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THE LEAST-SQUARE BISYMMETRIC SOLUTION TO A QUATERNION MATRIX EQUATION WITH APPLICATIONS

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Abstract. In this paper, we derive the necessary and sufficient conditions for the quaternionic matrix equation $XA = B$ to have the least-square bisymmetric solution and give the expression of such solution when the solvability conditions are met. Furthermore, we derive sufficient and necessary conditions for $XA = B$ to have the positive (nonnegative) definite least-square bisymmetric solution and the maximal (minimal) least-square bisymmetric solution.

1. **Introduction**

Let R *and* C, be the real, complex and quaternionicnumber fields. $H^{m\times n}$ stands for the set of all $m\times n$ quaternion matrices. $H_h^{n\times n}$ denotes the set of all $n\times n$ Hermitian quaternion matrices. A quaternion matrix A is called unitary if $A^*A = AA^* = I$, where I is the identity matrix. $A = (a_{ij}) \in H^{n\times n}$ is called bisymmetric if $a_{ij} = a_{n-i+1,n-j+1} = \overline{a_{ji}}$. Clearly, if a matrix A is bisymmetric, then A is Hermitian. For convenience, we denote the set of all $n\times n$ bisymmetric quaternionic matrices

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by B_n. For an $m \times n$ quaternionic matrix A, the Moore-Penrose inverse of matrix A, denoted by $A^\dagger$, is an $n \times m$ matrix which satisfy simultaneously

$$AA^\dagger A = A, A^\dagger AA^\dagger = A^\dagger, (AA^\dagger)^* = AA^\dagger, (A^\dagger A)^* = A^\dagger A,$$

where * denotes the conjugate transpose of quaternion matrices. The generalized inverse of matrix $A \in H^{m\times n}$, denoted by A^-, is an $n \times m$ matrix which satisfies $AA^-A = A$. Moreover, $E_A = I - AA^\dagger$, $F_A = I - A^\dagger A$ are two projectors induced by A.

It is well known that $a \in H$ can be uniquely expressed as $a = \alpha + \beta j, \alpha, \beta \in C$. Therefore for any quaternion matrix $A \in H^{m\times n}$, it can be uniquely written as $A = A_1 + A_2 j$ $(A_1, A_2 \in C^{m\times n})$. The complex representation of quaternion matrix $A = A_1 + A_2 j$ is defined to be

$$A^c = \begin{bmatrix} A_1 & -A_2 \\ \overline{A_2} & \overline{A_1} \end{bmatrix} \in C^{2m\times 2n}.$$

As we know, the complex matrix A^c is uniquely determined by the matrix A. For the complex representation of a quaternionic matrix, there are some properties as follows.

Lemma 1.1. *Let $A, B, C, D \in H^{m\times n}$, $a, b \in R$. Then*
(1) $(aA + bB)^c = aA^c + bB^c$;
(2) $(AB)^c = A^c B^c$;
(3) $(A^*)^c = (A^c)^*$;
(4) $((AB)^*)^c = (B^*)^c (A^*)^c$;
(5) $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^c = \begin{bmatrix} A^c & B^c \\ C^c & D^c \end{bmatrix}$.

According to [5], we introduce the norm of quaternion matrices.

Definition 1.2. *Suppose $A, B \in H^{m\times n}$ are arbitrary matrices and a is an arbitrary complex number. A function $\nu : H^{m\times n} \longrightarrow R$ is a quaternion norm if it satisfies the following statements:*

(1) $\nu(A) \geq 0$,
(2) $\nu(A) = 0$ *if and only if* $A = 0$,
(3) $\nu(aA) = |a|\nu(A)$,
(4) $\nu(A + B) \leq \nu(A) + \nu(B)$.

Let $\mu(M)$ be a Frobenius norm for any $M \in C^{2m\times 2n}$. For any $A \in H^{m\times n}$, we define

$$\nu(A) = \mu(A^c).$$

It is easy to verify that this matrix norm is unitary invariant. For convenience, we call it the *Frobenius norm* of quarternion matrix A, denoted by $\|A\|$. Let $A \in H^{n\times n}$. A quaternion λ is said to be a *right* eigenvalue of A if $Ax = x\lambda$ for some nonzero quaternionic column vector x. Similarly, λ is a *left* eigenvalue if $Ax = \lambda x$. It is evident that all right eigenvalues are real if A is Hermitian. Moreover, the right eigenvalue of $A \in H_h^{n\times n}$ must be left eigenvalue of A. The rank of $A \in H^{n\times n}$ is defined to be the maximum number of columns of A which are *right* linear independent [15]. Furthermore, the rank of A is equal to the number of positive singular values of A (see Theorem 7.2 in [15]). As shown in [15], matrices A, AA^* and A^*A are all of the same rank. So the rank of an Hermitian matrix is the number of nonzero right eigenvalues. For $A \in H_h^{n\times n}$, we define the inertia of A as follows:

$$IN(A) = \{IN_+(A), IN_-(A), IN_0(A)\},$$

where $IN_+(A), IN_-(A), IN_0(A)$ are the numbers of positive, negative, zero eigenvalues (including multiplicity), respectively.

