

NON-LINEAR ERGODIC THEOREMS IN COMPLETE NON-POSITIVE CURVATURE METRIC SPACES

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ABSTRACT. Hadamard (or complete $CAT(0)$) spaces are complete, non-positive curvature, metric spaces. Here, we prove a nonlinear ergodic theorem for continuous non-expansive semigroup in these spaces as well as a strong convergence theorem for the commutative case. Our results extend the standard non-linear ergodic theorems for non-expansive maps on real Hilbert spaces, to non-expansive maps on Hadamard spaces, which include for example (possibly infinite-dimensional) complete simply connected Riemannian manifolds with non-positive sectional curvature.

1. Introduction

The study of spaces of non-positive curvature originated with the discovery of hyperbolic spaces, and flourished by pioneering works of J. Hadamard and E. Cartan in first decades of the twentieth century. The idea of non-positive curvature geodesic metric spaces could be traced back to the work of H. Busemann and A.D. Alexandrov in the 50's. Later, M. Gromov restated features of global Riemannian geometry solely based on the so-called $CAT(0)$ inequality. Here, the letters C, A and T stand for Cartan, Alexandrov and Toponogov respectively.

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A metric space X is said to be a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as *thin* as its comparison triangle in the Euclidean plane. This latter property, which is what we referred to as the CAT(0) inequality, enables one to define the concept of non-positive curvature in this situation, generalizing the same concept in Riemannian geometry. Complete CAT(0) spaces (often called *Hadamard spaces*) have a remarkable geometric structures. In particular, in Hadamard spaces the distance function is convex, and it is possible to define orthogonal projection onto convex subsets. Also, the non-expansive mappings arise naturally in the study of isometries or more generally, local isometries. For more details on Hadamard spaces, we refer the reader to [4, 6, 7, 11, 14, 16]. Some aspects of the analysis on Hadamard spaces could be found in [1, 10, 13, 15, 16, 20].

On the other hand, as noticed by Reich and Shafrir [19] and Kirk [16], a class of *hyperbolic* metric spaces may be *an appropriate background for the study of non-linear operator theory, in general, and of iterative processes for non-expansive mappings in particular*; e.g., see [10, 13, 15, 16, 19].

Our work here devoted to ergodic theorems on Hadamard spaces. Throughout the over this paper, every Hilbert space is areal real Hilbert space. The first non-linear ergodic theorem for non-expansive mappings in a Hilbert space was proved by Baillon [3]. Let C be a non-empty closed convex subset of a Hilbert space H and T be a non-expansive mapping of C into itself. If the set $F(T)$, of fixed points of T is non-empty, then for each $x \in C$, the Cesàro mean $S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$ converges weakly to a point Px of $F(T)$, as $n \rightarrow \infty$. In this case, P is a non-expansive retraction of C onto $F(T)$ such that $PT = TP = P$ and $Px \in clco\{T^n x : n = 0, 1, 2, \dots\}$, for each $x \in C$, where $clcoA$ is the norm closure of the convex hull of A . The analogous results and various generalizations for non-expansive semi-groups of mappings on C could be found in many references; e.g., [2, 5, 12, 18]. If X is a Hadamard space which could not be embedded in any Hilbert space by isometries, then the classical nonlinear ergodic theorems fail. Here, we prove some nonlinear ergodic theorems for action of semigroups on Hadamard spaces. Our main results are Theorem 3.3 and Theorem 4.2 which generalize the well-known ergodic theorems for the actions of amenable and commutative semigroups of nonexpansive mappings on Hilbert spaces to Hadamard spaces.

