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MAXIMAL SUBSETS OF PAIRWISE NON-COMMUTING ELEMENTS OF SOME FINITE p-GROUPS

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ABSTRACT. Let G be a group. A subset X of G is a set of pairwise non-commuting elements if $xy \neq yx$ for any two distinct elements x and y in X. If $|X| \geq |Y|$ for any other set of pairwise non-commuting elements, Y in G, then X is said to be a maximal subset of pairwise non-commuting elements. Here, we determine the cardinality of a maximal subset of pairwise non-commuting elements in any non-abelian p-groups with central quotient of order less than or equal to p^3 for any prime number p. As an immediate consequence, we give this cardinality for any non-abelian group of order p^4.

1. Introduction

Let G be a non-abelian group and let X be a maximal subset of pairwise non-commuting elements of G. The cardinality of such a subset is denoted by $\omega(G)$. Also, $\omega(G)$ is the maximal clique size in the non-commuting graph of a group G. Let $Z(G)$ be the center of G. The non-commuting graph of a group G is a graph with $G \backslash Z(G)$ as the vertices and join two distinct vertices x and y, whenever $xy \neq yx$. By a famous result of Neumann [7], answering a question of Erdös, the

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finiteness of $\omega(G)$ in G is equivalent to the finiteness of the factor group $G/Z(G)$. Pyber [8] has shown that there is a constant c such that $|G : Z(G)| \leq c^{\omega(G)}$. Chin [4] obtained upper and lower bounds for $\omega(G)$ for an extra-special p-group G, where p is an odd prime number. For $p = 2$, Isaacs (see [3], p. 40) showed that $\omega(G) = 2n + 1$ for any extra-special group G of order 2^{2n+1}. Also, in [1, Lemma 4.4], it was proved that $\omega(GL(2, q)) = q^2 + q + 1$. Furthermore, in [2, Theorem 1.1], it was shown that $\omega(GL(3, q)) = q^6 + q^5 + 3q^4 + 3q^3 + q^2 - q - 1$, for $q \geq 4$, $\omega(GL(3, 2)) = 56$ and $\omega(GL(3, 3)) = 1067$.
Here, we show that $\omega(G) = p + 1$, for any finite p-group G with central quotient of order p^2, where p is a prime number (Lemma 3.1). Also, we find $\omega(G)$, for any finite p-group G with central quotient of order p^3 (Theorem 3.3). As an immediate consequence, we determine $\omega(G)$ for any non-abelian group of order p^4.

Throughout this paper, we use the following notation: p denotes a prime number, $\mathcal{C}_G(x)$ is the centralizer of an element x in a group G, the nilpotency class of a group G is shown by $\mathrm{cl}(G)$, and a p-group of maximal class is a non-abelian group G of order p^n with $\mathrm{cl}(G) = n - 1$.

2. **Basic results**

In this section, we give some basic results needed for are main results.

Lemma 2.1. *Let G be a finite group. Then,*

(i) *for any subgroup H of G, $\omega(H) \leq \omega(G)$, and*
(ii) *for any normal subgroup N of G, $\omega(G/N) \leq \omega(G)$.*

Proof. This is evident. □

A group G is called an AC-group, if the centralizer of every non-central element of G is abelian.

Lemma 2.2. *The followings on a group G are equivalent.*

(i) *G is an AC-group.*
(ii) *If $[x, y] = 1$, then $\mathcal{C}_G(x) = \mathcal{C}_G(y)$, where $x, y \in G\backslash Z(G)$.*
(iii) *If $[x, y] = [x, z] = 1$, then $[y, z] = 1$, where $x \in G\backslash Z(G)$.*
(iv) *If A and B are subgroups of G and $Z(G) < \mathcal{C}_G(A) \leq \mathcal{C}_G(B) < G$, then $\mathcal{C}_G(A) = \mathcal{C}_G(B)$.*

Proof. This is straightforward. See also [9], Lemma 3.2. □

Lemma 2.3. *Let G be an AC-group.*

(i) *If $a, b \in G\backslash Z(G)$ with distinct centralizers, then $\mathcal{C}_G(a) \cap \mathcal{C}_G(b) = Z(G)$.*