The least-square problem over complex number field has gained much attention. It has practical applications in information theory, theoretical physics, statistics, signal processing, system theory, automatic control, etc. In recent years, there are some good results about the least-square problem for matrix equation

$$XA = B \tag{1.1}$$

as follows. In [16], Zhou, Zhang and Hu investigated the least-square centrosymmetric solutions. Liu et al [7] considered the least-square centrohermitian solutions. Xie et al [10] studied the least-square symmetric and sub-anti-symmetric solutions. In [14], Xiu and Liao investigated the least-square anti-symmetric and persymmetric solutions. In [9], Sheng and Xie considered the least-square anti-symmetric solutions. In [13], Xie, Zhang and Hu investigated the least-square bisymmetric solutions. In recent years, quaternion matrices were extensively applied in quantum mechanics, rigid mechanics and control theory (see [1, 3, 4]). Thus, it is necessary to further study the theory and methods of quaternion matrices, especially the quaternion least-squares problem since this problem has been settled only a little. In this paper, we consider the least-squares bisymmetric solutions to quaternionic matrix equation (1.1).

The inertia theory is an ancient and useful context in matrix theory. As we know, the inertias of a Hermitian matrix can be used to characterize definiteness of the Hermitian matrix. The positive definite

Hermitian solutions to matrix equation is considered scarcely. In this paper, we use the inertia theory to derive necessary and sufficient conditions for quaternion matrix equations to have positive and nonnegative definite least-square bisymmetric solutions.

We organize the paper as follows. In section 2, we derive the necessary and sufficient conditions for quaternionic matrix equation (1.1) to have the least-square bisymmetric solution and give the expression of this solution. In section 3, we investigate the extreme inertias of the least-square bisymmetric solution and characterize the solution with extreme inertias. We get the sufficient and necessary conditions for (1.1) to have the positive and nonnegative least-square bisymmetric solution. In section 4, we give the definition of the maximal and minimal solutions to matrix equation. By means of inertia theory, we derive sufficient and necessary conditions for (1.1) to have the maximal and minimal least-square bisymmetric solutions.

2. **The least-square bisymmetric solutions to** (1.1)

For the construction of bisymmetric matrices, Wang [12] presents the following lemma.

Lemma 2.1. [12]

(1) *Suppose that*

$$D = \frac{1}{2}\begin{bmatrix} I_k & V_k \\ -V_k & I_k \end{bmatrix}$$

where I_k is $k \times k$ identity matrix and V_k is a $k \times k$ permutation matrix whose elements along the southwest-northeast diagonal are $1^{'}s$ and $0^{'}s$ otherwise. Then $X \in B_{2k}$ if and only if

$$X = D\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} D^*$$

where X_1, X_2 are $k \times k$ Hermitian matrices.

(2) *Suppose that*

$$D = \frac{1}{2}\begin{bmatrix} I_k & 0 & V_k \\ 0 & 1 & 0 \\ -V_k & 0 & I_k \end{bmatrix}$$

where I_k, V_k are the same as above. Then $X \in B_{2k+1}$ if and only if

$$X = D\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} D^*$$

where X_1, X_2 are Hermitian $K \times K$, $(K+1)\times(K+1)$ matrices, respectively.

By extending the result in [6] from the complex field to the quaternion field, we can get the following lemma.

Lemma 2.2. *Suppose that $A, B \in H^{n\times m}$, such that the matrix equation (1.1) is consistent. Then the equation (1.1) has a Hermitian solution if and only if B^*A is Hermitian. The general Hermitian solution is of the form*

$$(A^*)^\dagger B^* + BA^\dagger - (A^*)^\dagger A^* BA^\dagger + F_{A^*}KF_A,$$

where K is an arbitrary Hermitian quaternion matrix with appropriate size.

Now for $X \in B_{2k}$, we consider the necessary and sufficient conditions for the real quaternion matrix equation (1.1) to have least-square bisymmetric solutions and investigate the expression of these solutions.

Theorem 2.3. *Suppose that*

$$D = \frac{1}{2}\begin{bmatrix} I_k & V_k \\ -V_k & I_k \end{bmatrix}, D^*A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, D^*B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

$$A_1 = U \begin{bmatrix} \Sigma_{r_1} & 0 \\ 0 & 0 \end{bmatrix} V^* = U_1 \Sigma_1 V_1^*$$

and

$$A_2 = P \begin{bmatrix} \Sigma_{r_2} & 0 \\ 0 & 0 \end{bmatrix} = P_1 \Sigma_{r_2} Q_1^*$$

are the singular value decompositions of A_1, A_2, respectively, where A_1, A_2, $B_1, B_2 \in H^{k\times m}$, and $U = (U_1, U_2), V = (V_1, V_2), P = (P_1, P_2), Q = (Q_1, Q_2)$ are unitary matrices. Then:

(1) *The quaternion matrix equation (1.1) has least-square bisymmetric solutions if and only if $(2B_1V_1\Sigma_{r_1}^{-1})^*U_1$, $(2B_2Q_1\Sigma_{r_2}^{-1})^*P_1$ 1000 are Hermitian.*

(2) *The general expression of the least-square bisymmetric solutions to (1.1) is*

$$X = D\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} D^* + D\begin{bmatrix} F_{U_1^*}K_1F_{U_1^*} & 0 \\ 0 & F_{P_1^*}K_2F_{P_1^*} \end{bmatrix} D^*, \tag{2.1}$$

where

$$M_1 = 2B_1V_1\Sigma_{r_1}^{-1}U_1^\dagger + (2B_1V_1\Sigma_{r_1}^{-1}U_1^\dagger)^* - 2(U_1^*)^\dagger U_1^* B_1 V_1 \Sigma_{r_1}^{-1} U_1^\dagger,$$

$$M_2 = 2B_2Q_1\Sigma_{r_2}^{-1}P_1^\dagger + (2B_2Q_1\Sigma_{r_2}^{-1}P_1^\dagger)^* - 2(P_1^*)^\dagger P_1^* B_2 Q_1 \Sigma_{r_2}^{-1} P_1^\dagger$$

and K_1, K_2 are arbitrary $K \times K$ Hermitian quaternion matrices.

