MORPHIC PROPERTY OF GROUP

Q. WANG*, K. LONG AND L. FENG

Communicated by Ali Reza Ashrafi

Abstract. A group is called morphic, if for each normal endomorphism $\alpha \in end(G)$, there exists $\beta \in end(G)$ such that $ker(\alpha) = G\beta$ and $G\alpha = ker(\beta)$. Here, we consider the case that there exist normal endomorphisms β and γ such that $ker(\alpha) = G\beta$ and $G\alpha = ker(\gamma)$. We call G quasi-morphic, if this happens for any normal endomorphism $\alpha \in end(G)$. We get the following results: G is quasi-morphic if and only if, for any normal subgroup K such that $G/K \cong N \lhd G$, there exist $T, H \lhd G$ such that $G/T \cong K$ and $G/N \cong H$. Furthermore, we investigate the quasi-morphic property of finitely generated abelian group and get that a finitely generated abelian group is quasi-morphic if and only if it is finite.

1. **Introduction**

Nicholson and Sánchez first introduced morphic ring in [5], and investigated the morphic property of ring and module in [3–5] and [1]. In 2010, Li, et al. investigated the morphic property of group in [2]. A group G is called morphic, if every normal endomorphism α of G satisfies $G/G\alpha \cong ker(\alpha)$, or equivalently, for any normal endomorphism α there exists $\beta \in end(G)$ such that $ker(\alpha) = G\beta$ and $G\alpha = ker(\beta)$.

MSC(2010): Primary: 20K01; Secondary: 20k27, 20D35.
Keywords: Quasi-morphic group, finitely generated abelian group, normal endomorphism.
Received: 28 May 2011, Accepted: 4 September 2011.
*Corresponding author

Here, we investigate the quasi-morphic property of a group. Normal endomorphism α is called quasi-morphic, if there exist normal endomorphisms β and γ such that $ker(\alpha) = G\beta$ and $G\alpha = ker(\gamma)$. G is called quasi-morphic, if every normal endomorphism $\alpha \in end(G)$ is quasi-morphic.

We get the following results: G is quasi-morphic if and only if, for any normal subgroup K such that $G/K \cong N \triangleleft G$, there exist $T, H \triangleleft G$ such that $G/T \cong K$ and $G/N \cong H$. Furthermore, if G is a quasi-morphic abelian group, then $I(G) = \{G\alpha : \alpha \in end(G)\}$ is a lattice. Finally, we turn our attention to the finitely generated abelian group. We get that a finitely generated abelian group is quasi-morphic if and only if it is finite.

If G is a group, then we write $end(G)$ for the group endomorphism of G and write $aut(G)$ for the group automorphism. Endomorphisms are written on the right side of their arguments. We use $H \triangleleft G$ to indicate that H is a normal subgroup of G. We simply write C_n as a cyclic group of order n, and C_∞ as an infinite cyclic group.

2. Quasi-morphic endomorphism

We say that $\alpha \in end(G)$ is **normal** if $G\alpha \triangleleft G$. A normal endomorphism $\alpha \in end(G)$ is called **quasi-morphic**, if there exist normal endomorphisms β and γ such that $ker(\alpha) = G\gamma$ and $G\alpha = ker(\beta)$. It is clear that every morphic endomorphism is quasi-morphic.

Lemma 2.1. *Let G be a group. The followings are equivalent for a normal endomorphism α.*

(1) *α is quasi-morphic.*
(2) *$ker(\alpha)$ is an image of G and $G/G\alpha \cong K$, where K is a normal subgroup of G.*

Proof. (1)⇒(2) If α is quasi-morphic, then there exist normal endomorphisms β and γ such that $ker(\alpha) = G\beta$ and $G\alpha = ker(\gamma)$. Hence, $G/G\alpha = G/ker(\gamma) \cong G\gamma \triangleleft G$, and $ker(\alpha) = G\beta$ is an image of G.

(2)⇒(1) Given $\alpha \in end(G)$, $ker(\alpha)$ is an image of G and $G/G\alpha \cong K$, for some $K \triangleleft G$, by (2). We have isomorphism η: $G/G\alpha \to K$ and epic $\xi : G \to ker(\alpha)$. Defined normal endomorphism $\beta : G \to G$ and $\gamma : G \to G$, by $g\beta = (gG\alpha)\eta$ and $g\gamma = g\xi$, respectively. Then, $ker(\beta) = G\alpha$ and $G\gamma = ker(\alpha)$. Thus, α is quasi-morphic. □

By Lemma 2.1, every automorphism of group G is quasi-morphic, because $G/G\alpha \cong 1$ and $ker(\alpha) = 1$.

