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HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN A LORENTZIAN SPACE FORM

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ABSTRACT. We give some characterizations of n dimensional $(n \ge 2)$ hyperbolic cylinder, spherical cylinder or Euclidean cylinder in a Lorentzian space form. We show that the hyperbolic cylinder, spherical cylinder or Euclidean cylinder is the only complete space-like hypersurface in an (n+1) dimensional Lorentzian space form $M_1^{n+1}(c)$ with non-zero constant mean curvature and two distinct principal curvatures one of which is simple, if the norm square of the second fundamental form of M^n satisfies some pinching conditions, respectively.

1. Introduction

By an (n + 1) dimensional Lorentzian space form $M_1^{n+1}(c)$ we mean a Minkowski space R_1^{n+1} , a de Sitter space $S_1^{n+1}(c)$ or an anti-de Sitter space $H_1^{n+1}(c)$, according to c > 0, c = 0 or c < 0, respectively. That is, a Lorentzian space form $M_1^{n+1}(c)$ is a complete connected (n+1) dimensional Lorentzian manifold with constant curvature c. A hypersurface

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in a Lorentzian manifold is said to be space-like if the induced metric on the hypersurface is positive definite.

In connection with the negative settlement of the Bernstein problem due to Calabi [3], Cheng and Yau [4], and Chouque-Bruhat et al. [5] proved the following theorem independently.

Theorem 1.1 ([4, 5]). Let M^n be a complete space-like hypersurface in an (n+1) dimensional Lorentzian space form $M_1^{n+1}(c)$, $c \ge 0$. If M^n is maximal, then it is totally geodesic.

Ishihara [7] also proved the following well-known result.

Theorem 1.2 ([7]). If M^n is an *n* dimensional $(n \ge 2)$ complete maximal space-like hypersurface in anti-de Sitter space $H_1^{n+1}(-1)$, then,

$$(1.1) S \le n$$

and S = n if and only if $M^n = H^m(-\frac{n}{m}) \times H^{n-m}(-\frac{n}{n-m}), (1 \le m \le n-1)$, where S denotes the norm square of the second fundamental form of M.

As a generalization of Theorem 1.1, complete space-like hypersurfaces with constant mean curvature in a Lorentz manifold have been investigated by many mathematicians; see [2,6,8,10,12,13]. Ki et al. [9] proved the following result.

Theorem 1.3 ([9]). Let M^n be a complete space-like hypersurface with constant mean curvature in an (n+1) dimensional Lorentzian space form $M_1^{n+1}(c)$. If M^n satisfies one of the following properties,

(1) $c \leq 0$, (2) c > 0, $n \geq 3$ and $n^2 H^2 \geq 4(n-1)c$, (3) c > 0, n = 2 and $H^2 > c$,

then,

(1.2)
$$S \leq -nc + \frac{n^3 H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)} \sqrt{n^2 H^4 - 4(n-1)c H^2},$$

where S denotes the norm square of the second fundamental form of M.

From Ki et al. [9], we know that the well-known standard models of complete space-like hypersurfaces with non-zero constant mean curvature in an (n+1) dimensional Lorentzian space form $M_1^{n+1}(c)$ are the totally umbilical space-like hypersurfaces and the following product

manifolds:

$$H^{k}(c_{1}) \times R^{n-k}$$

= {(x, y) \epsilon R_{1}^{n+1} = R_{1}^{k+1} \times R^{n-k} : |x|^{2} = -\frac{1}{c_{1}} > 0},

where $c_1 < 0$ and $k = 1, \dots, n-1$. We note that $H^k(c_1) \times R^{n-k}$ in R_1^{n+1} has two distinct principal curvatures $\sqrt{-c_1}$ with multiplicity k and 0 with multiplicity n-k and $S = \frac{1}{k}n^2H^2$,

$$H^{k}(c_{1}) \times S^{n-k}(c_{2}) = \{(x, y) \in S_{1}^{n+1}(c) \\ \subset R_{1}^{n+2} = R_{1}^{k+1} \times R^{n-k+1} : |x|^{2} = -\frac{1}{c_{1}}, |y|^{2} = \frac{1}{c_{2}}\},$$

where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}, c_1 < 0, c_2 > 0$ and $k = 1, \dots, n-1$. We note that $H^k(c_1) \times S^{n-k}(c_2)$ in $S_1^{n+1}(c)$ has two distinct principal curvatures $\sqrt{c-c_1}$ with multiplicity k and $\sqrt{c-c_2}$ with multiplicity n-k and