Proof. From Lemma 2.1, it is easy to verify that $X \in B_{2k}$ if and only if there exists unitary matrix $\sqrt{2}D$ such that

$$2D^*X2D = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, \tag{2.2}$$

where

$$D = \frac{1}{2}\begin{bmatrix} I_k & V_k \\ -V_k & I_k \end{bmatrix}$$

and X_1, X_2 are $k \times k$ Hermitian matrices.

By the unitary invariance of the Frobenius norm and (2.2), we have

$$\begin{aligned}
\|XA - B\| &= \|\sqrt{2}D^*XA - \sqrt{2}D^*B\| \\
&= \|\sqrt{2}D^*X\sqrt{2}D\sqrt{2}D^*A - \sqrt{2}D^*B\| \\
&= \|\frac{1}{2}\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}\sqrt{2}D^*A - \sqrt{2}D^*B\| \\
&= \sqrt{2}\|\frac{1}{2}\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\| \\
&= \sqrt{2}\|\frac{1}{2}X_1A_1 - B_1\| + \sqrt{2}\|\frac{1}{2}X_2A_2 - B_2\| \\
&= \sqrt{2}\|\frac{1}{2}U^*X_1A_1V - U^*B_1V\| \\
&\quad +\sqrt{2}\|\frac{1}{2}P^*X_2A_2Q - P^*B_2Q\| \\
&= \sqrt{2}\|\frac{1}{2}U^*X_1UU^*A_1V - U^*B_1V\| \\
&\quad +\sqrt{2}\|\frac{1}{2}P^*X_2PP^*A_2Q - P^*B_2Q\| \\
&= \sqrt{2}\|\frac{1}{2}U_1^*X_1U_1\Sigma_{r_1} - U_1^*B_1V_1\| + \sqrt{2}\| - U_1^*B_1V_2\| \\
&\quad +\sqrt{2}\|\frac{1}{2}U_2^*X_1U_1\Sigma_{r_1} - U_2^*B_1V_1\| + \sqrt{2}\| - U_2^*B_1V_2\| \\
&\quad +\sqrt{2}\|\frac{1}{2}P_1^*X_2P_1\Sigma_{r_2} - P_1^*B_2Q_1\| + \sqrt{2}\| - P_1^*B_2Q_2\| \\
&\quad +\sqrt{2}\|\frac{1}{2}P_2^*X_2P_1\Sigma_{r_2} - P_2^*B_2Q_1\| + \sqrt{2}\| - P_2^*B_2Q_2\|.
\end{aligned}$$

Therefore, $\|XA - B\| = \min_{X\in B_{2k}}$ holds if and only if

$$\begin{aligned}
\frac{1}{2}U_1^* X_1 U_1 \Sigma_{r_1} &= U_1^* B_1 V_1,\\
\frac{1}{2}U_2^* X_1 U_1 \Sigma_{r_1} &= U_2^* B_1 V_1,\\
\frac{1}{2}P_1^* X_2 P_1 \Sigma_{r_2} &= P_1^* B_2 Q_1,\\
\frac{1}{2}P_2^* X_2 P_1 \Sigma_{r_2} &= P_2^* B_2 Q_1.
\end{aligned}$$

It is easy to find that

$$\begin{aligned}
U^* X_1 U &= \begin{bmatrix} U_1^* X_1 U_1 & U_1^* X_1 U_2 \\ U_2^* X_1 U_1 & U_2^* X_1 U_2 \end{bmatrix}\\
&= \begin{bmatrix} 2U_1^* B_1 V_1 \Sigma_{r_1}^{-1} & U_1^* X_1 U_2 \\ 2U_2^* B_1 V_1 \Sigma_{r_1}^{-1} & U_2^* X_1 U_2 \end{bmatrix}
\end{aligned}$$

and

$$U \begin{bmatrix} 2U_1^* B_1 V_1 \Sigma_{r_1}^{-1} \\ 2U_2^* B_1 V_1 \Sigma_{r_1}^{-1} \end{bmatrix} = 2B_1 V_1 \Sigma_{r_1}^{-1}, \quad U \begin{bmatrix} U_1^* X_1 U_2 \\ U_2^* X_1 U_2 \end{bmatrix} = X_1 U_2.$$

So we have

$$UU^* X_1 U = (2B_1 V_1 \Sigma_{r_1}^{-1}, X_1 U_2),$$

i.e.,

$$(X_1 U_1, X_1 U_2) = (2B_1 V_1 \Sigma_{r_1}^{-1}, X_1 U_2).$$

So we get a matrix equation

$$X_1 U_1 = 2B_1 V_1 \Sigma_{r_1}^{-1} \tag{2.3}$$

for Hermitian matrix X_1.
Similarly, for Hermitian matrix X_2, we have a matrix equation

$$X_2 P_1 = 2B_2 Q_1 \Sigma_{r_2}^{-1}. \tag{2.4}$$

Therefore, the equation (1.1) has least-square bisymmetric solution if and only if the matrix equations (2.3) and (2.4) have Hermitian solutions for X_1, X_2, respectively.