Theorem 2.2. *If α is quasi-morphic and $\rho \in aut(G)$, then $\alpha\rho$ and $\rho\alpha$ are quasi-morphic.*

Proof. If $\alpha \in end(G)$ is a quasi-morphic endomorphism, then there exist normal endomorphisms β and γ such that $ker(\alpha) = G\gamma$ and $G\alpha = ker(\beta)$. Since ρ is automorphism, we have $G/G\alpha\rho \cong G/G\alpha = G/ker(\beta) \cong G\beta \lhd G$ and $ker(\alpha\rho) = ker(\alpha) = G\gamma$ is an image of G. Hence, $\alpha\rho$ is quasi-morphic, by Lemma 2.1. Similarly, we have that $\rho\alpha$ is quasi-morphic. $\square$

In particular, every unit regular endomorphism is quasi-morphic. For more general endomorphisms, we have the following result.

Theorem 2.3. *If α and β are quasi-morphic endomorphisms, α is epic, and β is monic, then $\alpha\beta$ is quasi-morphic.*

Proof. Since $\alpha, \beta \in end(G)$ are quasi-morphic endomorphisms, we have normal endomorphisms γ and ρ such that $ker(\alpha) = G\gamma$ and $G\beta = ker(\rho)$. Then, $ker(\alpha\beta) = ker(\alpha) = G\gamma$ and $G\alpha\beta = G\beta = ker(\rho)$, because α is epic and β is monic. Thus, $\alpha\beta$ is quasi-morphic. $\square$

3. **Quasi-morphic group**

A group is called **quasi-morphic**, if every normal endomorphism $\alpha \in end(G)$ is quasi-morphic. By Lemma 2.1, a simple group and C_n are quasi-morpic, but C_∞ is not quasi-morphic.

Theorem 3.1. *The following are equivalent for a group G.*

(1) *G is quasi-morphic.*
(2) *For any normal subgroup K such that $G/K \cong N \lhd G$, there exists $T, H \lhd G$ such that $G/T \cong K$ and $G/N \cong H$.*

Proof. (1) $\Rightarrow$ (2) If $G/K \cong N$, then we have an isomorphism $\rho : G/K \to N$. Define normal endomorphism $\alpha : G \to G$ by $g\alpha = (gK)\rho$. Then, $ker(\alpha) = K$ and $G\alpha = N$. By (1), we have normal endomorphism β and γ such that $ker(\alpha) = G\beta$ and $G\alpha = ker(\gamma)$. Then, $G/ker(\beta) \cong G\beta = ker(\alpha) = K$ and $G/N = G/G\alpha = G/ker(\gamma) \cong G\gamma$.

(2) $\Rightarrow$ (1) For any normal endomorphism $\alpha \in end(G)$, we have $G/ker(\alpha) \cong G\alpha \lhd G$. Hence, there exist $T, H \lhd G$ such that $G/T \cong ker(\alpha)$ and $G/G\alpha \cong H$ by (2). Then, we have isomorphisms $\rho_1 : G/T \to ker(\alpha)$

and $\rho_2 : G/G\alpha \to H$. Define β and γ by $g\beta = (gT)\rho_1$, $g\gamma = (gG\alpha)\rho_2$. Then, $G\beta = ker(\alpha)$ and $G\alpha = ker(\gamma)$. Hence, α is quasi-morphic. Since α is arbitrary, we get that G is quasi-morphic. □

Theorem 3.2. *If G is quasi-morphic, then the followings are equivalent.*

(1) *Every normal subgroup of G is an image of G.*
(2) *Every image of G is isomorphism to a normal subgroup of G.*

Proof. (1) ⇒ (2) For any normal subgroup N of G, we have $G/K \cong N$, for some $K \lhd G$, by (1). Then, $G/N \cong T$, where $T \lhd G$, because G is quasi-morphic.

(2) ⇒ (1) For any normal subgroup K of G, there exists $N \lhd G$ such that $G/K \cong N$, by (2). Hence, $G/T \cong K$, for some T, because G is quasi-morphic. □

Let G be a group. If G satisfies (1) and (2), then G is quasi-morphic, by Theorem 3.1. C_∞ satisfies(1), but it is not quasi-morphic.

Recall that a group G is said to be uniserial, if the normal subgroup forms a finite chain, that is, it has the form: $G = G_0 \supset G_1 \supset \cdots G_n = 1$. We define the uniserial length of the normal subgroup $G_k \lhd G$ by $l_G(G_k) = n - k$ for each $k = 0, 1, \cdots, n$. We have the following lemma given in [2].