$$S = -nc + \frac{n^3 H^2}{2k(n-k)} \pm \frac{n(n-2k)}{2k(n-k)} H\sqrt{n^2 H^2 - 4k(n-k)c},$$

$$\begin{split} H^k(c_1) \times & H^{n-k}(c_2) = \{(x,y) \in H_1^{n+1}(c) \\ & \subset R_2^{n+2} = R_1^{k+1} \times R_1^{n-k+1}: \quad |x|^2 = -\frac{1}{c_1}, |y|^2 = -\frac{1}{c_2}\}, \end{split}$$

where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 < 0$ and $k = 1, \dots, n-1$. We note that $H^k(c_1) \times H^{n-k}(c_2)$ in $H_1^{n+1}(c)$ has two distinct principal curvatures $\pm \sqrt{c-c_1}$ with multiplicity k and $\pm \sqrt{c-c_2}$ with multiplicity n-k and

$$S = -nc + \frac{n^3 H^2}{2k(n-k)} \pm \frac{n(n-2k)}{2k(n-k)} H\sqrt{n^2 H^2 - 4k(n-k)c}.$$

From Ki et al. [9], $H^1(c_1) \times S^{n-1}(c_2)$, $H^1(c_1) \times R^{n-1}$ or $H^1(c_1) \times H^{n-1}(c_2)$ is, in particular, called a hyperbolic cylinder in $S_1^{n+1}(c)$, R_1^{n+1} or $H_1^{n+1}(c)$; $H^{n-1}(c_1) \times S^1(c_2)$ or $H^{n-1}(c_1) \times R^1$ is also called a spherical cylinder or Euclidean cylinder in $S_1^{n+1}(c)$ or R_1^{n+1} . The norm square of the second fundamental form of a hyperbolic cylinder $H^1(c_1) \times R^{n-1}$ or Euclidean cylinder $H^{n-1}(c_1) \times R^1$ in R_1^{n+1} satisfies:

(1.3)
$$S = n^2 H^2$$
, or $S = \frac{n^2 H^2}{n-1}$;

the norm square of the second fundamental form of a hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$ or spherical cylinder $H^{n-1}(c_1) \times S^1(c_2)$ in $S_1^{n+1}(c)$ or a hyperbolic cylinder $H^1(c_1) \times H^{n-1}(c_2)$ in $H_1^{n+1}(c)$ satisfies:

(1.4)
$$S = -nc + \frac{n^3 H^2}{2(n-1)} \pm \frac{n(n-2)}{2(n-1)} H \sqrt{n^2 H^2 - 4(n-1)c}.$$

Denote by $P_H(t)$ the following polynomial,

(1.5)
$$P_H(t) = (n-1)t^2 - nHt + c.$$

By a direct calculation, we know that (1.5) has two real roots:

$$t_1 = \frac{nH - \sqrt{n^2 H^2 - 4(n-1)c}}{2(n-1)}, \quad t_2 = \frac{nH + \sqrt{n^2 H^2 - 4(n-1)c}}{2(n-1)}.$$

For c = 0, $t_1 = 0$ and $t_2 > 0$; for c < 0, $t_1 < 0$ and $t_2 > 0$; for c > 0 and $H^2 \ge c$ (which implies $n^2 H^2 \ge 4(n-1)c$), $t_1 > 0$ and $t_2 > 0$. Therefore, we realize that (1.3) and (1.4) may be rewritten as follows:

(1.6) $S = n^2 H^2$, or $S = (n-1)t_2^2$, for c = 0,

(1.7) $S = (n-1)t_1^2 + c^2t_1^{-2}$, or $S = (n-1)t_2^2 + c^2t_2^{-2}$, for c < 0 and c > 0, and

(1.8)
$$(n-1)t_2^2 + c^2 t_2^{-2} \le (n-1)t_1^2 + c^2 t_1^{-2}.$$

Here, we investigate complete hypersurfaces with constant mean curvatures in a Lorentzian space form $M_1^{n+1}(c)$ and give some characterization of n dimensional $(n \ge 2)$ hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$, $H^1(c_1) \times R^{n-1}$ or $H^1(c_1) \times H^{n-1}(c_2)$ in $M_1^{n+1}(c)$ and spherical cylinder $H^{n-1}(c_1) \times S^1(c_2)$ or Euclidean cylinder $H^{n-1}(c_1) \times R^1$ in $S_1^{n+1}(c)$ or R_1^{n+1} . More precisely, we obtain the following result.