By Lemma 2.2, the matrix equation (2.3) has a Hermitian solution if and only if $(2B_1 V_1 \Sigma_{r_1}^{-1})^* U_1$ is Hermitian, in which case a general Hermitian solution is

$$X_1 = M_1 + F_{U_1^*} K_1 F_{U_1^*},$$

where

$$M_1 = 2B_1 V_1 \Sigma_{r_1}^{-1} U_1^{\dagger} + (2B_1 V_1 \Sigma_{r_1}^{-1} U_1^{\dagger})^* - 2(U_1^*)^{\dagger} U_1^* B_1 V_1 \Sigma_{r_1}^{-1} U_1^{\dagger}$$

and K_1 is an arbitrary $K \times K$ Hermitian matrix.

Similarly, the matrix equation (2.4) has a Hermitian solution if and only if $(2B_2Q_1\Sigma_{r_2}^{-1})^*P_1$ is Hermitian. The general Hermitian solution is

$$X_2 = M_2 + F_{P_1^*}K_2F_{P_1^*},$$

where

$$M_2 = 2B_2Q_1\Sigma_{r_2}^{-1}P_1^\dagger + (2B_2Q_1\Sigma_{r_2}^{-1}P_1^\dagger)^* - 2(P_1^*)^\dagger P_1^*B_2Q_1\Sigma_{r_2}^{-1}P_1^\dagger$$

and K_2 is an arbitrary $K \times K$ Hermitian matrix. Hence the expression of the least-square bisymmetric solutions can be written as

$$X = D\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}D^* + D\begin{bmatrix} F_{U_1^*}K_1F_{U_1^*} & 0 \\ 0 & F_{P_1^*}K_2F_{P_1^*} \end{bmatrix}D^*,$$

where K_1, K_2 are arbitrary $K \times K$ Hermitian quaternion matrices. $\square$

Theorem 2.4. *Suppose that*

$$D = \frac{1}{2}\begin{bmatrix} I_k & 0 & V_k \\ 0 & 1 & 0 \\ -V_k & 0 & I_k \end{bmatrix}, D^*A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, D^*B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

$A_1, B_1 \in H^{k\times m}$, $A_2, B_2 \in H^{(k+1)\times m}$.

$$A_1 = U\begin{bmatrix} \Sigma_{r_1} & 0 \\ 0 & 0 \end{bmatrix}V^* = U_1\Sigma_1V_1^*$$

and

$$A_2 = P\begin{bmatrix} \Sigma_{r_2} & 0 \\ 0 & 0 \end{bmatrix} = P_1\Sigma_{r_2}Q_1^*$$

are the singular value decompositions of A_1, A_2*, respectively, where* $U = (U_1, U_2), V = (V_1, V_2), P = (P_1, P_2), Q = (Q_1, Q_2)$ *are unitary matrices. Then:*

(1) *The quaternion matrix equation* (1.1) *has least-square bisymmetric solutions if and only if* $(2B_1V_1\Sigma_{r_1}^{-1})^*U_1$, $(2B_2Q_1\Sigma_{r_2}^{-1})^*P_1$ *are Hermitian.*
(2) *The general expression of a least-square bisymmetric solution to* (1.1) *is the same as* (2.1) *where* K_1, K_2 *are arbitrary Hermitian quaternion matrices with appropriate sizes.*

Remark 2.5. *Let*

$$D = \frac{1}{2}\begin{bmatrix} I_k & 0 & V_k \\ 0 & 1 & 0 \\ -V_k & 0 & I_k \end{bmatrix}$$

and $A_2, B_2 \in H^{(k+1)\times m}$, we find that the proof of Theorem 2.4 is similar to Theorem 2.3. Therefore, we skip it.

3. The least-square bisymmetric solutions to (1.1) with extreme inertias

In this section, we consider the extreme inertias of least-square bisymmetric solutions to (1.1) and characterize the least-square bisymmetric solutions with extreme inertias. Finally, we get some necessary and sufficient conditions for matrix equation (1.1) to have positive and non-negative least-square bisymmetric solutions.

First we give some equalities for the ranks of partitioned matrices which were derived by Marsaglia and Styan [8].

Lemma 3.1. [8] *Let $A \in H^{m\times n}$, $B \in H^{m\times k}$. Then*

$$r(A, B) = r(A) + r([I - AA^{-}]B) = r([I - BB^{-}]A) + r(A),$$

$$r\begin{bmatrix} A \\ B \end{bmatrix} = r(A) + r(B[I - A^{-}A]) = r(A[I - B^{-}B]) + r(B).$$

For the inertias of quaternion matrices, there are some simple results.

Lemma 3.2. *Let $A \in H_h^{m\times m}$, $B \in H_h^{n\times n}$ and $P \in H^{m\times m}$ be nonsingular. Then*

$$IN_{\pm}(A) = \frac{1}{2} in_{\pm}(A^c);$$

$$IN_{\pm}(PAP^*) = IN_{\pm}(A);$$

$$IN_{\pm}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = IN_{\pm}(A) + IN_{\pm}(B).$$

Recently, Tian [11] and Chu [2] present some equalities for inertias of block Hermitian complex matrices. In the following we extend two results [11] from complex matrix to quaternion matrix by Lemma 3.2.

