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ESSENTIAL NORM OF GENERALIZED COMPOSITION OPERATORS FROM WEIGHTED DIRICHLET OR BLOCH TYPE SPACES TO $\mathcal{Q}_K$ TYPE SPACES

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Abstract. We obtain lower and upper estimates for the essential norms of generalized composition operators from weighted Dirichlet spaces or Bloch type spaces to $\mathcal{Q}_K$ type spaces.

1. Introduction

We denote by $\mathcal{H}(\mathbb{D})$ the space of holomorphic functions on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space $\mathcal{A}^p_\alpha$ consists of those $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^p_{\mathcal{A}^p_\alpha} := (\alpha + 1)\int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) < \infty,$$

where $dA(z)$ is the normalized Lebesgue area measure on $\mathbb{D}$. Note that when $p \geq 1$ the space $\mathcal{A}^p_\alpha$ is a Banach space with the norm $\|\cdot\|_{\mathcal{A}^p_\alpha}$. We refer to [15] for the theory of this space. The weighted Dirichlet space

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$\mathcal{D}^p_\alpha$ consists of those $f \in \mathcal{H}(\mathbb{D})$ such that $f' \in \mathcal{A}^p_\alpha$. Hence, for $f \in \mathcal{D}^p_\alpha$ we have

$$\|f\|^p_{\mathcal{D}^p_\alpha} := |f(0)|^p + \|f'\|^p_{\mathcal{A}^p_\alpha} < \infty.$$

It is well known that $\mathcal{A}^p_\alpha = \mathcal{D}^p_{\alpha+p}$ [15, Theorem 2.16]. For $a \in \mathbb{D}$, $G(z,a) = \log \frac{1}{|\sigma_a(z)|}$ is Green's function on $\mathbb{D}$, where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius transformation of $\mathbb{D}$. For a right-continuous and nondecreasing function $K : [0,\infty) \to [0,\infty)$, and for $0 < p < \infty$, $-2 < \alpha < \infty$, the $\mathcal{Q}_K$ type space denoted by $\mathcal{Q}_K(p,\alpha)$ consists of $f \in \mathcal{H}(\mathbb{D})$, for which,

$$\|f\|^p_{K,p,\alpha} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^\alpha K(G(z,a)) dA(z) < \infty.$$

The space $\mathcal{Q}_K(p,\alpha)$ is a Banach space with the norm $\|f\|_{\mathcal{Q}_K(p,\alpha)} := |f(0)| + \|f\|_{K,p,\alpha}$, when $p \geq 1$. These spaces were introduced in [12]. If $\alpha+2 = p$, $\mathcal{Q}_K(p,\alpha)$ is the Möbius invariant, i.e., $\|f \circ \sigma_a\|_{K,p,\alpha} = \|f\|_{K,p,\alpha}$, for all $a \in \mathbb{D}$. We say that the space $\mathcal{Q}_K(p,\alpha)$ is trivial if it contains constant functions only. If $\int_0^1 (1-r^2)^\alpha K(\log \frac{1}{r}) r dr = \infty$, then $\mathcal{Q}_K(p,\alpha)$ is trivial [12]. Throughout the paper, we assume

$$\int_0^1 (1-r^2)^\alpha K(\log \frac{1}{r}) r dr < \infty.$$

Now, we recall some particular cases. If $p = 2$ and $\alpha = 0$, then we have that $\mathcal{Q}_K(p,\alpha) = \mathcal{Q}_K$. For more about the spaces of $\mathcal{Q}$ classes, see [2, 13]. For $0 < s < \infty$, if $K(t) = t^s$, then $\mathcal{Q}_K(p,\alpha) = F(p,\alpha,s)$ (see [14, 16]).

For $\alpha > 0$, the Bloch type space $\mathcal{B}^\alpha$ and the little Bloch type space $\mathcal{B}^\alpha_0$ are defined to be

$$\mathcal{B}^\alpha = \{f \in \mathcal{H}(\mathbb{D}) : b_\alpha(f) = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty\},$$

$$\mathcal{B}^\alpha_0 = \{f \in \mathcal{H}(\mathbb{D}) : \lim_{|z|\to 1} (1-|z|^2)^\alpha |f'(z)| = 0\}.$$

The space $\mathcal{B}^\alpha$ is a Banach space with the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + b_\alpha(f)$, and $\mathcal{B}^\alpha_0$ is a closed subspace of $\mathcal{B}^\alpha$. By [12, Theorem 2.1], we have $\mathcal{Q}_K(p,\alpha) \subseteq \mathcal{B}^{\frac{\alpha+2}{p}}$ and $\mathcal{Q}_K(p,\alpha) = \mathcal{B}^{\frac{\alpha+2}{p}}$ if and only if

$$\int_0^1 (1-r^2)^{-2} K(\log \frac{1}{r}) r dr < \infty.$$

Recall that the essential norm $\|T\|_e$ of a bounded linear operator T is its distance (in the operator norm) from the compact operators, that is,

$$\|T\|_e = \inf_K \|T - K\|,$$

where the infimum is taken over all compact operators K. Note that $\|T\|_e = 0$ if and only if T is compact, so that estimates on $\|T\|_e$ lead to conditions for the operator T to be compact.

