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ε-SIMULTANEOUS APPROXIMATIONS OF DOWNWARD SETS

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Abstract. We prove some results on characterization of ε-simultaneous approximations of downward sets in vector lattice Banach spaces. Also, we give some results about simultaneous approximations of normal sets.

1. Introduction

The theory of best simultaneous approximation has been studied by many authors (for example, [2, 9]). Singer [8] introduced the concept of ε-simultaneous approximation. Best simultaneous approximation is a generalization of best approximation and ε-simultaneous approximation in a sense is a generalization of best simultaneous approximation. Most studies about best simultaneous approximation have been done on convex sets. However, convexity is sometimes a very restrictive assumption. Here, we shall prove some results on characterization of ε-simultaneous approximations of downward sets in vector lattice Banach spaces.

There are many spaces along with an order $\leq$. The L^p and $C(X)$ spaces are some examples. The notion of an order in a vector space facilitates the study of the spaces in an abstract setting. First, let us give some basic preliminaries concerning vector lattices (see [1, 3]).

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Definition 1.1. *A lattice $(L, \leq)$ is said to be conditionally complete if it satisfies one of the following equivalent conditions:*

(1) *Every non-empty lower bounded set admits an infimum.*
(2) *Every non-empty upper bounded set admits an supremum.*
(3) *There exists a complete lattice $\bar{L} := L \cup \{\perp, \top\}$, which we call the minimal completion of L, with bottom element $\perp$ and top element $\top$, such that L is a sublattice of $\bar{L}$, $infL = \perp$ and $supL = \top$.*

A (real) vector lattice $(X, \leq, +, .)$ is a set X endowed with a partial order $\leq$ such that $(X, \leq)$ is a lattice, with a binary operation $+$ and a scalar product. A vector lattice $(X, \leq, +, .)$ such that $(X, \leq)$ is a conditionally complete lattice is called conditionally complete vector lattice. A conditionally complete lattice Banach space X is a real Banach space that is a conditionally complete vector lattice and $|x| \leq |y|$ implies $\|x\| \leq \|y\|$, for all $x, y \in X$.

Let X be a normed space. For a non-empty subset W of X and a non-empty bounded set S in X, define $d(S, W) = \inf_{w \in W} \sup_{s \in S} \|s - w\|$. An element $w_0 \in W$ is called a best simultaneous approximation to S from W, if $d(S, W) = \sup_{s \in S} \|s - w_0\|$. The set of all best simultaneous approximation to S from W will be denoted by $S_W(S)$.

Definition 1.2. *Let X be a normed space, W a subset of X and S a bounded set in X. An element $w_0 \in W$ is called ε-simultaneous approximation, if*

$$\sup_{s \in S} \|s - w_0\| \leq d(S, W) + \varepsilon.$$

The set of all ε-simultaneous approximations to S from W will be denoted by $S_{W,\varepsilon}(S)$.

One advantage of considering the set $S_{W,\varepsilon}(S)$, instead of the set $S_W(S)$, is that the set $S_{W,\epsilon}(S)$ is always nonempty, for all $\varepsilon > 0$.

If for each bounded set S in X there exists at least one best simultaneous approximation to S from W, then W is called a simultaneous proximinal subset of X. If for each bounded set S in X there exists a unique best simultaneous approximation to S from W, then W is called a simultaneous Chebyshev subset of X.

Here, we study best simultaneous approximations in conditionally complete lattice Banach spaces with a strong unit $\mathbf{1}$. Recall that an element $\mathbf{1} \in X$ is called a strong unit, if for each $x \in X$ there exists $\lambda > 0$ such that $x \leq \lambda\mathbf{1}$ (see [1]). We assume that X contains a strong unit $\mathbf{1}$. By using the strong unit $\mathbf{1}$, we can define a norm on X by $\|x\| = inf\{\lambda > 0 : |x| \leq \lambda\mathbf{1}\}$, for all $x \in X$. Also, we define