Lemma 3.3. *Let $A \in H_h^{m\times m}$, $B \in H^{m\times n}$ and*

$$M = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}.$$

Then the following equality holds for the inertia of M.

$$IN_{\pm}(M) = r(B) + IN_{\pm}(E_B A E_B).$$

Lemma 3.4. *Let $A \in H_h^{m\times m}$, $B \in H^{m\times n}$,*

$$M = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, S = \begin{bmatrix} 0 \\ I_n \end{bmatrix}, S_1 = S - MM^\dagger S.$$

Then the following statements hold:

(1) *The extreme inertias of $A - BXB^*$ are given by*

$$\max_{X\in H_h^{n\times n}} IN_\pm(A - BXB^*)) = IN_\pm(M),$$

$$\min_{X\in H_h^{n\times n}} IN_\pm(A - BXB^*) = r(A, B) - IN_\mp(M). \tag{3.1}$$

(2) *The general expressions of X satisfying $IN_+(A - BXB^*) = IN_+(M)$ can be written as*

$$X = -S^*M^\dagger S - UU^*,$$

where $U \in H^{n\times k}$ is chosen such that $IN_-(-F_{S_1}UU^*F_{S_1}) = r(F_{S_1})$; the general expressions of X satisfying $IN_-(A-BXB^*) = IN_-(M)$ can be written as

$$X = -S^*M^\dagger S + UU^*,$$

where $U \in H^{n\times k}$ is chosen such that $IN_+(F_{S_1}UU^*F_{S_1}) = r(F_{S_1})$.

(3) *The general expressions of X satisfying $IN_+(A - BXB^*) = r(A, B) - IN_-(M)$ can be written as*

$$X = -S^*M^\dagger S - UU^* + V - F_{S_1}VF_{S_1},$$

where $U \in H^{n\times k}, V \in H_h^{n\times n}$ are arbitrary Hermitian matrices.

(4) *The general expressions of X satisfying $IN_-(A - BXB^*) = r(A, B) - IN_+(M)$ can be written as*

$$X = -S^*M^\dagger S + UU^* + V - F_{S_1}VF_{S_1},$$

where $U \in H^{n\times k}, V \in H_h^{n\times n}$ are arbitrary Hermitian matrices.

In the following, we investigate the extreme inertias of the least-square bisymmetric solutions to the quaternion matrix equation (1.1) and characterize the least-square bisymmetric solutions with extreme inertias.

Theorem 3.5. *With the assumption of Theorem 2.3, let*

$$\widehat{M_1} = \begin{bmatrix} M_1 & F_{U_1^*} \\ (F_{U_1^*})^* & 0 \end{bmatrix}, \widehat{M_2} = \begin{bmatrix} M_2 & F_{P_1^*} \\ (F_{P_1^*})^* & 0 \end{bmatrix}, S = \begin{bmatrix} 0 \\ I_k \end{bmatrix},$$

$$S_i = S - \widehat{M_i}\widehat{M_i}^\dagger S, (i = 1, 2)$$

and S_X be the set of the least-square bisymmetric solutions to (1.1). *If the quaternion matrix equation* (1.1) *has least-square bisymmetric solutions, then the extreme inertias of the least-square bisymmetric solutions to* (1.1) *are*

$$\begin{aligned}\max_{X\in S_X} IN_{\pm}(X) &= 2k-(r_1+r_2)+IN_{\pm}(2U_1U_1^{\dagger}B_1V_1\Sigma_{r_1}^{-1}U_1^{\dagger}) \\ &\quad +IN_{\pm}(2P_1P_1^{\dagger}B1000_2Q_1\Sigma_{r_2}^{-1}P_1^{\dagger}).\end{aligned}$$

$$\begin{aligned}\min_{X\in S_X} IN_{\pm}(X) &= r(B_1V_1)+r(B_2Q_1)-IN_{\mp}(2U_1U_1^{\dagger}B_1V_1\Sigma_{r_1}^{-1}U_1^{\dagger}) \\ &\quad -IN_{\mp}(2P_1P_1^{\dagger}B_2Q_1\Sigma_{r_2}^{-1}P_1^{\dagger}).\end{aligned}$$

Proof. By Lemmas 3.2 3.4, the extreme inertias of the least-square bisymmetric solutions to (1.1) satisfy the following equations

$$\begin{aligned}\max_{X\in S_X} IN_{\pm}(X) &= \max_{K_1=K_1^*} IN_{\pm}(M_1+F_{U_1^*}K_1E_{U_1}) \\ &\quad +\max_{K_2=K_2^*} IN_{\pm}(M_2+F_{P_1^*}K_2E_{P_1}) \\ &= IN_{\pm}(\widehat{M_1})+IN_{\pm}(\widehat{M_2}). \\ \min_{X\in S_X} IN_{\pm}(X) &= \min_{K_1K_1^*} IN_{\pm}(M_1+F_{U_1^*}K_1E_{U_1}) \\ &\quad +\min_{K_2=K_2^*} IN_{\pm}(M_2+F_{P_1^*}K_2E_{P_1}) \\ &= r(M_1,F_{U_1^*})+r(M_2,F_{P_1^*})-IN_{\mp}(\widehat{M_1})-IN_{\mp}(\widehat{M_2}). \qquad (3.2)\end{aligned}$$