Let φ be an analytic self-map of $\mathbb{D}$ and $g \in \mathcal{H}(\mathbb{D})$. The generalized composition operator C_φ^g is defined by

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi, \quad f \in \mathcal{H}(\mathbb{D}),\ z \in \mathbb{D},$$

as introduced in [4]. When $g = \varphi'$, this operator is essentially the composition operator C_φ which is defined by $C_\varphi f = f \circ \varphi$. Li and Stevic [4] studied the compactness problem for the generalized composition operators between Zygmund spaces and between Bloch type spaces and between these two spaces. Zhu gave characterization of the compact generalized composition operators from the generalized weighted Bergman spaces to the Bloch type spaces in [17], also boundedness and compactness of composition operators from Bloch type spaces to $F(p, \alpha, s)$ spaces in [16] and weighted composition operators from $F(p, \alpha, s)$ spaces to H_μ^∞ spaces in [18]. The compactness of the generalized composition operators and the products of the Volterra type operators and composition operators between Q_K spaces were studied in [9]. Rättyä [8] studied the essential norm of composition operators from $\mathcal{D}_\alpha^p$ to the $\mathcal{Q}_s$-spaces in terms of the Nevanlinna counting function. Lindstrom et al.[5] obtained lower and upper estimates for the essential norm of a composition operator from the Bloch space $\mathcal{B} := \mathcal{B}^1$ into $\mathcal{Q}_p$. Ueki and Luo [11] characterized the essential norm of weighted composition operators between the weighted Bergman spaces on the ball of $\mathbb{C}^n$. Our aim here is to study the essential norms of bounded generalized composition operators from the weighted Dirichlet spaces or the Bloch type spaces to $\mathcal{Q}_K(p, \alpha)$ spaces. We remark that if $C_\varphi^g f \in \mathcal{Q}_K(q, \beta)$ for every f in $\mathcal{B}^\alpha$ or $\mathcal{D}_\alpha^p$, then by the closed Graph Theorem, the generalized composition operator C_φ^g from $\mathcal{B}^\alpha$ or $\mathcal{D}_\alpha^p$ into $\mathcal{Q}_K(q, \beta)$ is bounded.

Throughout this paper, constants are denoted by C, they are positive and not necessarily the same in all occurrences. The notation $a \simeq b$

means that there exist positive constants C_1 and C_2 such that $C_1 b \leq a \leq C_2 b$.

A positive Borel measure μ on $\mathbb{D}$ is called a bounded t-Carleson measure, if

$$\|\mu\|_t := \sup_I \frac{\mu(S(I))}{|I|^t} < \infty, \quad t \in (0, \infty),$$

where $|I|$ denotes the arc length of a subarc I of the boundary of $\mathbb{D}$,

$$S(I) = \{z \in \mathbb{D} : \frac{z}{|z|} \in I, 1 - |I| \leq |z|\},$$

is the Carleson box based on I, and the supremum is taken over all subarcs I with $|I| \leq 1$.

To prove one of our main results (Theorem 2.1), we need the following lemma which characterizes the Carleson measure in terms of functions in the weighted Bergman spaces.

Lemma 1.1. [7] *Let μ be a positive measure on $\mathbb{D}$, and let $0 < p \leq q < \infty$. Then, μ is a bounded $(\alpha+2)\frac{q}{p}$-Carleson measure if and only if there is a positive constant C, depending only on p, q and α, such that*

$$\int_{\mathbb{D}} |f(z)|^q d\mu(z) \leq C \|\mu\|_{(\alpha+2)\frac{q}{p}} \|f\|^q_{\mathcal{A}^p_\alpha},$$

for all $f \in \mathcal{A}^p_\alpha$.