$$B(S,r) := \{y \in X : supS - r\mathbf{1} \leq y \leq infS + r\mathbf{1}\}, \tag{1.1}$$

where $r > 0$ and S is a bounded set in X. It is clear that $B(S,r)$ is a closed convex subset of X. We also have

$$|x| \leq \|x\|\mathbf{1}, \text{ for all } x \in X. \tag{1.2}$$

It is well known that X equipped with this norm is a conditionally complete lattice Banach space. Recall that a subset W of an ordered set X is said to be downward whenever for each $w \in W$ and $x \in X$ with $x \leq w$, we can conclude that $x \in W$. For each subset W of a normed space X, define the polar set of W by

$$W^0 := \{f \in X^* : f(w) \leq 0, \text{ for all } w \in W\},$$

where X^* is the dual space of X. If X is a lattice and there exists the least element of W, then we denote it by $minW$. Let $\varphi : X \times X \longrightarrow \mathbb{R}$ be a function defined by

$$\varphi(x,y) := sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq x+y,\} \text{ for all } x,y \in X. \tag{1.3}$$

Since $\mathbf{1}$ is a strong unit, the set $\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq x+y\}$ is non-empty and bounded from above by $\|x+y\|$. Clearly, this set is closed. It follows from the definition of φ that the function enjoys the following properties:

$$
\begin{aligned}
&(1.4) & -\infty &< \varphi(x,y) \leq \|x+y\|, \text{ for all } x,y \in X \\
&(1.5) & \varphi(x,y)\mathbf{1} &\leq x+y, \text{ for all } x,y \in X \\
&(1.6) & \varphi(x,y) &= \varphi(y,x), \text{ for all } x,y \in X \\
&(1.7) & \varphi(x,-x) &= sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq x-x=0\} = 0, \text{ for all } x \in X.
\end{aligned}
$$

For each $y \in X$, define the function $\varphi_y : X \to \mathbb{R}$ by

$$\varphi_y(x) := \varphi(x,y), \text{ for all } x \in X. \tag{1.8}$$

A function$f : X \to \mathbb{R}$ is called topical if it is increasing. The function φ_y defined by (1.8) is topical and Lipschitz continuous (see [5]). In fact, we have

$$|\varphi_y(x) - \varphi_y(z)| \leq \|x-z\|, \text{ for all } x,z \in X. \tag{1.9}$$

Also, the function φ, defined by (1.3), is continuous.

2. ε-simultaneous approximations of downward sets

Let X be a conditionally complete lattice Banach space with a strong unit $\mathbf{1}$. In this section, we prove some results about ε-simultaneous approximation of downward sets. We start with the following results for easy citation.

Lemma 2.1. [4] *Let W be a downward subset of X and $x \in X$. Then, the following statements are true:*

(1) *If $x \in W$, then $x - \epsilon\mathbf{1} \in intW$, for all $\varepsilon > 0$,*
(2) *We have $intW = \{x \in X : x + \varepsilon\mathbf{1} \in W, \text{ for some } \varepsilon > 0\}$.*

Lemma 2.2. [4] *Let W be a downward subset of X and S be an arbitrary bounded subset of X. If $r = d(S, W)$, then $w_0 = \sup S - r1 \in S_W(S)$ and is the least element of $S_W(S)$. Thus, W is a simultaneous proximinal subset of X.*

Lemma 2.3. [5] *Let W be a closed downward subset of X, $y_0 \in bdW$ and φ be the function defined by (1.3). Then, $\varphi(w, -y_0) \leq 0$, for all $w \in W$.*