Lemma 3.3. *Let G be uniserial with normal subgroup lattice $G = G_0 \supset G_1 \supset \cdots G_n = 1$.*

(1) *If $H \lhd G$ is also uniserial, then $l_G(H) = l_H(H)$.*
(2) *In particular, if $G_i \cong G/G_k$, then $i = n - k$.*

Theorem 3.4. *If G is a uniserial group, then the followings are equivalent.*

(1) *G is quasi-morphic.*
(2) *If $G/G_k \cong G_{n-k}, k = 1, 2, \ldots, n$, then $G/G_{n-k} \cong G_k$.*
(3) *G is morphic.*

Proof. (1) ⇒ (2) If $G/G_k \cong G_{n-k}$, then we have $N = G_i$ such that $G/G_{n-k} \cong N = G_i$, by Theorem 3.1. By Lemma 3.3, we have $i = k$. Hence, $G/G_{n-k} \cong G_k$.

(2) ⇒ (3) Let $\alpha \in end(G)$ be a normal endomorphism, and write $\ker(\alpha) = G_k$ and $G\alpha = G_i$. Then, $G/G_k \cong G_i$, $i = n - k$, by Lemma 3.3. Hence, we have $G/G_{n-k} \cong G_k$, by (2). G is morphic, by [2, Lemma 5].

(3) $\Rightarrow$ (1) is clear. $\square$

Next, we investigate the group which have a composition series.

Theorem 3.5. *Let G be a group which has a composition series. G is quasi-morphic, if G satisfies the following conditions: (a) every subgroup of G is isomorphic to an image of G; and (b) $G/K \cong G/K_1$, for arbitrary $K, K_1 \lhd G$ with the same length.*

Proof. Let K and N be normal subgroups of G, write $n = length(G)$ and $t = length(K)$. Assume $G/K \cong N$, and there exists $T \lhd G$ such that $G/T \cong K$, by (a). Then, $length(T) = length(N) = n - t$, and so $G/N \cong G/T \cong K$, by (b). Then, G is quasi-morphic. $\square$

Example 3.6. *If G has a composition series, and the length of the composition series is 2, then G is quasi-morphic.*

Proof. If $G/N \cong K \lhd G$, then we show that $G/K \cong T$, and $G/H \cong N$, for some $T, H \lhd G$. If $N = G$ or 1, or $K = 1$, $G/K \cong N$ is clear. If $K = G$, then we show that N must be 1. In fact, when $1 \neq N \subset G$, G/N is simple, but G is not simple, we have a contradiction. Next, suppose N and K are nontrivial subgroups. Since N, K are normal subgroups, then we have that composition series $G \supseteq K \supseteq 1$ is isomorphic to composition series $G \supseteq N \supseteq 1$, by the hypothesis. Then, $G/K \cong N$, because $G/N \cong K$. $\square$

$C_2 \times C_2$ is quasi-morphic, because the length of its composition series is 2.

4. **Abelian group**

If G is an abelian group, then every $\alpha \in end(G)$ is a normal endomorphism. Let $I(G) = \{G\alpha : \alpha \in end(G)\}$ be the set of images and $K(G) = \{ker(\alpha) : \alpha \in end(G)\}$ be the set of kernels.

Theorem 4.1. *Let G be an abelian group. Then, G is quasi-morphic if and only if $I(G) = K(G)$.*

Proof. $\Rightarrow$) Suppose G is quasi-morphic. For any $G\alpha \in I(G)$, there exists β such that $G\alpha = ker(\beta)$. Hence $G\alpha \in K(G)$, $I(G) \subset K(G)$. If $ker(\alpha) \in K(G)$, then there exists γ such that $ker(\alpha) = G\gamma$. Hence, $K(G) \subseteq I(G)$. We have $I(G) = K(G)$.

$\Leftarrow$) For any $\alpha \in end(G)$, we have $G\alpha \in I(G) = K(G)$. Hence, there exists β such that $G\alpha = ker(\beta)$. We also have $ker(\alpha) \in K(G) = I(G)$.