Let X and Y be Banach spaces of functions in $\mathcal{H}(\mathbb{D})$ and $C^g_\varphi : X \to Y$ be a generalized composition operator. For an analytic function $f(z) = \sum_{k=0}^\infty a_k z^k$ in X and any integer $n \geq 1$, define

$$R_n f(z) = \sum_{k=n+1}^\infty a_k z^k,$$

and $T_n = id - R_n$, where id is the identity operator on X. By compactness of $T_n : X \to X$, we have

$$\|C^g_\varphi\|_e = \|C^g_\varphi(T_n + R_n)\|_e = \|C^g_\varphi R_n\|_e \leq \|C^g_\varphi R_n\|,$$

for each $n \in \mathbb{N}$. Hence,

$$\|C^g_\varphi\|_e \leq \liminf_{n\to\infty} \|C^g_\varphi R_n\|. \tag{1.1}$$

The following lemma also plays an important role in the proof of Theorem 2.1.

Lemma 1.2. [11, Corollary 2.1] *Let $\alpha > -1$ and $1 < p < \infty$. Then, $\|R_n f\|_{\mathcal{A}^p_\alpha} \to 0$, as $n \to \infty$, for each $f \in \mathcal{A}^p_\alpha$.*

Using the uniform boundedness principle, Lemma 1.2 implies that the sequence $(R_n)_{n\in N}$ is a norm bounded sequence of operators on $\mathcal{A}^p_\alpha$.

2. Main result

For $0 < q < \infty, -2 < \beta < \infty$, $a \in \mathbb{D}$ and $g \in \mathcal{H}(\mathbb{D})$, we define the generalized weighted Lebesgue measure $\mu_{a,g}$ by

$$d\mu_{a,g}(z) = |g(z)|^q (1-|z|^2)^\beta K(G(z,a))dA(z).$$

Also, for any analytic self-map φ of $\mathbb{D}$, we define another measure $\mu_{a,g,\varphi}$ by $\mu_{a,g,\varphi} = \mu_{a,g} \circ \varphi^{-1}$ and call it the pull-back measure of $\mu_{a,g}$ induced by φ. By the measure theory, for any function $f \in \mathcal{H}(\mathbb{D})$, we can write

$$\int_{\mathbb{D}} |f(\varphi(z))| d\mu_{a,g}(z) = \int_{\mathbb{D}} |f(z)| d\mu_{a,g,\varphi}(z).$$

Using the above notation we now establish the following result.

Theorem 2.1. *Let $1 < p \le q < \infty$, $\alpha > -1, \beta > -2$, and let $C^g_\varphi : \mathcal{D}^p_\alpha \to \mathcal{Q}_K(q,\beta)$ be bounded. Then,*

$$\|C^g_\varphi\|^q_e \simeq \limsup_{|b|\to 1^-} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\sigma'_b(z)|^{(\alpha+2)\frac{q}{p}} d\mu_{a,g,\varphi}(z).$$

In particular, C^g_φ is compact if and only if

$$\lim_{|b|\to 1^-} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\sigma'_b(z)|^{(\alpha+2)\frac{q}{p}} d\mu_{a,g,\varphi}(z) = 0.$$

Proof. Let $b \in \mathbb{D}$ and consider the functions

$$f_b(z) = \int_0^z (-\sigma'_b(\xi))^{\frac{\alpha+2}{p}} d\xi.$$

We have $\|f_b\|_{\mathcal{D}^p_\alpha} = 1$ and $f_b \to 0$ uniformly on compact subsets of $\mathbb{D}$, as $|b| \to 1$. By [15, Theorem 2.12], the space $\mathcal{D}^p_\alpha$ is reflexive when $1 < p < \infty$. Hence, the closed unit ball of $\mathcal{D}^p_\alpha$ is weakly compact. Therefore, $(f_b)_{b\in\mathbb{D}}$ contains a weakly convergent net, say again, $(f_b)_{b\in\mathbb{D}}$. Since $(f_b)_{b\in\mathbb{D}}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|b| \to 1$, we see that $f_b \to 0$ weakly, as $|b| \to 1$. Thus, we have $\|Tf_b\|_{\mathcal{Q}_K(q,\beta)} \to 0$, as $|b| \to 1$, for every compact operator $T : \mathcal{D}^p_\alpha \to \mathcal{Q}_K(q,\beta)$. Hence,

$$\begin{aligned}\|C_\varphi^g - T\| &\geq \limsup_{|b|\to 1} \|C_\varphi^g f_b - T f_b\|_{\mathcal{Q}_K(q,\beta)} \\ &\geq \limsup_{|b|\to 1} (\|C_\varphi^g f_b\|_{\mathcal{Q}_K(q,\beta)} - \|T f_b\|_{\mathcal{Q}_K(q,\beta)}) \\ &= \limsup_{|b|\to 1} \|C_\varphi^g f_b\|_{\mathcal{Q}_K(q,\beta)},\end{aligned}$$

for every compact operator $T : \mathcal{D}_\alpha^p \to \mathcal{Q}_K(q,\beta)$. Therefore,

$$\begin{aligned}\|C_\varphi^g\|_e^q \geq &\limsup_{|b|\to 1} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\sigma_b'(\varphi(z))|^{(\alpha+2)\frac{q}{p}} |g(z)|^q \times \\ &(1-|z|^2)^\beta K(G(z,a))dA(z) \\ = &\limsup_{|b|\to 1} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\sigma_b'(z)|^{(\alpha+2)\frac{q}{p}} d\mu_{a,g,\varphi}(z).\end{aligned}$$