Let W be a closed subset of X and S be a bounded subset of X such that $S \cap W = \phi$. In addition, suppose that $w_0 \in intW \cap S_{W,\varepsilon}(S)$. Thus, there exists $\alpha > 0$ such that

$$V = \{y \in X : \|y - w_0\| < \alpha\} \subset W.$$

Lemma 2.4. *Let α be as above. Then, $\alpha \leq \varepsilon$.*

Proof. Assume that $r = d(S, W)$ and $\varepsilon < \alpha$. Let $\varepsilon_0 = \frac{\alpha}{r+\alpha}, s \in S$ and

$$w_s = w_0 + \varepsilon_0(s - w_0).$$

Note that $\|w_s - w_0\| = \varepsilon_0\|s - w_0\| \leq \varepsilon_0(r + \varepsilon) = \alpha\frac{r+\varepsilon}{r+\alpha} < \alpha$, because $\frac{r+\varepsilon}{r+\alpha} < 1$ and $\sup_{s\in\ S} \|s - w_0\| \leq r + \varepsilon$. Then, $w_s \in V$, for all $s \in S$ and

$$r = d(S, W) \leq \sup_{t\in S} \|t - w_s\|, \text{ for all } s \in S.$$

Thus, $r \leq \inf_{s\in S} \sup_{t\in S} \|t - w_s\|$. On the other hand, we have

$$\|t - w_s\| = \|(t - w_0) - \varepsilon_0(s - w_0)\|, \text{ for all } t, s \in S.$$

This implies that

$$r \leq \inf_{s\in S} \sup_{t\in S} \|t - w_s\| = \inf_{s\in S} \sup_{t\in S} \|(t - w_0) - \varepsilon_0(s - w_0)\|$$

$\leq \sup_{t\in S} \|(t-w_0)-\varepsilon_0(t-w_0)\| = (1-\varepsilon_0)\sup_{t\in S}\|t-w_0\| \leq (1-\varepsilon_0)(r+\varepsilon) < r.$
This contradiction completes the proof. □

By using Lemma 2.4, it is easy to prove the following result.

Proposition 2.5. *Let W be a closed subset of X and S be a bounded subset of X such that $S \cap W = \phi$. Then, $S_{W,\varepsilon}(S) \subset V = \{w - \alpha\mathbf{1} :$ for some $w \in bdW$ and $0 \leq \alpha \leq \varepsilon\}$.*

Corollary 2.6. *Let W be a closed subset of X and S be a bounded subset of X such that $S \cap W = \phi$. Then, $S_W(S) \subset bdW$.*

Proposition 2.7. *Let W be a closed downward subset of X, and S be a bounded subset of X. Then, there exists the least element $w_0 := minS_{W,\varepsilon}(S)$.*

Proof. Put $r := d(S,W)$ and $w_0 = \sup S - (r+\varepsilon)\mathbf{1} \leq \sup S - r\mathbf{1}$. By Lemma 2.2, $\sup S - r\mathbf{1} \in W$. Since W is a downward set, $\sup S - (r+\epsilon)\mathbf{1} \in W$. Therefore, $w_0 \in S_{W,\varepsilon}(S)$, and so $\sup_{s\in S}\|s-w_0\| \leq r+\varepsilon$. Thus, $w \leq w_0$, for all $w \in S_{W,\varepsilon}(S)$. Hence, $w_0 := minS_{W,\varepsilon}(S)$. □

Proposition 2.8. *Let W be a closed downward subset of X, S be a bounded subset of X such that $S \cap W = \phi$, $w_0 \in S_{w,\varepsilon}(S)$ and φ be the function defined by (1.3). Then, $\varphi(w,-w_0) \leq \varepsilon$, for all $w \in W$.*

Proof. By Proposition 2.5, there exist $y_0 \in bdW$ and $0 \leq \alpha \leq \varepsilon$ such that $w_0 = y_0 - \alpha\mathbf{1}$. By Lemma 2.3, we have

$$\begin{aligned}
\varphi(w,-w_0) &= \varphi(w,\alpha\mathbf{1}-y_0)\\
&= \sup\{\lambda\in\mathbb{R} : \lambda\mathbf{1} \leq w+\alpha\mathbf{1}-y_0\}\\
&= \sup\{\lambda\in\mathbb{R} : (\lambda-\alpha)\mathbf{1} \leq w-y_0\}\\
&= \sup\{\beta+\alpha\in\mathbb{R} : \beta\mathbf{1} \leq w-y_0\}\\
&= \sup\{\beta\in\mathbb{R} : \beta\mathbf{1} \leq w-y_0\}+\alpha\\
&= \varphi(w,-y_0)+\alpha \leq \varepsilon.
\end{aligned}$$