Then, there exists γ such that $ker(\alpha) = G\gamma$. Thus, α is quasi-morphic. Since α is arbitrary, G is quasi-morphic. □

Theorem 4.2. *If G is a quasi-morphic abelian group, and $N = G\alpha$, $H = G\beta (\alpha, \beta \in end(G))$, then there exists $\gamma \in end(G)$ such that $NH = G\gamma$.*

Proof. Since G is quasi-morphic, we have endomorphism $\varphi \in end(G)$ such that $G\alpha = ker(\varphi)$, and we also have $\rho \in end(G)$ such that $G\beta\varphi = ker(\rho)$. It suffices to show that $NH = G\alpha G\beta = ker(\varphi\rho)$. Let $z = xy \in G\alpha G\beta$, where $x \in G\alpha, y \in G\beta$. Then, $(z)\varphi\rho = (xy)\varphi\rho = (x\varphi\rho)(y\varphi\rho) = 0$, and hence $G\alpha G\beta \subseteq ker(\varphi\rho)$.

For the other inclusion, let $s \in ker(\varphi\rho)$. Then, $s\varphi \in ker(\rho) = G\beta\varphi$, say $s\varphi = y\varphi$, where $y \in G\beta$, which shows $sy^{-1} \in ker(\varphi) = G\alpha$. Hence, $s \in G\alpha G\beta$ and $ker(\varphi\rho) \subseteq G\alpha G\beta$. Now, we have $G\alpha G\beta = ker(\varphi\rho)$. Since $\varphi\rho$ is a normal endomorphism, we have $\gamma \in end(G)$ such that $ker(\varphi\rho) = G\gamma$, and hence $NH = G\gamma$. □

Corollary 4.3. *If G is quasi-morphic and abelian, $\alpha_1, \ldots, \alpha_n \in end(G)$ and $N_1 = G\alpha_1, N_2 = G\alpha_2, \cdots, N_n = G\alpha_n$, then $N_1N_2 \ldots N_n = G\gamma$, for some $\gamma \in end(G)$.*

Similarly, it is easy to see that if G is a quasi-morphic abelian group, and $N_1 = ker(\alpha_1), N_2 = ker(\alpha_2), \cdots, N_n = ker(\alpha_n)(\alpha_1, \ldots, \alpha_n \in end(G))$, then we also have

$$N_1N_2 \cdots N_n = ker(\gamma),$$

where $\gamma \in end(G)$.

Theorem 4.4. *If G is a quasi-morphic abelian group, and $N = G\alpha, H = G\beta$, for some $\alpha, \beta \in end(G)$, then there exists $\gamma \in end(G)$ such that $N \cap H = G\gamma$.*

Proof. Since G is quasi-morphic, we have an endomorphism $\varphi \in end(G)$ such that $G\alpha = ker(\varphi)$, and we also have ρ such that $ker(\beta\varphi) = G\rho$. It suffices to show that $N \cap H = G\rho\beta$. For any $x \in G\alpha \cap G\beta$, say $x = (g_1)\alpha = (g_2)\beta$. Then, $g_1\alpha\varphi = g_2\beta\varphi = 0$, and hence $g_2 \in ker(\beta\varphi) = G\rho$, $x \in G\rho\beta$. We have $N \cap H \subseteq G\rho\beta$. On the other hand, let $y \in G\rho\beta$, say $y = (g)\rho\beta$. Since $y\varphi = (g)\rho\beta\varphi = 0$, we have $y \in ker(\varphi) = G\alpha$. Hence, $G\rho\beta \subseteq G\alpha \cap G\beta$. Then, $N \cap H = G\rho\beta$. □

Corollary 4.5. *If G is abelian and quasi-morphic, and $N_1 = G\alpha_1, N_2 = G\alpha_2, \cdots, N_n = G\alpha_n$ $(\alpha_1, \ldots, \alpha_n \in end(G))$, then $N_1 \cap N_2 \cap \cdots \cap N_n = G\gamma$, where γ is an endomorphism of $end(G)$.*

Similarly, suppose G is a quasi-morphic abelian group, and $N_1 = ker(\alpha_1), N_2 = ker(\alpha_2), \cdots, N_n = ker(\alpha_n)$ $(\alpha_1, \ldots, \alpha_n \in end(G))$. Then, $N_1 \cap N_2 \cap \cdots \cap N_n = ker(\gamma)$ holds for some $\gamma \in end(G)$.

Theorem 4.6. *Let G be an abelian group. If G is quasi-morphic group, then $I(G)$ and $K(G)$ are lattices.*

Proof. We can get it by Theorem 4.1, Theorem 4.2 and Theorem 4.4. □

The converse fails. For example, $I(C_\infty)$ and $K(C_\infty)$ are lattices, but C_∞ is not quasi-morphic.