Thus, the lower bound for $\|C_\varphi^g\|_e$ is obtained. We now estimate the upper bound for the essential norm. To do this, using (1.1) we get

$$\begin{aligned}\|C_\varphi^g\|_e^q \leq &\liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{D}_\alpha^p}\leq 1} \|C_\varphi^g(R_n f)\|^q_{\mathcal{Q}_K(q,\beta)} \\ = &\liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{D}_\alpha^p}\leq 1} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(R_n f)'(\varphi(z))|^q |g(z)|^q \times \\ &(1-|z|^2)^\beta K(G(z,a))dA(z) \\ = &\liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{D}_\alpha^p}\leq 1} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(R_n f)'(\varphi(z))|^q d\mu_{a,g}(z) \\ = &\liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{D}_\alpha^p}\leq 1} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(R_n f)'(z)|^q d\mu_{a,g,\varphi}(z). \qquad (2.1)\end{aligned}$$

Let $r \in (0,1)$. For each $a \in \mathbb{D}$ and $f \in \mathcal{D}_\alpha^p$, by [10, Lemma 5] we have

$$\begin{aligned}&\int_{|z|\leq r} |(R_{n-1} f')(z)|^q d\mu_{a,g,\varphi}(z) \\ &\leq \|f\|^q_{\mathcal{D}_\alpha^p} \int_{|z|\leq r} (\sum_{k=n-1}^\infty \frac{\Gamma(k+\alpha+2)}{k!\Gamma(\alpha+2)} |z|^k)^q d\mu_{a,g,\varphi}(z) \\ &\leq \|f\|^q_{\mathcal{D}_\alpha^p} (\sum_{k=n-1}^\infty \frac{\Gamma(k+\alpha+2)}{k!\Gamma(\alpha+2)} r^k)^q \int_{|z|\leq r} d\mu_{a,g,\varphi}(z).\end{aligned}$$

By the boundedness of C_φ^g, for the coordinate function $f_0(z) = z$ in $\mathcal{D}_\alpha^p$ we have

$$\begin{aligned}\sup_{a\in\mathbb{D}} \int_{|z|\le r} d\mu_{a,g,\varphi}(z) &\le \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g(z)|^q (1-|z|^2)^\beta K(G(z,a))dA(z)\\ &= \|C_\varphi^g f_0\|^q_{\mathcal{Q}_K(q,\beta)} < \infty.\end{aligned}$$

It is well known that the series $\sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha+2)}{k!\Gamma(\alpha+2)} r^k$ converges to $(1-r)^{-\alpha-2}$ for any $r \in (0,1)$. Hence,

$$\lim_{n\to\infty} \sup_{\|f\|_{\mathcal{D}_\alpha^p}\le 1} \sup_{a\in\mathbb{D}} \int_{|z|\le r} |(R_{n-1}f')(z)|^q d\mu_{a,g,\varphi}(z) = 0, \tag{2.2}$$

for any $r \in (0,1)$.

We now estimate $\int_{|z|>r} |(R_{n-1}f')(z)|^q d\mu_{a,g,\varphi}(z)$. By boundedness of C_φ^g, for every $a, b \in \mathbb{D}$ we have

$$\int_{\mathbb{D}} |\sigma_b'(z)|^{(\alpha+2)\frac{q}{p}} d\mu_{a,g,\varphi}(z) \le \|C_\varphi^g f_b\|^q_{\mathcal{Q}_K(q,\beta)} \le \|C_\varphi^g\|^q,$$

since $\|f_b\|_{\mathcal{D}_\alpha^p} = 1$. Thus, $\mu_{a,g,\varphi}$ is a bounded $(\alpha+2)\frac{q}{p}-$Carleson measure, by [1, Theorem A].