By Lemma 3.3, we obtain

$$\begin{aligned}IN_{\pm}(\widehat{M_1}) &= r(F_{U_1^*})+IN_{\pm}(U_1U_1^{\dagger}M_1U_1U_1^{\dagger}) \\ &= k-r_1+IN_{\pm}(2U_1U_1^{\dagger}B_1V_1\Sigma_{r_1}^{-1}U_1^{\dagger}). \\ IN_{\pm}(\widehat{M_1}) &= k-r_2+IN_{\pm}(2P_1P_1^{\dagger}B_2Q_1\Sigma_{r_2}^{-1}P_1^{\dagger}).\end{aligned}$$

By Lemma 3.1 and elementary block matrix operations, we can get

$$\begin{aligned}
r\begin{bmatrix} M_1 & I_k \\ 0 & U_1^* \end{bmatrix} &= r(M_1, F_{U_1^*}) + r(U_1) \\
&= r(M_1, F_{U_1^*}) + r_1, \\
r\begin{bmatrix} M_1 & I_k \\ 0 & U_1^* \end{bmatrix} &= r\begin{bmatrix} M_1 & I_k \\ -U_1^* M_1 & 0 \end{bmatrix} \\
&= r\begin{bmatrix} 0 & I_k \\ -2(B_1 V_1 \Sigma_{r_1}^{-1})^* & 0 \end{bmatrix} \\
&= k + r(B_1 V_1).
\end{aligned}$$

The above equalities, gives that

$$r(M_1, F_{U_1^*}) = k + r(B_1 V_1) - r_1. \tag{3.3}$$

Similarly, we can verify that

$$r(M_2, F_{P_1^*}) = k + r(B_2 Q_1) - r_2. \tag{3.4}$$

By Lemma 3.3 and substituting (3.3), (3.4) into (3.2), we get the result. $\square$

By Lemma 3.4 and Theorem 3.5, we easily can get the following Theorem.

Theorem 3.6. *With the assumption of Theorem 3.5,*

(1) *if $X \in S_X$ satisfies*

$$\begin{aligned}
\max_{X \in S_X} IN_+(X) &= 2k - (r_1 + r_2) + IN_+(2U_1 U_1^\dagger B_1 V_1 \Sigma_{r_1}^{-1} U_1^\dagger) \\
&\quad + IN_+(2P_1 P_1^\dagger B_2 Q_1 \Sigma_{r_2}^{-1} P_1^\dagger),
\end{aligned}$$

then X can be expressed as

$$X = D \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} D^*,$$

where

$$X_1 = M_1 + F_{U_1^*}(-S^* \widehat{M_1}^\dagger S - N_1 N_1^*) F_{U_1^*},$$

$$X_2 = M_2 + F_{P_1^*}(-S^* \widehat{M_2}^\dagger S - N_2 N_2^*) F_{P_1^*}$$

and $N_i (i = 1, 2)$ are chosen such that

$$IN_-(-F_{S_1} N_1 N_1^* F_{S_1}) = r(F_{S_1}),$$
$$IN_-(-F_{S_2} N_2 N_2^* F_{S_2}) = r(F_{S_2}).$$

(2) *If $X \in S_X$ satisfies*

$$\begin{aligned}\max_{X\in S_X} IN_-(X) &= 2k-(r_1+r_2)+IN_-(2U_1U_1^\dagger B_1V_1\Sigma_{r_1}^{-1}U_1^\dagger)\\ &\quad +IN_-(2P_1P_1^\dagger B_2Q_1\Sigma_{r_2}^{-1}P_1^\dagger),\end{aligned}$$

then X can be written as

$$X = D\begin{bmatrix} X_1 & 0\\ 0 & X_2\end{bmatrix}D^*,$$

where

$$X_1 = M_1 + F_{U_1^*}(-S^*\widehat{M_1}^\dagger S + N_1N_1^*)F_{U_1^*},$$

$$X_2 = M_2 + F_{P_1^*}(-S^*\widehat{M_2}^\dagger S + N_2N_2^*)F_{P_1^*}$$

and $N_i (i=1,2)$ are chosen such that

$$IN_+(F_{S_1}N_1N_1^*F_{S_1}) = r(F_{S_1}),$$
$$IN_+(F_{S_2}N_2N_2^*F_{S_2}) = r(F_{S_2}).$$

(3) *If $X \in S_X$ satisfies*

$$\begin{aligned}\min_{X\in S_X} IN_+(X) &= r(B_1V_1)+r(B_2Q_1)-IN_-(2U_1U_1^\dagger B_1V_1\Sigma_{r_1}^{-1}U_1^\dagger)\\ &\quad -IN_-(2P_1P_1^\dagger B_2Q_1\Sigma_{r_2}^{-1}P_1^\dagger),\end{aligned}$$

then of X can be written as

$$X = D\begin{bmatrix} X_1 & 0\\ 0 & X_2\end{bmatrix}D^*,$$

where

$$X_1 = M_1 + F_{U_1^*}(-S^*\widehat{M_1}^\dagger S - N_1N_1^* + W_1 - F_{S_1}W_1F_{S_1})F_{U_1^*},$$

$$X_2 = M_2 + F_{P_1^*}(-S^*\widehat{M_2}^\dagger S - N_2N_2^* + W_2 - F_{S_2}W_2F_{S_2})F_{P_1^*}$$

and $N_i (i=1,2)$ are arbitrary quaternion matrices and $W_i (i=1,2)$ are arbitrary Hermitian quaternion matrices.