Lemma 4.7. *Suppose $G = \oplus_{i=1}^{n} < u_i >$ is an abelian p-group, and $ord(u_i) = p^{a_i}$, $a_i \leq a_{i+1}$, $i = 1, 2, \cdots, n-1$. Let $H = \oplus_{j=1}^{m} < v_j >$ be a subgroup of G, and $ord(v_j) = p^{b_j}$, $b_j \leq b_{j+1}, j = 1, \cdots, m-1$. Then, $m \leq n$.*

Proof. Let $G_p = \{g \in G : pg = 0\}$ be a set of G. If $x, y \in G_p$, then $px = py = 0$, $p(x-y) = 0$. Hence, G_p is a subgroup of G.

Next, we show that the order of G_p is p^n. For any $x \in G$, we have $x = b_1u_1 + b_2u_2 + \cdots + b_nu_n$, where $0 \leq b_i \lneq p^{a_i}$. If x is an element of G_p, then $px = pb_1u_1 + \cdots + pb_nu_n = 0$. Hence, $pb_1u_1 = \cdots = pb_nu_n = 0$, because $G = \oplus_{i=1}^{n} < u_i >$. Then, $p^{a_i} \mid pb_i$, and $b_i = p^{a_i-1}c_i$. Since $0 \leq b_i \lneq p^{a_i}$, we have $0 \leq c_i \lneq p$. Moreover, G_p can be represented by $G_p = \{\sum_{i=1}^{n} c_ip^{a_i-1}u_i | 0 \leq c_i < p, i = 1, \ldots, n\}$. Hence, the order of G_p is p^n.

Similarly, we can define H_p. H_p is a subgroup of G_p, and $ord(H_p) = p^m$. Then, we have $m \leq n$. □

Proposition 4.8. *Suppose $G = \oplus_{i=1}^{n} < u_i >$ is an abelian p-group, and $ord(u_i) = p^{a_i}$, $a_i \leq a_{i+1}$, $i = 1, 2, \cdots, n-1$. Let $H = \oplus_{j=1}^{m} < v_j >$ be a subgroup of G, and $ord(v_j) = p^{b_j}$, $b_j \leq b_{j+1}, j = 1, \cdots, m-1$. Then, $p^{b_{m-i}} \leq p^{a_{n-i}}$, where $i = 0, 1, 2, \cdots, m-1$.*

Proof. We prove it by induction. If $m = 1$, then $H =< v_1 >$, where $v_1 \in G$. For any element $\alpha \in G$, we have $ord(\alpha) \leq ord(u_n)$. Hence, $ord(v_1) \leq ord(u_n)$.

Now, let $H_2 = \oplus_{k=1}^{t} < v_k > (t \lneq m)$ be a subgroup of $G_2 = \oplus_{s=1}^{w} < \lambda_s >$ and assume that H_2 satisfies $ord(v_{t-i}) \leq ord(\lambda_{w-i})$, where $i = 0, 1, 2, \cdots, t-1$.

First, for the H given in this proposition, we show that $ord(v_1) \leq ord(u_{n-m+1})$. Suppose $ord(u_{n-m+1}) \lneq ord(v_1)$, and define $p^{a_{n-m+1}}G = \{g \in G : g = p^{a_{n-m+1}}h$, where $h \in G\}$. We assert that $p^{a_{n-m+1}}G =$

$\oplus_{i=t}^{n} < p^{a_{n-m+1}} u_i >$, t is the least value such that $p^{a_{n-m+1}} \lneq p^{a_t}$(if we cannot find t, then $p^{a_{n-m+1}} G = 0$). In fact, if $g \in p^{a_{n-m+1}} G$, then we have $g = p^{a_{n-m+1}} h$, where $h = a_1 u_1 + \cdots + a_n u_n$. Hence, $g = a_1 p^{a_{n-m+1}} u_1 + \cdots + a_n p^{a_{n-m+1}} u_n = a_t p^{a_{n-m+1}} u_t + \cdots + a_n p^{a_{n-m+1}} u_n \in \sum_{i=t}^{n} < p^{a_{n-m+1}} u_i >$, $p^{a_{n-m+1}} G \subseteq \sum_{i=t}^{n} < p^{a_{n-m+1}} u_i >$, and clearly $p^{a_{n-m+1}} G = \oplus_{i=t}^{n} < p^{a_{n-m+1}} u_i >$. Similarly, we can define $p^{a_{n-m+1}} H$. Then, $p^{a_{n-m+1}} H = \oplus_{j=1}^{m} < p^{a_{n-m+1}} v_j >$ and $p^{a_{n-m+1}} H \leq p^{a_{n-m+1}} G$. This is a contradiction to Lemma 4.7, and thus $ord(v_1) \leq ord(u_{n-m+1})$. If k is the largest value such that $p^{b_k} = p^{b_1}$, then we have $ord(v_i) = ord(v_1) \leq ord(u_{n-m+i}), i = 1, 2, \cdots, k$.