Take $D_r = \{z \in \mathbb{D} : |z| < r\}$, for $0 < r < 1$, and suppose $\mu_{a,g,\varphi,r}$ denotes the restriction of the measure $\mu_{a,g,\varphi}$ to the set $\mathbb{D}\backslash D_r$. Define

$$M_r = \sup_{|I|\le 1-r} \frac{\mu_{a,g,\varphi}(S(I))}{|I|^{(\alpha+2)\frac{q}{p}}}.$$

We show that

$$\|\mu_{a,g,\varphi,r}\|_{(\alpha+2)\frac{q}{p}} \le 2M_r. \tag{2.3}$$

For any arc $I \subset \partial\mathbb{D}$ with $|I| \le 1$, take a constant $\lambda > 0$ such that $|I| = \lambda(1-r)$. If $0 < \lambda \le 1$, then obviously $S(I)\backslash D_r = S(I)$, and so $\mu_{a,g,\varphi,r}(S(I)) = \mu_{a,g,\varphi}(S(I)) \le M_r|I|^{(\alpha+2)\frac{q}{p}}$. In the case $\lambda > 1$, let $I = \{e^{i\theta} : \phi \le \theta \le \phi + |I|\}$. For $n = [\lambda] + 1$, take

$$I_k = \{e^{i\theta} : \phi + (k-1)(1-r) \le \theta \le \phi + k(1-r)\},$$

for $k=1,2,...,n$. It follows that

$$\begin{aligned}\mu_{a,g,\varphi,r}(S(I)) = \mu_{a,g,\varphi}(S(I)\backslash D_r) &\leq \sum_{k=1}^{n} \mu_{a,g,\varphi}(S(I_k)) \\ &\leq \sum_{k=1}^{n} M_r |I_k|^{(\alpha+2)\frac{q}{p}} \leq nM_r(1-r)^{(\alpha+2)\frac{q}{p}} \\ &\leq 2M_r(\lambda(1-r))^{(\alpha+2)\frac{q}{p}} \leq 2M_r|I|^{(\alpha+2)\frac{q}{p}},\end{aligned}$$

which implies (2.3).

For any arc $I \subset \partial\mathbb{D}$ with $|I| \leq 1-r$, take $b=(1-|I|)e^{i\theta}$, where $e^{i\theta}$ is the center of I. It can be proved that for any $z \in S(I)$, $|\sigma_b'(z)| \geq C|I|^{-1}$. Thus,

$$M_r \leq C \sup_{|b|\geq r} \int_{\mathbb{D}} |\sigma_b'(z)|^{(\alpha+2)\frac{q}{p}} d\mu_{a,g,\varphi}(z), \quad r \in (0,1).$$

Therefore, by Lemma 1.1, the note after Lemma 1.2 and the above estimates for each $a \in \mathbb{D}$ and every $f \in \mathcal{D}_\alpha^p$, we have

$$\begin{aligned}\int_{|z|>r} |(R_n f)'(z)|^q d\mu_{a,g,\varphi}(z) &\leq C\|R_n f\|_{\mathcal{D}_\alpha^p}^q \|\mu_{a,g,\varphi,r}\|_{(\alpha+2)\frac{q}{p}} \\ &\leq C\|R_n f\|_{\mathcal{D}_\alpha^p}^q M_r \\ &\leq C\|R_n f\|_{\mathcal{D}_\alpha^p}^q \sup_{|b|\geq r} \int_{\mathbb{D}} |\sigma_b'(z)|^{(\alpha+2)\frac{q}{p}} d\mu_{a,g,\varphi}(z) \\ &\leq C\|f\|_{\mathcal{D}_\alpha^p}^q \sup_{|b|\geq r} \int_{\mathbb{D}} |\sigma_b'(z)|^{(\alpha+2)\frac{q}{p}} d\mu_{a,g,\varphi}(z). \qquad (2.4)\end{aligned}$$

Using (2.1), (2.2) and (2.4), for any $r \in (0,1)$, we get

$$\begin{aligned}\|C_\varphi^g\|_e^q &\leq \liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{D}_\alpha^p}\leq 1} \sup_{a\in\mathbb{D}} \int_{|z|>r} |(R_n f)'(z)|^q d\mu_{a,g,\varphi}(z) \\ &\leq C \sup_{a\in\mathbb{D}} \sup_{|b|\geq r} \int_{\mathbb{D}} |\sigma_b'(z)|^{(\alpha+2)\frac{q}{p}} d\mu_{a,g,\varphi}(z).\end{aligned}$$

Taking limit, as $r \to 1$, implies

$$\|C_\varphi^g\|_e^q \leq C \limsup_{|b|\to 1} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\sigma_b'(z)|^{(\alpha+2)\frac{q}{p}} d\mu_{a,g,\varphi}(z).$$

□

As mentioned before, the lower and upper estimates for the essential norm of a composition operator from the Bloch space $\mathcal{B}$ ($\alpha = 1$) into $\mathcal{Q}_p$ space were obtained in [5]. Here, similarly we obtain lower and upper estimates for the essential norm of a bounded generalized composition operator $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{Q}_K(q, \beta)$, when $\alpha > 1$. To do this, we need the following test functions.