2. Preliminaries

Among many equivalent definitions for a Hadamard space, a Hadamard space is a *complete* metric space (X, d) which is satisfied in the following condition (see [8, Lemma 2.4]):

CAT(0)-INEQUALITY: For every two points $x_0, x_1 \in X$ and for every $0 < t < 1$, there exists some $x_t \in X$ such that

$$(2.1) \quad d^2(y, x_t) \leq (1-t)d^2(y, x_0) + td^2(y, x_1) - t(1-t)d^2(x_0, x_1) \quad (y \in X).$$

For other definitions and important properties, one can see the standard texts such as [4, 6, 7, 14]. Now, let $\{x_\alpha; \alpha \in I\}$ be a bounded net in the Hadamard space (X, d) . For $x \in X$, set

$$r(x, \{x_\alpha; \alpha \in I\}) = \limsup_{\alpha} d(x, x_\alpha).$$

The asymptotic radius of $\{x_\alpha; \alpha \in I\}$ is given by

$$r(\{x_\alpha; \alpha \in I\}) = \inf\{r(x, \{x_\alpha; \alpha \in I\}); x \in X\},$$

and the asymptotic center of $\{x_\alpha; \alpha \in I\}$ is the set

$$A(\{x_\alpha; \alpha \in I\}) = \{x \in X; r(x, \{x_\alpha; \alpha \in I\}) = r(\{x_\alpha; \alpha \in I\})\}.$$

It is known that in a Hadamard space, $A(\{x_\alpha; \alpha \in I\})$ consists of exactly one point (see e.g., [17]).

As in the Hilbert space case, one can show that the asymptotic center belongs to the $clco\{x_\alpha; \alpha \in I\}$, the closed convex hull of the net $\{x_\alpha; \alpha \in I\}$. In fact, if z is the nearest point of $clco\{x_\alpha; \alpha \in I\}$ to the unique asymptotic center x_0 of $\{x_\alpha; \alpha \in I\}$, then $d(z, y) \leq d(x_0, y)$, for all y in $clco\{x_\alpha; \alpha \in I\}$. Hence,

$$\limsup_{\alpha} d(z, x_\alpha) \leq \limsup_{\alpha} d(x_0, x_\alpha) = r(\{x_\alpha; \alpha \in I\}).$$

Since the asymptotic center is unique, we have $x_0 = z \in clco\{x_\alpha; \alpha \in I\}$.

Recall that a semigroup S is called a *semitopological semigroup* if S is a Hausdorff topological space such that the maps $s \mapsto st$, ($s \in S$) and $s \mapsto ts$, ($s \in S$) are continuous maps for any $t \in S$. If S is a semitopological semigroup and $C_{\mathbb{R}}(S)$ is the Banach space of all bounded real-valued maps on S with the supremum norm, a continuous and linear

functional $\mu \in C_{\mathbb{R}}(S)^*$ is called a *mean*, if $\|\mu\| = \mu(1) = 1$. For any $f \in C_{\mathbb{R}}(S)$, we use the following notations:

$$\mu(f) = \mu_s(f(s)) = \int_S f(s) d\mu(s).$$

If, moreover, for all $f \in C_{\mathbb{R}}(S)$ and $s \in S$, we have $\int_S f(ts) d\mu(t) = \int_S f(st) d\mu(t)$, we say μ is an *invariant mean*. We call S to be an *amenable* semigroup, if $C_{\mathbb{R}}(S)$ admits an invariant mean. It is well-known that every abelian semigroup is amenable.

Lemma 2.1. *If μ is a mean on S and $u : S \rightarrow X$ is a continuous and bounded map into a Hadamard space X , then the map $\varphi : X \rightarrow \mathbb{R}$, defined by $\varphi(x) = \int d^2(u(s), x) d\mu(s)$, for $x \in X$, attains its unique minimum at a point of $C_u := \text{clco}\{u(s) | s \in S\}$, the closed convex hull of $u(S)$.*

Proof. It can be easily verified that the map φ is a continuous and strictly convex function. More precisely,

$$\varphi\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y) - \frac{1}{4}d^2(x, y), \quad \forall x, y \in X.$$

Hence, it attains its unique minimum at a point $x_0 \in X$ [20, Prop. 1.7]. If x_0 does not belong to C_u , then we have

$$d^2(u(s), x_0) \geq d^2(u(s), \pi x_0) + d^2(\pi x_0, x_0),$$

for each $s \in S$, where $\pi : X \rightarrow C_u$ is the nearest point projection map [6, pp. 176-177]. But, by definition of x_0 , we have

$$\int_S d^2(u(s), x_0) d\mu(s) \leq \int_S d^2(u(s), \pi x_0) d\mu(s),$$

and hence $d^2(\pi x_0, x_0) = 0$ and so $x_0 = \pi x_0 \in C_u$, which is a contradiction.