(4) *If $X \in S_X$ satisfies*

$$\begin{aligned}\min_{X\in S_X} IN_-(X) &= r(B_1V_1)+r(B_2Q_1)-IN_+(2U_1U_1^\dagger B_1V_1\Sigma_{r_1}^{-1}U_1^\dagger)\\ &\quad -IN_+(2P_1P_1^\dagger B_2Q_1\Sigma_{r_2}^{-1}P_1^\dagger),\end{aligned}$$

then X can be written as

$$X = D\begin{bmatrix} X_1 & 0\\ 0 & X_2\end{bmatrix}D^*,$$

where

$$X_1 = M_1 + F_{U_1^*}(-S^*\widehat{M_1}^{\dagger}S + N_1N_1^* + W_1 - F_{S_1}W_1F_{S_1})F_{U_1^*},$$

$$X_2 = M_2 + F_{P_1^*}(-S^*\widehat{M_2}^{\dagger}S + N_2N_2^* + W_2 - F_{S_2}W_2F_{S_2})F_{P_1^*}.$$

and $N_i (i = 1, 2)$ *are arbitrary quaternion matrices and* $W_i (i = 1, 2)$ *are arbitrary Hermitian quaternion matrices.*

Theorem 3.7. *With the notations as in Theorem 2.3, assume that the matrix equation* $XA = B$ *has least-square bisymmetric solutions and* S_X *is the set of the least-square bisymmetric solutions. Then:*

(1) *There exists a positive definite least-square bisymmetric solution to* $XA = B$ *if and only if*

$$r_1 = IN_+(2U_1U_1^{\dagger}B_1V_1\Sigma_{r_1}^{-1}U_1^{\dagger}),$$
$$r_2 = IN_+(2P_1P_1^{\dagger}B_2Q_1\Sigma_{r_2}^{-1}P_1^{\dagger}).$$

(2) *There exists a nonnegative definite least-square bisymmetric solution to* $XA = B$ *if and only if*

$$r(B_1V_1) = IN_+(2U_1U_1^{\dagger}B_1V_1\Sigma_{r_1}^{-1}U_1^{\dagger}),$$
$$r(B_2Q_1) = IN_+(2P_1P_1^{\dagger}B_2Q_1\Sigma_{r_2}^{-1}P_1^{\dagger}).$$

Proof. It is obvious that

$$X = D\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}D^* \in S_X$$

is positive definite if and only if $\max_{X\in S_X} IN_+(X) = 2k$.

By Theorem 3.5, we obtain

$$\begin{aligned}\max_{X\in S_X} IN_+(X) = 2k \iff & \max_{X_1} IN_+(X_1) = k, \max_{X_2} IN_+(X_2) = k \\ \iff & r_1 = IN_+(2U_1U_1^{\dagger}B_1V_1\Sigma_{r_1}^{-1}U_1^{\dagger}), \\ & r_2 = IN_+(2P_1P_1^{\dagger}B_2Q_1\Sigma_{r_2}^{-1}P_1^{\dagger}).\end{aligned}$$

Similarly, $X \in S_X$ is nonnegative definite if and only if $\min_{X\in S_X} IN_-(X) = 0$.
Moreover,

$$\begin{aligned}\min_{X\in S_X} IN_-(X) = 0 \iff & \min_{X_1} IN_+(X_1) = 0, \min_{X_2} IN_+(X_2) = 0 \\ \iff & r(B_1V_1) = IN_+(2U_1U_1^{\dagger}B_1V_1\Sigma_{r_1}^{-1}U_1^{\dagger}), \\ & r(B_2Q_1) = IN_+(2P_1P_1^{\dagger}B_2Q_1\Sigma_{r_2}^{-1}P_1^{\dagger}).\end{aligned}$$

□

Remark 3.8. *Any*

$$X = D \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} D^* \in S_X$$

is positive (nonnegative) definite if and only if $\min_{X\in S_X} IN_+(X) = 2k$ $(\max_{X\in S_X} IN_-(X) = 0)$. *So we can derive the sufficient and necessary conditions for all least-square bisymmetric solutions to* $XA = B$ *to be positive (non-negative) definite.*

4. The extremal least-square bisymmetric solutions

In this section, we consider the maximal and minimal least-square bisymmetric solutions by means of inertia theory. For convenience, we give some notations. The notation $X \geq 0$ ($X \leq 0$) means that the matrix X is non-negative definite (non-positive definite). Let $F(X) = 0$ be a matrix equation. A hermitian solution P_0 of $F(X) = 0$ will be called maximal (respectively minimal) if $P_0 - P \geq 0 (respectively P_0 - P \leq 0)$ for every other hermitian solution P of $F(X) = 0$. Clearly, the extremal solution is unique if it exists.