Let $\alpha > 1, \lambda_m \in (\frac{1}{2}, 1)$ be such that $\lambda_m \to 1$, when $m \to \infty$, and let

$$f_{n,m,\theta}(z) = \sum_{k=0}^{\infty} 2^{k(\alpha-1)} (\lambda_m e^{i\theta} z)^{2^k+n}, \quad n, m \in \mathbb{N}, \theta \in [0, 2\pi), z \in \mathbb{D}.$$

Then, each $f_{n,m,\theta}$ is analytic in a neighborhood of $\overline{\mathbb{D}}$, so $f_{n,m,\theta} \in \mathcal{B}_0^\alpha$. Moreover, by a direct computation, one can show $\|f_{n,m,\theta}\|_{\mathcal{B}^\alpha} \leq C$, where C is a constant independent of n, m and θ.

For any $s > -1$, $f \in \mathcal{B}_0^\alpha$ and $h \in \mathcal{A}_\beta^1$, we write

$$< f, h >_s = \lim_{r \to 1^-} (s+1) \int_{\mathbb{D}} f(rz) \overline{h(rz)} (1 - |z|^2)^s dA(z).$$

Suppose $\alpha > 1$ and $\beta > -1$. If $s = \alpha + \beta - 1$, then $(\mathcal{B}_0^\alpha)^* = \mathcal{A}_\beta^1$ under the integral pairing $<, >_s$, where the equality holds with equivalent norms (see [15, Theorem 7.5]).

Lemma 2.2. *Let $\alpha > 1$. For every $F \in (\mathcal{B}_0^\alpha)^*$, we have*

$$\lim_{n\to\infty} \sup_{m,\theta} |F(f_{n,m,\theta})| = 0.$$

Proof. Let $\alpha > 1, \beta > -1$. For given $F \in (\mathcal{B}_0^\alpha)^*$, by the above fact, there exists $h \in \mathcal{A}_\beta^1$ such that

$$F(f) = < f, h >_s, \quad f \in \mathcal{B}_0^\alpha,$$

where $s = \alpha + \beta - 1$. Therefore, using the inequality on page 436 of [3], we get

$$\begin{aligned}
|F(f_{n,m,\theta})| =& |< f_{n,m,\theta}, h >_s| \\
\leq& \lim_{r\to 1^-}(s+1)\int_{\mathbb{D}} |f_{n,m,\theta}(rz)||h(rz)|(1-|z|^2)^s dA(z) \\
\leq& \lim_{r\to 1^-}(\alpha+\beta)\int_{\mathbb{D}} (r\lambda_m|z|)^n \sum_{k=0}^{\infty} 2^{k(\alpha-1)}(r\lambda_m|z|)^{2^k}\times \\
& |h(rz)|(1-|z|^2)^s dA(z) \\
\leq& \lim_{r\to 1^-}(\alpha+\beta)\int_{\mathbb{D}} |rz|^n \sum_{k=0}^{\infty} 2^{k(\alpha-1)}|z|^{2^k}\times \\
& |h(rz)|(1-|z|^2)^s dA(z) \\
\leq& \lim_{r\to 1^-}(\alpha+\beta)\int_{\mathbb{D}} |rz|^n \frac{C(\alpha-1)}{(1-|z|)^{\alpha-1}}|h(rz)|(1-|z|)^s dA(z) \\
\leq& C\lim_{r\to 1^-}\int_{\mathbb{D}} |rz|^n|h(rz)|(1-|z|)^{\beta} dA(z) \\
=& C\lim_{r\to 1^-} r^2\int_{\mathbb{D}} |z|^n|h(z)|(1-|z|)^{\beta} dA(z) \\
=& C\int_{\mathbb{D}} |z|^n|h(z)|(1-|z|)^{\beta} dA(z),
\end{aligned}$$

for every m, n and θ. Since $h \in \mathcal{A}^1_\beta$, the Lebesgue Dominated Convergence Theorem gives

$$\lim_{n\to\infty}\sup_{m,\theta}|F(f_{n,m,\theta})| = 0.$$

$\square$

Theorem 2.3. *Let $\alpha > 1, q \geq 1, \beta > -2$, and let $C^g_\varphi : \mathcal{B}^\alpha \to \mathcal{Q}_K(q,\beta)$ be bounded. Then,*

$$\|C^g_\varphi\|^q_e \simeq \limsup_{r\to 1^-}\sup_{a\in\mathbb{D}}\int_{|z|>r}\frac{d\mu_{a,g,\varphi}(z)}{(1-|z|^2)^{\alpha q}}.$$