This completes the proof. □

Theorem 2.9. *Let W be a closed downward subset of X, S be a bounded subset of X such that $S \cap W = \phi$, $y_0 \in W$, $r_0 = \sup_{s\in S}\|s-y_0\|$ and φ be the function defined by (1.3). Then, the following statements are equivalent:*

(1) $y_0 \in S_{W,\varepsilon}(S)$.
(2) *There exists* $l \in X$ *such that*

$$\varphi(w,l) \leq \varepsilon \leq \varphi(y,l), \text{ for all } w \in W, \ y \in B(S,r_0). \tag{2.1}$$

Moreover, if (2.1) holds with $l = -y_0$*, then* $y_0 = \min S_{W,\varepsilon}(S)$.

Proof. (1) $\Longrightarrow$ (2). Suppose that $y_0 \in S_{W,\epsilon}(S)$. Then, $r_0 = \sup_{s\in S} \|s - y_0\| \leq r + \epsilon$, where $r = d(S,W)$. Since W is a closed downward subset of X, by Lemma 2.2, the least element $\sup S - r\mathbf{1}$ of $S_W(S)$ exists. Let $w_0 := \sup S - (r_0 + \varepsilon)\mathbf{1}$. Note that

$$r \leq r_0 \Rightarrow (-r_0 - \varepsilon)\mathbf{1} \leq -r_0\mathbf{1} \leq -r\mathbf{1} \Rightarrow \sup S - (r_0 + \varepsilon)\mathbf{1} \leq \sup S - r\mathbf{1}.$$

By Lemma 2.2, we get $w_0 \in W$. Let $l = -w_0$ and $y \in B(S,r_0)$ be arbitrary. Thus, by using (1.1), we have $-r_0\mathbf{1} \leq y - \sup S$. This implies

$$-r_0 \in \{\alpha \in \mathbb{R} : \alpha\mathbf{1} \leq y - \sup S\}.$$

Hence, we obtain

$$\begin{aligned}
\varphi(y,l) &= \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq y + l\} \\
&= \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq y - w_0\} \\
&= \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq y - (\sup S - (r_0 + \varepsilon)\mathbf{1})\} \\
&= \sup\{\lambda \in \mathbb{R} : (\lambda - r_0 - \varepsilon)\mathbf{1} \leq y - \sup S\} \\
&= \sup\{\alpha + r_0 + \varepsilon \in \mathbb{R} : \alpha\mathbf{1} \leq y - \sup S\} \\
&= \sup\{\alpha \in \mathbb{R} : \alpha\mathbf{1} \leq y - \sup S\} + r_0 + \varepsilon \\
&\geq -r_0 + \varepsilon + r_0 = \varepsilon.
\end{aligned}$$

On the other hand, since $w_0 \in S_{W,\varepsilon}(S)$, by using Proposition 2.8, we get $\varphi(w, -w_0) \leq \varepsilon$, for all $w \in W$. Therefore, $\varphi(w,l) \leq \varepsilon$.

(2) $\Longrightarrow$ (1). Assume that there exists $l \in X$ such that $\varphi(w,l) \leq \varepsilon \leq \varphi(y,l)$, for all $w \in W$ and $y \in B(S,r_0)$. Since $B(S,r_0) = \{y \in X : supS - r_0\mathbf{1} \leq y \leq infS + r_0\mathbf{1}\}$, $supS - r_0\mathbf{1} \in B(S,r_0)$. Thus, we get $\varphi(supS - r_0\mathbf{1}, l) \geq \varepsilon \geq 0$. By definition of φ, we have $\varphi(\sup S, l) \geq r_0$. Hence, by using (1.5), we have