(ii) *If $G = \cup_{i=1}^{k} \mathcal{C}_G(a_i)$, where $\mathcal{C}_G(a_i)$ and $\mathcal{C}_G(a_j)$ are distinct for $1 \leq i < j \leq k$, then $\{a_1 \dots a_k\}$ is a maximal set of pairwise non-commuting elements.*

Proof. (i) We see that $Z(G) \leq \mathcal{C}_G(a) \cap \mathcal{C}_G(b)$. If $Z(G) < \mathcal{C}_G(a) \cap \mathcal{C}_G(b)$, then there exists an element x in $\mathcal{C}_G(a) \cap \mathcal{C}_G(b)$ such that $x \notin Z(G)$. This means that $\mathcal{C}_G(a) = \mathcal{C}_G(x)$ and $\mathcal{C}_G(b) = \mathcal{C}_G(x)$, by Lemma 2.2 (ii), which is impossible.
(ii) By Lemma 2.2 (ii), $\{a_1, a_2, \dots a_k\}$ is a set of pairwise non-commuting elements. Suppose to the contrary that $\{b_1, b_2, \dots, b_t\}$ is another set of non-commuting elements of G with $t > k$. Then, we see that there exist positive integers r, s and i with $r \neq s$, $1 \leq r, s \leq t$ and $1 \leq i \leq k$, such that $b_r, b_s \in \mathcal{C}_G(a_i)$. This yields that $\mathcal{C}_G(b_r) = \mathcal{C}_G(b_s)$, by Lemma 2.2 (ii), or equivalently $b_r b_s = b_s b_r$, which is a contradiction. □

3. **Main results**

In this section, we determine the cardinality of a maximal subset of pairwise non-commuting elements in any p-groups with central quotient of order less than or equal to p^3. Then, we give this cardinality for any non-abelian group of order p^4.

Lemma 3.1. *Let G be a group of order p^n with the central quotient of order p^2, where p is a prime number. Then, $\omega(G) = p + 1$.*

Proof. First, we show that G ia an AC-group. Suppose that a is a non-central element of G. So, $Z(G) < \mathcal{C}_G(a)$. Therefore, $|\mathcal{C}_G(a)| = p^{n-1}$. Since $\mathcal{C}_G(a) = \langle Z(G), a\rangle$, we see that $\mathcal{C}_G(a)$ is abelian and so G is an AC-group. Now, since G is finite, we may write $G = \cup_{i=1}^{k} \mathcal{C}_G(a_i)$, where $\mathcal{C}_G(a_i)$ and $\mathcal{C}_G(a_j)$ are distinct for $1 \leq i < j \leq k$. Therefore, $X = \{a_1, a_2, \dots, a_k\}$ is a maximal subset of pairwise non-commuting elements of G, by Lemma 2.3 (ii). Thus, by Lemma 2.3 (i),

$$|G| = \sum_{i=1}^{k} (|\mathcal{C}_G(a_i)| - |Z(G)|) + |Z(G)|.$$

This yields that $p^n = k \times (p^{n-1} - p^{n-2}) + p^{n-2}$, and so $k = p + 1$. □

Lemma 3.2. *Let G be a group of order p^n with the central quotient of order p^3, where p is a prime number.*

(i) *G is an AC-group.*
(ii) *If G possesses an abelian maximal subgroup, then there exists an element x in $G \backslash Z(G)$ such that $\mathcal{C}_G(x)$ is of order p^{n-1} and $\mathcal{C}_G(x)$ is uniquely determined.*

Proof. (i) Let $x \in G \setminus Z(G)$. Then, $Z(G) < Z(\mathcal{C}_G(x)) \leq \mathcal{C}_G(x) < G$. This yields that $|\mathcal{C}_G(x) : Z(\mathcal{C}_G(x))|$ divides p, and so $\mathcal{C}_G(x)$ is abelian.
(ii) Let M be an abelian maximal subgroup of G and $x \in M \setminus Z(G)$. We see that $\mathcal{C}_G(x) = M$, since $M \leq \mathcal{C}_G(x) < G$. Now, if $\mathcal{C}_G(y)$ is of order p^{n-1} with $\mathcal{C}_G(x) \neq \mathcal{C}_G(y)$, then $\mathcal{C}_G(x) \cap \mathcal{C}_G(y) = Z(G)$, by Lemma 2.3 (i). Moreover, $|G : \mathcal{C}_G(x) \cap \mathcal{C}_G(y)| \leq |G : \mathcal{C}_G(x)||G : \mathcal{C}_G(y)| = p^2$, which is impossible. □