In particular, C^g_φ is compact if and only if

$$\lim_{r\to 1^-}\sup_{a\in\mathbb{D}}\int_{|z|>r}\frac{d\mu_{a,g,\varphi}(z)}{(1-|z|^2)^{\alpha q}} = 0.$$

Proof. By Lemma 2.2, we have $f_{n,m,\theta} \to 0$ weakly in $\mathcal{B}_0^\alpha$ and then in $\mathcal{B}^\alpha$, as $n \to \infty$. Thus, we have $\|Tf_{n,m,\theta}\|_{\mathcal{Q}_K(q,\beta)} \to 0$, as $n \to \infty$, for every compact operator $T : \mathcal{B}^\alpha \to \mathcal{Q}_K(q, \beta)$. Then, one can conclude that

$$\begin{aligned}\|C_\varphi^g - T\| \geq & C \limsup_{n\to\infty} \|(C_\varphi^g - T)f_{n,m,\theta}\|_{\mathcal{Q}_K(q,\beta)} \\ \geq & C \limsup_{n\to\infty} \|C_\varphi^g f_{n,m,\theta}\|_{\mathcal{Q}_K(q,\beta)},\end{aligned}$$

for every compact operator $T : \mathcal{B}^\alpha \to \mathcal{Q}_K(q, \beta)$. Therefore,

$$\|C_\varphi^g\|_e \geq C \limsup_{n\to\infty} \|C_\varphi^g f_{n,m,\theta}\|_{\mathcal{Q}_K(q,\beta)},$$

for every m and θ. Now, we estimate $\|C_\varphi^g f_{n,m,\theta}\|_{\mathcal{Q}_K(q,\beta)}$. For each $a \in \mathbb{D}$, we have

$$\begin{aligned}J &= \int_{\mathbb{D}} |f'_{n,m,\theta}(\varphi(z))|^q |g(z)|^q (1 - |z|^2)^\beta K(G(z, a))dA(z) \\ &= \int_{\mathbb{D}} |\sum_{k=0}^{\infty}(2^k + n)2^{k(\alpha-1)}(\lambda_m e^{i\theta})(\lambda_m e^{i\theta}\varphi(z))^{2^k+n-1}|^q d\mu_{a,g}(z) \\ &= \lambda_m^q \int_{\mathbb{D}} |\lambda_m z|^{(n-1)q} |\sum_{k=0}^{\infty}(2^k + n)2^{k(\alpha-1)}(\lambda_m e^{i\theta} z)^{2^k}|^q d\mu_{a,g,\varphi}(z).\end{aligned}$$

Integrating with respect to θ and using Fubini's Theorem, by [19, p.203] we have

$$\begin{aligned}J &\geq C \int_{\mathbb{D}} |\lambda_m z|^{(n-1)q} (\sum_{k=0}^{\infty}(2^k + n)^2 2^{2k(\alpha-1)} |\lambda_m z|^{2^{k+1}})^{\frac{q}{2}} d\mu_{a,g,\varphi}(z) \\ &\geq C \int_{\mathbb{D}} |\lambda_m z|^{nq} (\sum_{k=0}^{\infty} 2^{2k\alpha} |\lambda_m z|^{2(2^k-1)})^{\frac{q}{2}} d\mu_{a,g,\varphi}(z).\end{aligned}$$

By the formula (3.8) in [6], we have

$$J \geq C \int_{\mathbb{D}} \frac{|\lambda_m z|^{nq}}{(1 - |\lambda_m z|^2)^{\alpha q}} d\mu_{a,g,\varphi}(z),$$

for each m. Using Fatou's Lemma, we get

$$\begin{aligned}J &\geq C \liminf_{m\to\infty} \int_{\mathbb{D}} \frac{|\lambda_m z|^{nq}}{(1 - |\lambda_m z|^2)^{\alpha q}} d\mu_{a,g,\varphi}(z) \\ &\geq C \int_{\mathbb{D}} \frac{|z|^{nq}}{(1 - |z|^2)^{\alpha q}} d\mu_{a,g,\varphi}(z).\end{aligned}$$

Therefore,

$$\begin{aligned}
\|C_\varphi^g\|_e^q &\geq C \limsup_{n\to\infty} \sup_{a\in\mathbb{D}} J \geq C \limsup_{n\to\infty} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|z|^{nq}}{(1-|z|^2)^{\alpha q}} d\mu_{a,g,\varphi}(z) \\
&\geq C \limsup_{n\to\infty} \sup_{a\in\mathbb{D}} \int_{|z|>1-\frac{1}{nq}} \frac{|z|^{nq}}{(1-|z|^2)^{\alpha q}} d\mu_{a,g,\varphi}(z) \\
&\geq C \limsup_{n\to\infty} \sup_{a\in\mathbb{D}} \int_{|z|>1-\frac{1}{nq}} \frac{1}{(1-|z|^2)^{\alpha q}} d\mu_{a,g,\varphi}(z) \\
&\geq C \limsup_{r\to 1} \sup_{a\in\mathbb{D}} \int_{|z|>r} \frac{d\mu_{a,g,\varphi}(z)}{(1-|z|^2)^{\alpha q}}.
\end{aligned}$$