Now, we can define the mean of u by

$$(2.2) \quad \mu_s(u(s)) := \arg \min \{x \mapsto \int d^2(u(s), x) d\mu(s)\}.$$

By Lemma 2.1, when $u : S \rightarrow X$ is continuous and bounded, the mean value $\mu_s(u(s))$ is well defined and belongs to $C_u = \text{clco}\{u(s) | s \in S\}$. $\square$

3. Nonlinear Ergodic Theorems for Amenable Semigroups of Nonexpansive Mappings

All over this section, X is a Hadamard space, S is a semitopological semigroup such that the space $C_{\mathbb{R}}(S)$ of continuous bounded real-valued functions on S has an invariant mean μ . We say that the set $\mathcal{S} := \{T_s : X \rightarrow X | s \in S\}$ is a *continuous nonexpansive semigroup* (compare to [9, Theorem 4.4]) if

- (i) $T_{ts}x = T_t T_s x$, for all $s, t \in S$ and for all $x \in X$,
- (ii) the map $s \mapsto T_s x$ is a continuous map from S to X , for all $x \in X$,
- (iii) $d(T_s x, T_s y) \leq d(x, y)$, for all $s \in S$ and for all $x, y \in X$.

Let $\mathcal{F}(\mathcal{S}) := \{x \in X | T_s x = x, (s \in S)\}$ be the set of all common fixed points of $\mathcal{S}$. According to the following lemma, it is an empty set or a Hadamard space.

Lemma 3.1. *If (X, d) is a Hadamard space and $T : X \rightarrow X$ is a nonexpansive map, then the subset $\mathcal{F}(T) = \{x \in X; Tx = x\}$ of X is a closed and convex subset.*

Proof. By continuity of T , it is obvious that $\mathcal{F}(T)$ is closed. For convexity, let $x, y \in \mathcal{F}(T)$. Then,

$$d(T(\frac{x+y}{2}), T(x)) \leq d(\frac{x+y}{2}, x) = \frac{1}{2}d(x, y)$$

and

$$d(T(\frac{x+y}{2}), T(y)) \leq d(\frac{x+y}{2}, y) = \frac{1}{2}d(x, y).$$

But, $T(x) = x$ and $T(y) = y$, and so by triangular inequality,

$$d(x, y) \leq d(T(\frac{x+y}{2}), x) + d(T(\frac{x+y}{2}), y) \leq d(x, y).$$

Therefore, all of the above inequalities are in fact equality, that is,

$$d(T(\frac{x+y}{2}), x) = d(T(\frac{x+y}{2}), y) = \frac{1}{2}d(x, y),$$

and this yields $T(\frac{x+y}{2}) = \frac{x+y}{2}$, by the uniqueness of the midpoint. $\square$

By Lemma 3.1, $\mathcal{F}(\mathcal{S}) = \bigcap_{s \in S} \mathcal{F}(T_s)$ is a closed convex subset of the Hadamard space (X, d) , and therefore, when it is not an empty set, the nearest projection map $\pi : X \rightarrow \mathcal{F}(\mathcal{S})$ exists and it is well-defined (e.g., see [6], p.176). Let $\mathcal{S}(x) := \{T_s x \mid s \in S\}$ be the orbit of $x \in X$.

Proposition 3.2. *The followings are equivalent:*

- (i) $\mathcal{S}(x)$ is bounded, for some $x \in X$.
- (ii) $\mathcal{S}(x)$ is bounded, for all $x \in X$.
- (iii) $\mathcal{F}(\mathcal{S})$ is non-empty.