$$r_0\mathbf{1} \leq \varphi(\sup S, l)\mathbf{1} \leq \sup S + l. \tag{2.2}$$

Therefore, $-\sup S \leq l - r_0\mathbf{1}$. Now, let $w \in W$ and $t_w = \varphi(w, -\sup S)\mathbf{1} + \sup S \in X$. By (1.5), $\varphi(w, -\sup S)\mathbf{1} \leq w - \sup S$. Since W is a downward set and $w \in W$, $t_w \in W$ and so $\varphi(t_w, l) \leq \varepsilon$. Since $\varphi(t_w, .)$ is topical, by using (2.2), we have

$$\varphi(t_w, -\sup S) \leq \varphi(t_w, l - r_0\mathbf{1}) = \varphi(t_w, l) - r_0 \leq \varepsilon - r_0.$$

Since $\varphi(., -\sup S)$ is topical and $t_w = \varphi(w, -\sup S)\mathbf{1} + \sup S$, from (1.7) we get

$$\begin{aligned} \varepsilon - r_0 \geq \varphi(t_w, -\sup S) &= \varphi(\varphi(w, -\sup S)\mathbf{1} + \sup S, -\sup S) \\ &= \varphi(w, -\sup S) + \varphi(\sup S, -\sup S) = \varphi(w, -\sup S). \end{aligned}$$

Now, by using (1.7) and Lipschitz continuity of $\varphi_{-\sup S} := \varphi(., -\sup S)$, we obtain

$$\begin{aligned} \varepsilon + r_0 \quad \leq \quad & |\varphi(w, -\sup S)| \\ & = |\varphi(\sup S, -\sup S) - \varphi(w, -\sup S)| \\ & \leq \| \sup S - w\|. \end{aligned}$$

Thus, $-\varepsilon + r_0 \leq |\sup S - w\| \leq \sup_{s\in S} |s - w\|$, for all $w \in W$ and so we obtain $-\varepsilon + r_0 \leq r = d(S, W)$. Consequently, $r_0 \leq r + \varepsilon$ and $y_0 \in S_{W,\varepsilon}(S)$. Finally, suppose that (2.1) holds with $l = -y_0$. Then, by the implication (1) $\Longrightarrow$ (2), we have $y_0 \in S_{W,\varepsilon}(S)$. Let $w_1 \in S_{W,\varepsilon}(S)$ be arbitrary. If $r_1 = \sup_{s\in S} \|s - w_1\|$, then by the implication (1) $\Longrightarrow$ (2) we have $\varphi(w, l) \leq \varepsilon \leq \varphi(y, l)$, for all $w \in W$ and $y \in B(S, r_1)$, where $l = -\sup S + (r_1 + \varepsilon)\mathbf{1}$. Since $y_0 \in W$, $\varphi(y_0, l) = \varphi(y_0, -\sup S + (r_1 + \varepsilon)\mathbf{1}) \leq \varepsilon$. Thus, from definition of φ, we get $y_0 - \sup S + (r_1 + \varepsilon)\mathbf{1} \leq \varepsilon\mathbf{1}$. Hence, $y_0 \leq \sup S - r_1\mathbf{1}$. Therefore, $\sup S - r_1\mathbf{1} \leq w_1$ and so $y_0 \leq w_1$. Thus, $y_0 = \min S_{W,\varepsilon}(S)$. □

Here, we recall that a downward set W is called strictly downward, if for each boundary point w_0 of W, the inequality $w_0 < w$ implies $w \notin W$. For example, the level sets of a continuous strictly increasing real function give rise to strictly downward sets ([5, 6, 7]).