That is, the lower bound for the essential norm is obtained. To get the upper bound, let r_m be a sequence in $(0,1)$ such that $r_m \to 1$, as $m \to \infty$. We define the operator T_m on $\mathcal{H}(\mathbb{D})$ by $T_m f(z) = f(r_m z)$. It is clear that $T_m f \in \mathcal{B}_0^\alpha$, for each $f \in \mathcal{H}(\mathbb{D})$, and the operator T_m is compact on $\mathcal{B}^\alpha$. So, $\|C_\varphi^g\|_e \leq \|C_\varphi^g - C_\varphi^g T_m\|$, for each m. On the other hand, for every $f \in \mathcal{B}^\alpha$, we have

$$\begin{aligned}
&\|(C_\varphi^g - C_\varphi^g T_m) f\|_{\mathcal{Q}_K(q,\beta)}^q \\
&\quad= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(f - T_m f)'(\varphi(z))|^q |g(z)|^q (1-|z|^2)^\beta K(G(z,a)) dA(z) \\
&\quad= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z) - r_m f'(r_m z)|^q d\mu_{a,g,\varphi}(z) \\
&\quad= \sup_{a\in\mathbb{D}} \Big(\int_{|z|\leq r} |f'(z) - r_m f'(r_m z)|^q d\mu_{a,g,\varphi}(z) \\
&\qquad + \int_{|z|>r} |f'(z) - r_m f'(r_m z)|^q d\mu_{a,g,\varphi}(z) \Big) \\
&\quad\leq \sup_{a\in\mathbb{D}} \Big(M \int_{|z|\leq r} d\mu_{a,g,\varphi}(z) + \int_{|z|>r} (|f'(z)| + |f'(r_m z)|)^q d\mu_{a,g,\varphi}(z) \Big) \\
&\quad\leq \sup_{a\in\mathbb{D}} \Big(M \int_{|z|\leq r} d\mu_{a,g,\varphi}(z) + 2^{q+1} \|f\|_{\mathcal{B}^\alpha}^q \int_{|z|>r} \frac{d\mu_{a,g,\varphi}(z)}{(1-|z|^2)^{\alpha q}} \Big),
\end{aligned}$$

where $M = \sup_{|z|\leq r} |f'(z) - r_m f'(r_m z)|^q$ converges to zero, as $m \to \infty$, by uniform continuity of f on every compact subset of $\mathbb{D}$. Also, by the boundedness of C_φ^g, for the coordinate function $f_0(z) = z$ in $\mathcal{B}^\alpha$ we have

$$\begin{aligned}\sup_{a\in\mathbb{D}} \int_{|z|\le r} d\mu_{a,g,\varphi}(z) &\le \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g(z)|^q (1-|z|^2)^\beta K(G(z,a))\,dA(z)\\ &= \|C_\varphi^g f_0\|^q_{\mathcal{Q}_K(q,\beta)} < \infty.\end{aligned}$$

Hence,

$$\begin{aligned}\|C_\varphi^g\|_e^q &\le \lim_{m\to\infty} \|C_\varphi^g - C_\varphi^g T_m\|^q\\ &= \lim_{m\to\infty} \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \|(C_\varphi^g - C_\varphi^g T_m) f\|^q_{\mathcal{Q}_K(q,\beta)}\\ &\le \lim_{m\to\infty} \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \sup_{a\in\mathbb{D}} (M \int_{|z|\le r} d\mu_{a,g,\varphi}(z)\\ &\quad + 2^{q+1}\|f\|^q_{\mathcal{B}^\alpha} \int_{|z|>r} \frac{d\mu_{a,g,\varphi}(z)}{(1-|z|^2)^{\alpha q}})\\ &\le 2^{q+1} \sup_{a\in\mathbb{D}} \int_{|z|>r} \frac{d\mu_{a,g,\varphi}(z)}{(1-|z|^2)^{\alpha q}},\end{aligned}$$

for any $r \in (0,1)$. Therefore,

$$\|C_\varphi^g\|_e^q \le 2^{q+1} \limsup_{r\to 1} \sup_{a\in\mathbb{D}} \int_{|z|>r} \frac{d\mu_{a,g,\varphi(z)}}{(1-|z|^2)^{\alpha q}}.$$

□

Note that this theorem is also valid for $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Q}_K(q,\beta)$.

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