Proof. ($i \Rightarrow iii$) Let $x \in X$ and $\mathcal{S}(x)$ be bounded, and consider the map $s \mapsto T_s x$. This is a continuous and bounded map and so it has a mean value $\mu_s(T_s x)$ as in (2.2). Since $\mathcal{S}$ is nonexpansive and μ is invariant,

$$\begin{aligned} \int_S d^2(T_r x, T_t \mu_s(T_s x)) d\mu(r) &= \int_S d^2(T_{tr} x, T_t \mu_s(T_s x)) d\mu(r) \\ &\leq \int_S d^2(T_r x, \mu_s(T_s x)) d\mu(r), \end{aligned}$$

for each $t \in S$. By uniqueness of argmin in (2.2), we deduce that $T_t \mu_s(T_s x) = \mu_s(T_s x)$, for each $t \in S$, which means that $\mu_s(T_s x) \in \mathcal{F}(\mathcal{S})$.

($iii \Rightarrow ii$) Let $x_0 \in \mathcal{F}(\mathcal{S})$. Then, for each $x \in X$ and $s \in S$, we have

$$d(T_s x, x_0) = d(T_s x, T_s x_0) \leq d(x, x_0),$$

and so $\mathcal{S}(x)$ is bounded.

($ii \Rightarrow i$) This is straight forward. □

Now, we can prove a nonlinear ergodic theorem for amenable semi-groups.

Theorem 3.3. *Let $\mathcal{S} := \{T_s : X \rightarrow X \mid s \in S\}$ be a nonexpansive semigroup with $\mathcal{F}(\mathcal{S}) \neq \emptyset$. Then, there exists a retraction $P : X \rightarrow \mathcal{F}(\mathcal{S})$ with the following properties:*

- (i) $T_s P = P T_s = P^2 = P$, for all $s \in S$,
- (ii) $Px \in \bigcap_{t \in S} \text{clco}\{T_{st} x : s \in S\} \cap \mathcal{F}(\mathcal{S})$, for all $x \in X$.

Proof. For any $x \in X$, let $Px = \mu_s(T_s x)$. (i) It is clear that $P^2 = P$. Also, we have already proved that P maps X into $\mathcal{F}(\mathcal{S})$ (Proposition

3.2, (i) $\Rightarrow$ (iii), and hence $T_s P = P$, for each $s \in S$. Now, for $x \in X$, put

$$\varphi_x(y) := \int d^2(T_s x, y) d\mu(s).$$

Then,

$$\begin{aligned} \varphi_x(y) &= \int d^2(T_{st} x, y) d\mu(s) \\ &= \int d^2(T_s(T_t x), y) d\mu(s) \\ &= \varphi_{T_t x}(y), \end{aligned}$$

for each $t \in S$ and $y \in X$. But, $P(x) = \arg \min_{y \in X} \varphi_x(y)$ and $P(T_t x) = \arg \min_{y \in X} \varphi_{T_t x}(y)$. Hence, $P = PT_t$.

(ii) Given $x \in X$ and $t \in S$, define $u : S \rightarrow X$ by $u(s) = T_{st} x$, for $s \in S$. Then,

$$\begin{aligned} Px &= \arg \min \{y \mapsto \int_S d^2(T_t x, y) d\mu(t)\} \\ &= \arg \min \{y \mapsto \int_S d^2(T_{st} x, y) d\mu(t)\} \\ &= \arg \min \{y \mapsto \int_S d^2(u(s), y) d\mu(t)\}. \end{aligned}$$

Therefore, $Px \in C_u = \text{clco}\{u(s), s \in S\}$ by Lemma 2.1, and we are done. $\square$

4. Nonlinear Ergodic Theorems for Commutative Semigroups of Nonexpansive Mappings

In this section, suppose that S as a commutative semigroup. We prove that in this case the map P must be a nonexpansive retraction (Theorem 4.2). It is well-known that the following is a partial order on S :

$$s \geq t \text{ if and only if there exists } u \in S \text{ satisfying } s = ut.$$

Now, by the method employed in [3], one can prove the following proposition.

Proposition 4.1. *Let (X, d) be a Hadamard space, $\mathcal{S} = \{T_s; s \in S\}$ be a commutative semigroup of nonexpansive mappings on X with $\mathcal{F}(\mathcal{S}) \neq \emptyset$. Then, for any $s \in S$, the net $\{\pi T_s x\}_{s \in S}$ converges to a point in $\mathcal{F}(\mathcal{S})$, where $\pi : X \rightarrow \mathcal{F}(\mathcal{S})$ is the nearest point projection. Moreover, $Px := \lim_s \pi T_s x$ is the unique asymptotic center of the net $\mathcal{S}(x) := \{T_s x; s \in S\}$.*

Proof. By Lemma 3.1, the map π is well-defined. Now, let $x \in X$ and $s, t \in S$, with $s \geq t$. Then, there exists some $u \in S$ such that $s = ut$.