If $k \lneq m$, then we can define $p^{b_1} H$ and $p^{b_1} G$ as above. Then, $p^{b_1} H = \oplus_{j=k+1}^{m} < p^{b_1} v_j >$, $p^{b_1} G = \oplus_{i=h}^{n} < p^{b_1} u_i >$ (h is the least value such that $p^{b_1} \lneq p^{a_h}$), and $p^{b_1} H$ is the subgroup of $p^{b_1} G$. We have $m - k \leq n - h + 1, h \leq n - (m - k - 1)$, by Lemma 4.7. Hence, $ord(p^{b_1} v_{m-i}) \leq ord(p^{b_1} u_{n-i})$, where $i = 0, 1, \cdots, m-k-1$, by assumption, and it follows that $ord(v_{m-i}) \leq ord(u_{n-i}) (i = 0, 1, \cdots, m-k-1)$. □

Lemma 4.9. *Suppose $G = \oplus_{i=1}^{n} < u_i >$ is an abelian p-group, and $ord(u_i) = p^{a_i}$, $a_i \leq a_{i+1}$, $i = 1, 2, \cdots, n-1$. Let $H = \oplus_{j=1}^{m} < v_j >$ be an image of G, and $ord(v_j) = p^{b_j}$, $b_j \leq b_{j+1}, j = 1, \cdots, m-1$. Then, $m \leq n$.*

Proof. We have an epic $\theta : G \to H$ by the hypothesis. Since $G = \oplus_{i=1}^{n} < u_i >$, we have $\mathbb{Z}^n / T \cong G$, where $T \lhd \mathbb{Z}^n$ and $rank(T) = n$. There exists an isomorphism $\alpha : \mathbb{Z}^n / T \to G$. Define $\beta : \mathbb{Z}^n / T \to H$ by $(x + T)\beta = (x + T)\alpha\theta$. Then, $ker(\beta) = K/T$, $T \leq K \leq \mathbb{Z}^n$. Hence, $\mathbb{Z}^n / K \cong (\mathbb{Z}^n / T)/(K/T) \cong H$. Since $rank(T) \leq rank(K)$, by [6, Theorem 10.17], we have $rank(K) = n$. There exist bases $\{y_1, y_2, \cdots, y_n\}$ of $\mathbb{Z}^n$ such that

$$K = < d_1 y_1, \cdots, d_n y_n >, d_i \mid d_{i+1} (i = 1, \cdots, n-1).$$

Then, $\mathbb{Z}^n / K = \oplus_{i=1}^{n} < \bar{y_i} > \cong H, \bar{y_i} = y_i + K$. If t is the least value such that $1 \lneq d_t$, then we have $\mathbb{Z}^n / K = \oplus_{i=t}^{n} < \bar{y_i} > \cong H$, and $m = n - t + 1 \leq n$. □

Proposition 4.10. *Suppose $G = \oplus_{i=1}^{n} < u_i >$ is an abelian p-group, and $ord(u_i) = p^{a_i}$, $a_i \leq a_{i+1}$, $i = 1, 2, \cdots, n-1$. Let $H = \oplus_{j=1}^{m} < v_j >$ be an image of G, and $ord(v_j) = p^{b_j}$, $b_j \leq b_{j+1}, j = 1, \cdots, m-1$. Then, $p^{b_{m-i}} \leq p^{a_{n-i}}$, where $i = 0, 1, 2, \cdots, m-1$.*

Proof. We use induction on m. It is clearly true for $m = 1$. Now, let $H_1 = \oplus_{k=1}^{t} < v_k > (t \lneq m)$ be an image of $G_1 = \oplus_{s=1}^{w} < \lambda_s >$ and

assume that H_1 satisfies $ord(v_{t-i}) \leq ord(\lambda_{w-i})$, where $i = 0, 1, 2, \cdots, t-1$.
For the H given in this proposition, there exists an epic $\alpha : G \to H$.