Theorem 4.1. *With notations as in Theorem 2.3, assume that the matrix equation* (1.1) *has least-square bisymmetric solutions. Then:*

(1) *There exists the maximal least-square bisymmetric solution*

$$P_0 = D \begin{bmatrix} P_{01} & 0 \\ 0 & P_{02} \end{bmatrix} D^*$$

for (1.1) *if and only if*

$$IN_+(U_1U_1^\dagger P_{01} - 2U_1U_1^\dagger B_1V_1\Sigma_{r_1}^{-1}U_1^\dagger) = r(U_1U_1^\dagger P_{01} - U_1U_1^\dagger 2B_1V_1\Sigma_{r_1}^{-1}U_1^\dagger),$$

$$IN_+(P_1P_1^\dagger P_{02} - 2P_1P_1^\dagger B_2Q_1\Sigma_{r_2}^{-1}P_1^\dagger) = r(P_1P_1^\dagger P_{02} - P_1P_1^\dagger 2B_2Q_1\Sigma_{r_2}^{-1}P_1^\dagger).$$

(2) *There exists the minimal least-square bisymmetric solution*

$$P_0 = D \begin{bmatrix} P_{01} & 0 \\ 0 & P_{02} \end{bmatrix} D^*$$

for (1.1) *if and only if*

$$IN_-(U_1U_1^\dagger P_{01} - 2U_1U_1^\dagger B_1V_1\Sigma_{r_1}^{-1}U_1^\dagger) = r(U_1U_1^\dagger P_{01} - U_1U_1^\dagger 2B_1V_1\Sigma_{r_1}^{-1}U_1^\dagger),$$

$$IN_-(P_1P_1^\dagger P_{02} - 2P_1P_1^\dagger B_2Q_1\Sigma_{r_2}^{-1}P_1^\dagger) = r(P_1P_1^\dagger P_{02} - P_1P_1^\dagger 2B_2Q_1\Sigma_{r_2}^{-1}P_1^\dagger).$$

Proof. Suppose that

$$X = D \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} D^*$$

is a least-square bisymmetric solution to (1.1). By Theorem 2.3, X can be expressed as (2.1).

As we know

$$\begin{aligned} P_0 \geq X &\iff \min_{XA=B, X\in H_h^{2k\times 2k}} IN_-(P_0 - X) = 0 \\ &\iff \min_{X_1} IN_-(P_{01} - X_1) = 0, \min_{X_2} IN_-(P_{02} - X_2) = 0. \\ P_0 \leq X &\iff \min_{XA=B, X\in H_h^{2k\times 2k}} IN_+(P_0 - X) = 0 \\ &\iff \min_{X_1} IN_+(P_{01} - X_1) = 0, \min_{X_2} IN_+(P_{02} - X_2) = 0. \end{aligned}$$

By (3.1), Lemmas 3.1 and 3.4, we get

$$\begin{aligned} \min_{X_1} IN_\pm(P_{01} - X_1) &= r\big(P_{01} - M_1, F_{U_1^*}\big) - IN_\mp \begin{bmatrix} P_{01} - M_1 & F_{U_1^*} \\ F_{U_1^*} & 0 \end{bmatrix} \\ &= r(F_{U_1^*}) + r(U_1U_1^\dagger(P_{01} - M_1)) \\ &\quad - r(F_{U_1^*}) - IN_\mp(U_1U_1^\dagger(P_{01} - M_1)U_1U_1^\dagger) \\ &= r(U_1U_1^\dagger(P_{01} - 2B_1V_1\Sigma_{r_1}^{-1}U_1^\dagger)) \\ &\quad - IN_\mp(U_1U_1^\dagger(P_{01} - 2B_1V_1\Sigma_{r_1}^{-1}U_1^\dagger)U_1U_1^\dagger). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \min_{X_2} IN_\pm(P_{02} - X_2) &= r(P_1P_1^\dagger(P_{02} - 2B_2Q_1\Sigma_{r_2}^{-1}P_1^\dagger)) \\ &\quad - IN_\mp(P_1P_1^\dagger(P_{02} - 2B_2Q_1\Sigma_{r_2}^{-1}P_1^\dagger)P_1P_1^\dagger). \end{aligned}$$

□

5. **Conclusion**

In this paper, we investigate the least-square bisymmetric solutions to quaternion matrix equation (1.1) and give the expression of these solutions. We investigate the extreme inertias of these least-square bisymmetric solutions and characterize the solutions with extreme inertias. By the inertia theory, we get sufficient and necessary conditions for the matrix equation (1.1) to have positive and nonnegative least-square bisymmetric solutions. Finally, we derive sufficient and necessary conditions for (1.1) to have maximal and minimal least-square bisymmetric solutions.

From the proof of Theorem 2.3, we can get a necessary and sufficient condition for (1.1) to have bisymmetric solutions and give the expression of these bisymmetric solutions as follows.

Theorem 5.1. *With the assumption as in Theorem 2.3, the quaternion matrix equation* (1.1) *has bisymmetric solutions if and only if* $(2B_1V_1\Sigma_{r_1}^{-1})^*U_1$, $(2B_2Q_1\Sigma_{r_2}^{-1})^*P_1$ *are Hermitian and* $B_1V_2 = 0, B_2Q_2 = 0$.

The expression of the bisymmetric solutions is the same as (2.1).

So we can investigate the extreme inertias of bisymmetric solutions and get some necessary and sufficient conditions for matrix equation (1.1) to have positive and nonnegative bisymmetric solutions and to have the maximal and minimal bisymmetric solutions. so we can consider the system

$$\begin{cases} AX = C \\ XB = D \end{cases}, \tag{5.1}$$

where $A, C \in H^{m\times n}$, $B, D \in H^{n\times p}$. As we know, system (5.1) is equivalent to

$$X(A^*, B) = (C^*, D).$$

So we can get the results about matrix equation system (5.1) as in Theorem 5.1.

In Section 4 we derive necessary and sufficient condition for matrix equation (1.1) to have the maximal and minimal least-square bisymmetric solutions. But to our knowledge, there have been little information on characterizing the extreme solutions. So we propose the following question:

How one characterize the maximal and minimal (least-square) bisymmetric solutions to (1.1)?

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