By the CAT(0)-inequality and definition of π , we have

$$\begin{aligned} d^2(\pi T_s x, T_s x) &\leq d^2\left(\frac{\pi T_t x + \pi T_s x}{2}, T_s x\right) \\ &\leq \frac{1}{2}d^2(\pi T_t x, T_s x) + \frac{1}{2}d^2(\pi T_s x, T_s x) - \frac{1}{4}d^2(\pi T_t x, \pi T_s x). \end{aligned}$$

But

$$d(\pi T_t x, T_s x) = d(T_u \pi T_t x, T_u T_t x) \leq d(\pi T_t x, T_t x),$$

and hence

$$\frac{1}{2}d^2(\pi T_t x, \pi T_s x) \leq d^2(\pi T_t x, T_t x) - d^2(\pi T_s x, T_s x),$$

for any $s, t \in S$ with $s \geq t$.

Therefore, the net $\{d(\pi T_s x, T_s x); s \in S\}$ is a decreasing net, and thus converges to a real number and this implies that $\{\pi T_s x\}_{s \in S}$ is a Cauchy net in the closed subset $\mathcal{F}(\mathcal{S})$ of the complete metric space X . Hence, it converges to some point $Px \in \mathcal{F}(\mathcal{S})$.

Since S is an amenable semigroup and $\mathcal{F}(\mathcal{S}) \neq \emptyset$, by Proposition 3.2, $\mathcal{S}(x)$ is a bounded set. So, let $P_0 x$ be its asymptotic center. By the property of π , we have

$$\begin{aligned} d(Px, T_s x) &\leq d(Px, \pi T_s x) + d(\pi T_s x, T_s x) \\ &\leq d(Px, \pi T_s x) + d(P_0 x, T_s x), \end{aligned}$$

for any $s \in S$. But, $\lim_s d(Px, \pi T_s x) = 0$, and therefore,

$$\limsup_s d(Px, T_s x) \leq \limsup_s d(P_0 x, T_s x).$$

Since the asymptotic center is unique, $Px = P_0 x$. □

Finally, we can prove the following ergodic theorem for commutative semigroups of nonexpansive mappings on a Hadamard space.

Theorem 4.2. *Let (X, d) be a Hadamard space, $\mathcal{S} := \{T_s; s \in S\}$ be a commutative semigroup of nonexpansive mappings on X with $\mathcal{F}(\mathcal{S}) \neq \emptyset$. Then, there exists a nonexpansive retraction $P : X \rightarrow \mathcal{F}(\mathcal{S})$ which has the following properties:*

- (i) $T_s P = P T_s = P^2 = P$, for all $s \in S$,
- (ii) $Px \in \text{clco}\{T_s x; s \in S\}$ for all $x \in X$.

Proof. For any $x \in X$, let $Px = \lim_s \pi T_s x$ as before.

(i) In Proposition 4.1, we have proved that $Px \in \mathcal{F}(\mathcal{S})$, and hence $T_s Px = Px$ and

$$PT_s x = \lim_t \pi T_t T_s x = \lim_t \pi T_{ts} x = \lim_t \pi T_t x = Px,$$

and

$$P^2 x = \lim_s \pi T_s Px = \pi Px = Px$$

for any $s \in S$.

(ii) We saw in Proposition 4.1 that Px is the asymptotic center of $\mathcal{S}(x)$, and so $Px \in \text{clco}\mathcal{S}(x)$.

Finally, we shall prove that P is nonexpansive. Since the projection map π is nonexpansive, we have

$$\begin{aligned} d(Px, Py) &= \lim_s d(\pi T_s x, \pi T_s y) \\ &\leq \lim_s d(T_s x, T_s y) \leq d(x, y) \end{aligned}$$

for any $x, y \in X$. □

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