First, for the H given in this proposition, we show that $ord(v_1) \leq ord(u_{n-m+1})$. Suppose $ord(u_{n-m+1}) \lneq ord(v_1)$. We can define $p^{a_{n-m+1}}G$ and $p^{a_{n-m+1}}H$ as in Proposition 4.8.Then, $p^{a_{n-m+1}}G = \oplus_{i=t}^{n} < p^{a_{n-m+1}}u_i >$, where t is the least value such that $p^{a_{n-m+1}} \lneq p^{a_t}$ ($p^{a_{n-m+1}}G = 0$, if we cannot find t) and $p^{a_{n-m+1}}H = \oplus_{j=1}^{m} < p^{a_{n-m+1}}v_j >$. Then, α induces a homomorphism $\alpha_1 : p^{a_{n-m+1}}G \to H$. For any $y \in (p^{a_{n-m+1}}G)\alpha_1$, we have $y \in p^{a_{n-m+1}}(G)\alpha_1 = p^{a_{n-m+1}}H$. Then, α induces a homomorphism $\alpha_2 :\ p^{a_{n-m+1}}G \to p^{a_{n-m+1}}H$. Since α is epic, then α_2 is epic. Since $n - t + 1 \lneq m$, we have a contradiction to Lemma 4.9. Hence, $ord(v_1) \leq ord(u_{n-m+1})$. If k is the largest value such that $p^{b_1} = p^{b_k}$, then $ord(v_i) = ord(v_1) \leq ord(u_{n-m+i})$ $(i = 1, \cdots, k)$.

If $k \lneq m$, then we can define $p^{b_1}G$ and $p^{b_1}H$ as in Proposition 4.8. Then, $p^{b_1}H = \oplus_{j=k+1}^{m} < p^{a_1}v_j >$, $p^{b_1}G = \oplus_{i=w}^{n} < p^{b_1}u_i >$ (w is the least value such that $p^{b_1} \lneq p^{a_w}$) and α induces an epic $\alpha_3 : p^{b_1}G \to p^{b_1}H$. We have $m - k \leq n - w + 1, w \leq n - (m - k - 1)$, by Lemma 4.9. Hence, $ord(p^{b_1}v_{m-i}) \leq ord(p^{b_1}u_{n-i})$ $(i = 0, 1, \ldots, m - k - 1)$, by assumption, and it follows that $ord(v_{m-i}) \leq ord(u_{n-i})$ $(i = 0, 1, \cdots, m - k - 1)$.

□

Theorem 4.11. *If G is an abelian p-group, then G is quasi-morphic.*

Proof. Let $G = \oplus_{i=1}^{n} < u_i >$, where $ord(u_i) = p^{a_i}$ and let $a_i \leq a_{i+1}$ $(i = 1, 2, \cdots, n-1)$ be an abelian p-group. For any $H, K \lhd G$, if $G/H \cong K$, then we show that $G/K \cong L$ and $G/T \cong H$, for some $T, L \lhd G$. By Proposition 4.10, we have $G/K = \oplus_{j=1}^{m} < v_j >$, where $m \leq n$ and $ord(v_{m-i}) \leq ord(u_{n-i})$, $i = 0, 1, \cdots, m - 1$. Write $ord(v_j) = p^{b_j}, j = 1, 2, \cdots, m$. Let $L = \oplus_{i=n-m+1}^{n} < p^{a_i - b_i}u_i >$ be a subgroup of G. Then, $G/K \cong L$.

By Proposition 4.8, we have $H = \oplus_{k=1}^{t} < w_k >$, $t \leq n$ and $ord(w_{t-i}) \leq ord(u_{n-i})$ $(i = 0, 1, \cdots, t-1)$. Write $ord(w_k) = p^{c_k}$, $k = 1, 2, \cdots, t$. Let $T = \oplus_{j=1}^{n-t} < u_j > \oplus_{i=1}^{t} < p^{c_i}u_{n-t+i} >$ be a subgroup of G. Then, $G/T \cong H$. Hence, G is a quasi-morphic group, by Theorem 3.1. □

Theorem 4.12. *Let $G = G_1 \times G_2 \times \cdots \times G_n$, where the G_i are the groups such that $Hom(G_i, G_j) = \{0\}$, whenever $i \neq j$. Then, G is quasi-morphic if and only if G_i is quasi-morphic.*