Theorem 2.10. *Let W be a closed downward subset of X and S be a bounded subset of X such that $S \cap W = \phi$. Then, the following statements are equivalent:*

(1) *W is a strictly downward subset of X.*
(2) *W is a simultaneous Chebyshev subset of X.*

Proof. (1) $\Rightarrow$ (2). Since W is downward set, by using Lemma 2.2, W is simultaneous proximinal. We claim $S_W(S) = \{\sup S - r'\mathbf{1}\}$, where $r' = d(S, W)$. Let there exist $w_0 \in S_W(S)$ such that $w_0 \neq \sup S - r'\mathbf{1}$. In this case, by Corollary 2.6, $\sup S - r'\mathbf{1} \in bdW$. Also, by Lemma

2.2, $\sup S - r'\mathbf{1} < w_0$. Since W is a strictly downward set, this implies that $w_0 \notin W$, which is a contradiction. Therefore, W is a simultaneous Chebyshev set of X.

(2) $\Rightarrow$ (1). Let W be a simultaneous Chebyshev subset of X. If W is not a strictly downward, then there exists $w_0 \in bdW$, such that $w_0 < w$, for all $w \in W$. Let $r \geq \|w - w_0\| > 0$. It follows from (1.2) that

$$w - w_0 \leq |w - w_0| \leq \|w - w_0\|\mathbf{1} \leq r\mathbf{1},$$

and so $w \leq w_0 + r\mathbf{1}$. Let $S = \{w_0 + r\mathbf{1}\}$. Then, $\sup_{s\in S} \|s - w_0\| = \|r\mathbf{1}\| = r$. We claim that $d(S, W) = r$. Suppose that this does not hold. Then, there exists $y \in W$ such that $\|w_0 + r\mathbf{1} - y\| < r$ (note that $w_0 + r\mathbf{1} \neq y$, because if $w_0 + r\mathbf{1} = y \in W$, then by Lemma 2.1, $w_0 \in intW$, which is a contradiction). Thus, there exists $r_0 \in (0, r)$ such that $\|w_0 + r\mathbf{1} - y\| \leq r_0$. Hence, by using (1.2), we have $w_0 + r\mathbf{1} \leq y + r_0\mathbf{1}$, and so

$$w_0 + \lambda_0\mathbf{1} \leq y, \text{ where } \lambda_0 = (r - r_0) > 0.$$

Since W is a downward set and $y \in W$, $w_0 + \lambda_0\mathbf{1} \in W$. Hence, by Lemma 2.1, $w_0 \in intW$. This is a contradiction. Therefore, $d(S, W) = r = \sup_{s\in S} \|s - w_0\|$, that is, $w_0 \in S_W(S)$. On the other hand, we have $w < w_0 + r\mathbf{1}$. Since $w_0 < w$, we have $0 \leq (w_0 + r\mathbf{1}) - w < w_0 + r\mathbf{1} - w_0 = r\mathbf{1}$. Hence,

$$\sup_{s\in S} \|s - w\| = \|w_0 + r\mathbf{1} - w\| \leq \|r\mathbf{1}\| = r = d(S, W) \leq \sup_{s\in S} \|s - w\|.$$

Thus, $\sup_{s\in S} \|s - w\| = d(S, W)$, and so $w \in S_W(S)$, where $w \neq w_0$. This is impossible, because W is a simultaneous Chebyshev subset of X. $\square$

3. Downward hulls and simultaneous approximation

As known, the downward hull U_* of the set $U \subseteq X$ is the intersection of all downward sets containing U. Recall that a subset G of the positive cone

$$X^+ = \{x \in X : x \geq 0\}$$

is called normal whenever $g \in G, x \in X^+$ and $x \leq g$ imply that $x \in G$. For a subset A, we shall use the notation $A^+ = \{a^+ : a \in A\}$, where $a^+ = \sup(a, 0)$. We also use the notation $a^- = -\inf(a, 0)$.

Remark 3.1. *Let W be a downward set, S be a bounded subset such that $S \cap W = \phi$, $w \in W$ and $s \in S$. If $0 \leq w - s$, then $s < w$, and so*

$s \in W$, which is a contradiction. After here, we suppose that $0 \leq s - w$, for all $w \in W$ and $s \in S$.

We start with the following result for easy citation.