Theorem 3.3. *Let G be a group of order p^n with the central quotient of order p^3, where p is a prime number.*
(i) *If G possesses no abelian maximal subgroup, then $\omega(G) = p^2 + p + 1$.*
(ii) *If G possesses an abelian maximal subgroup, then $\omega(G) = p^2 + 1$.*

Proof. (i) For any non-central element x in G, we have $Z(G) < \mathcal{C}_G(x) < G$. Therefore, $|\mathcal{C}_G(x)| = p^{n-2}$, since G is an AC-group. Now, we may write
$G = \cup_{i=1}^{k} \mathcal{C}_G(a_i)$, where $\mathcal{C}_G(a_i)$ and $\mathcal{C}_G(a_j)$ are distinct, for $1 \leq i < j \leq k$. Therefore, $X = \{a_1, a_2, \ldots, a_k\}$ is a maximal subset of pairwise non-commuting elements of G, by Lemma 2.3 (ii). Thus, by Lemma 2.3(i),

$$|G| = \sum_{i=1}^{k} (|\mathcal{C}_G(a_i)| - |Z(G)|) + |Z(G)|.$$

This yields that $p^n = k \times (p^{n-2} - p^{n-3}) + p^{n-3}$, and so $k = p^2 + p + 1$.
(ii) By Lemma 3.2 (ii), there exists $a \in G \setminus Z(G)$ such that $\mathcal{C}_G(a)$ is of order p^{n-1} and this is the only centralizer of order p^{n-1}. Now, we may write $G = \cup_{i=1}^{k} \mathcal{C}_G(b_i)$ such that the elements of the union are distinct. Since $a \in G$, there exists $1 \leq i \leq k$ such that $a \in \mathcal{C}_G(b_i)$, and so $ab_i = b_i a$. Therefore, $\mathcal{C}_G(b_i) = \mathcal{C}_G(a)$, by Lemma 2.2 (ii). This means that $\mathcal{C}_G(a)$ is one of the elements of the union. We may assume that $\mathcal{C}_G(a) = \mathcal{C}_G(b_1)$. Hence, $G = \mathcal{C}_G(a) \cup \mathcal{C}_G(b_2) \cup \cdots \cup \mathcal{C}_G(b_k)$, where $|\mathcal{C}_G(b_i)| = p^{n-2}$, for $2 \leq i \leq k$. So, by using Lemma 2.3 (i), we deduce that $|G| = |\mathcal{C}_G(a)| + \sum_{i=2}^{k} (|\mathcal{C}_G(b_i)| - |Z(G)|)$, or equivalently $p^n = p^{n-1} + (k-1)(p^{n-2} - p^{n-3})$, and hence $k = p^2 + 1$. □

Corollary 3.4. *Let G be a non-abelian group of order p^4.*

(i) *If G is of maximal class, then $\omega(G) = 1 + p^2$.*
(ii) *If G is of class two, then $\omega(G) = 1 + p$.*

Proof. (i) By Lemma 3.2, we see that G is an AC-group, since $|Z(G)| = p$. Now, by considering class equation, there exists $x \in G \setminus Z(G)$ such that $|\mathcal{C}_G(x)| = p^3$. The rest follows from Theorem 3.3 (ii).
(ii) We claim that $|Z(G)| = p^2$. For otherwise, $|Z(G)| = p$, and so, by [6, Lemma 04], we have $\exp(G/Z(G)) = \exp(G') = p$. Therefore, G is an extra special group, which is a contradiction, by [10, Theorem 4.18]. Now, we can complete the proof by Lemma 3.1. □

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