Proof. $\Rightarrow$) If $\alpha \in end(G)$, then there exist $\alpha_i \in end(G_i)$ such that

$$(g_1, g_2, \cdots, g_n)\alpha = (g_1\alpha_1, g_2\alpha_2, \cdots, g_n\alpha_n),$$

for all $(g_1, g_2, \cdots, g_n) \in G$, since $Hom(G_i, G_j) = \{0\}$, if $i \neq j$. Thus, $ker(\alpha) = \prod_{i=1}^n ker(\alpha_i)$ and $im(\alpha) = \prod_{i=1}^n im(\alpha_i)$. For any normal endomorphism $\alpha_i \in end(G_i)$, define $\alpha : G \to G$ by $(g_1, \cdots, g_{i-1}, g_i, g_{i+1}, \cdots, g_n)\alpha = (g_1, \cdots, g_{i-1}, (g_i)\alpha_i, g_{i+1}, \ldots, g_n)$, for all $(g_1, \cdots, g_n) \in G$. There exist normal endomorphisms β and γ such that $ker(\alpha) = G\beta$ and $G\alpha = ker(\gamma)$, because G is quasi-morphic. Hence, $ker(\alpha_i) = G\beta_i$ and $G\alpha_i = ker(\gamma_i)$. Then, G_i is quasi-morphic.

$\Leftarrow$) Suppose $\alpha = \prod_{i=1}^n \alpha_i$ is any normal endomorphism of G. Since G_i is quasi-morphic, we have normal endomorphisms β_i and γ_i ($i = 1, 2, \ldots, n$), such that $ker(\alpha_i) = G\beta_i$ and $G\alpha_i = ker(\gamma_i)$. Let $\beta = \prod_{i=1}^n \beta_i$ and $\gamma = \prod_{i=1}^n \gamma_i$. Then, $ker(\alpha) = G\beta$ and $G\alpha = ker(\gamma)$. Hence, G is quasi-morphic.

□

Theorem 4.13. *If G is a finite abelian group, then G is quasi-morphic.*

Proof. If G is a finite abelian group, then $G = P_1 \times \cdots P_2 \times \cdots \times P_n$, where P_i is p_i-group. Since $Hom(P_i, P_j) = 0$, $i \neq j$, G is quasi-morphic, by Theorem 4.11 and Theorem 4.12. □

Theorem 4.14. *A finitely generated abelian group is quasi-morphic if and only if it is finite.*

Proof. $\Rightarrow$) If G is a finitely generated abelian group, then

$$G = G_{p_1} \oplus G_{p_2} \oplus \cdots \oplus G_{p_n} \oplus G^1 \oplus \cdots G^m,$$

where G_{p_i} is the p_i-primary component and G^j is the infinite cyclic group. If $1 \leq m$, then let $G^m =< u_m >$ be an infinite cyclic group. Let p be a prime, and $p_1, \cdots, p_n \lneq p$. We have

$$G/0 \cong G \cong G_{p_1} \oplus G_{p_2} \oplus \cdots \oplus G_{p_n} \oplus G^1 \oplus \cdots G^{m-1} \oplus < pu_m >= K.$$

But, G/K is not isomorphic to a subgroup of G. Hence, G is not quasi-morphic, by Theorem 3.1.

$\Leftarrow$) This is clear by Theorem 4.13. □

Acknowledgments

The work of the author was supported by the National Natural Science Foundation of China (No. 10926183) and the Foundation of National University of Defense Technology (No. JC08-2-03).

References

[1] V. Camillo and W. K. Nichlson, Quasi-morphic rings, *J. Algebra Appl.* **6** (2007), no. 5, 789–799.
[2] Y. Li and W. K. Nicholson and L. Zan, Morphic groups, *J. Pure Appl. Algebra* **214** (2010), no. 10, 1827–1834.
[3] W. K. Nicholson and E. Sánchez Campos, Morphic modules, *Comm. Algebra* **33** (2005), no. 8, 2629–2647.
[4] W. K. Nicholson and E. Sánchez Campos, Principal rings with the dual of the isomorphism theorem, *Glasg. Math. J.* **46** (2004), no. 1, 181–191.
[5] W. K. Nicholson and E. Sánchez Campos, Rings with the dual of the isomorphism theorem, *J. Algebra* **271** (2004), no. 1, 391–406.
[6] J. J. Rotman, An Introduction to the Theory of Groups, Fourth edition, Grad. Texts in Math., 148, Springer-Verlag, New York, 1995.

Qichuan Wang
Department of Mathematics and Systems Science, National University of Defense Technology, P. R. China 410073, Changsha, China
Email: `wangqichuan1026@163.com.cn`

Kai Long
Department of Mathematics and Systems Science, National University of Defense Technology, P. R. China 410073, Changsha, China
Email: `lkkkkkkkk@hotmail.com.cn`

Lianggui Feng
Department of Mathematics and Systems Science, National University of Defense Technology, P. R. China 410073, Changsha, China
Email: `fenglg2002@sina.com.cn`