Proposition 3.2. [4] *Let G be a a normal subset of X^+ and $G_* \subset X$ be the downward hull of G. Then, the following statements hold:*

(1) $G_* = \{x \in X : x^+ \in G\}$.
(2) $G = G_* \cap X^+$.
(3) *G is closed if and only if G_* is closed.*
(4) $(G_*)^+ = G$.

Proposition 3.3. *Let S be a bounded set of X, G be a normal subset of X^+ and G_* be the downward hull of the set G. If $S \cap G_* = \phi$, then, for each $g \in G_*$, we have*

$$\sup_{s\in S} \|s - g^+\| \leq \sup_{s\in S} \|s - g\|.$$

Proof. Let $s \in S$ and $g = g^+ - g^-$. Then, $s - g^+ \leq s - g$ and by Remark 3.1, $s - g^+ \geq 0$. Therefore, $s - g^+ = |s - g^+| \leq |s - g|$. It follow that $\|s - g^+\| \leq \|s - g\|$, for all $s \in S$. Hence, for each $g \in G_*$, we obtain $\sup_{s\in S} \|s - g^+\| \leq \sup_{s\in S} \|s - g\|$. □

Proposition 3.4. *Let G be a normal subset of X^+. Then, G is a simultaneous proximinal subsets of X.*

Proof. By Lemma 2.2, G_* is simultaneous proximinal. Thus, $S_{G_*}(S) \neq \phi$ for all bounded subsets S with $S \cap G_* = \phi$. If $g_0 \in S_{G_*}(S)$, then $g_0 \in G_*$ and $g_0^+ \in G$, by Proposition 3.2. By using Proposition 3.3, for each $g \in G_*$, we have

$$\sup_{s\in S} \|s - g_0^+\| \leq \sup_{s\in S} \|s - g_0\| \leq \sup_{s\in S} \|s - g\|.$$

Since $g_0 \in S_{G_*}(S)$ and $G \subset G_*$, $\sup_{s\in S} \|s - g_0^+\| \leq \sup_{s\in S} \|s - g\|$, for all $g \in G$. Therefore, $g_0^+ \in S_G(S)$. □

In the following corollaries, G is a normal subset of X^+, S is a bounded subset of X such that $S \cap G_* = \phi$, where G_* is the downward hull of G.

Corollary 3.5. $S_{G_*}(S) = S_G(S)$.

Proof. Let $g \in S_{G_*}(S)$. By Proposition 3.3, $\sup_{s\in S} \|s - g^+\| \leq \sup_{s\in S} \|s - g\|$, for all $g \in G_*$. Since $G \subseteq G_*$, by using Proposition 3.2, we have $g^+ \in G_*$. Thus, again by using Proposition 3.2, $g^+ = g$, and so $S_{G_*}(S) \subseteq S_G(S)$. Now, let $g_0 \in G_*$ and $g_0 \notin S_{G_*}(S)$. Then, there exists $g \in G_*$ such that $\sup_{s\in S} \|s - g\| \leq \sup_{s\in S} \|s - g_0\|$. By using Proposition 3.3, we obtain $\sup_{s\in S} \|s - g^+\| \leq \sup_{s\in S} \|s - g\|$. Hence, we get $\sup_{s\in S} \|s - g^+\| \leq \sup_{s\in S} \|s - g_0\|$. Since $g^+ \in G$, $g_0 \notin S_G(S)$. $\square$

Corollary 3.6. $d(S, G_*) = d(S, G)$.

Proof. Since $G \subseteq G_*$, $d(S, G_*) \leq d(S, G)$. The equality holds by Proposition 3.3. $\square$

Corollary 3.7. $\min S_{G_*}(S) = \min S_G(S)$.

Proof. By Lemma 2.2, $w_0 = \min S_{G_*}(S)$ exists. Now, the equality follows from Corollary 3.5. $\square$

Corollary 3.8. *G is simultaneous proximinal.*

Proof. The result follows from Lemma 2.2 and Corollary 3.5. $\square$

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