Main Theorem. Let M^n be an n dimensional $(n \ge 2)$ complete space-like hypersurface in an (n+1) dimensional Lorentzian space form $M_1^{n+1}(c)$ with non-zero constant mean curvature and two distinct principal curvatures, one of which λ is simple and $\lim_{s\to\infty} \lambda \neq H$. Then,

(1) for c = 0, (i) M^n is isometric to the Euclidean cylinder $H^{n-1}(c_1) \times R^1$, $c_1 < 0$, if $S \le (n-1)t_2^2$, and (ii) M^n is isometric to the hyperbolic cylinder $H^1(c_1) \times R^{n-1}$ or Euclidean cylinder $H^{n-1}(c_1) \times R^1$, $c_1 < 0$ if $(n-1)t_2^2 \le S \le n^2 H^2$,

where t_2 is the positive real root of (1.5).

(2) For c < 0, M^n is isometric to the hyperbolic cylinder $H^1(c_1) \times H^{n-1}(c_2)$, $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 < 0$ if one of the following conditions is satisfied:

(i) $S \le (n-1)t_2^2 + c^2t_2^{-2}$, or (ii) $(n-1)t_2^2 + c^2t_2^{-2} \le S \le (n-1)t_1^2 + c^2t_1^{-2}$,

where t_1 is the negative real root and t_2 the positive real root of (1.5). (3) For c > 0 and $H^2 \ge c$, M^n is isometric to the hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$ or spherical cylinder $H^{n-1}(c_1) \times S^1(c_2)$, $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$,

 $c_1 < 0, c_2 > 0 \text{ if one of the following conditions is satisfied:}$ $(i) <math>S \le (n-1)t_2^2 + c^2t_2^{-2}, \text{ or}$ $(ii) <math>(n-1)t_2^2 + c^2t_2^{-2} \le S \le (n-1)t_1^2 + c^2t_1^{-2}, \text{ where } t_1 \text{ and } t_2 \text{ are the two positive real roots of } (1.5).$

2. Preliminaries

Let M^n be an *n* dimensional space-like hypersurface in an (n+1) dimensional Lorentzian space form $M_1^{n+1}(c)$. We choose a local field of semi-Riemannian orthonormal frames $\{e_1, \dots, e_{n+1}\}$ in $M_1^{n+1}(c)$ such that at each point of M^n , $\{e_1, \dots, e_n\}$ span the tangent space of M^n and form an orthonormal frame there. We use the following convention on the range of indices:

 $1 < A, B, C, \dots < n+1, \quad 1 < i, j, k, \dots < n.$

Let $\{\omega_1, \dots, \omega_{n+1}\}$ be the dual frame field so that the semi-Riemannian metric of $M_1^{n+1}(c)$ is given by: $d\bar{s}^2 = \sum_i \omega_i^2 - \omega_{n+1}^2 = \sum_A \epsilon_A \omega_A^2$, where $\epsilon_i = 1$ and $\epsilon_{n+1} = -1$.

The structure equations of $M_1^{n+1}(c)$ are given by

(2.1)
$$d\omega_A = \sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

(2.2)
$$d\omega_{AB} = \sum_{C} \epsilon_{C} \omega_{AC} \wedge \omega_{CB} + \Omega_{AB},$$

where.

(2.3)
$$\Omega_{AB} = -\frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D,$$

(2.4)
$$K_{ABCD} = \epsilon_A \epsilon_B c (\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}).$$

Restricting these forms to M^n , we have,

(2.5)
$$\omega_{n+1} = 0.$$

Cartan's Lemma implies:

(2.6)
$$\omega_{n+1,i} = \sum_{j} h_{ij} \omega_j, \quad h_{ij} = h_{ji}.$$

The structure equations of M^n are:

(2.7)
$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0,$$

(2.8)
$$d\omega_{ij} = \sum_{k} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,$$

(2.9)
$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}),$$

where the R_{ijkl} are the components of the curvature tensor of M and

(2.10)
$$h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j$$

is the second fundamental form of M. From the above equation, we have,

(2.11)
$$n(n-1)(R-c) = S - n^2 H^2,$$

where n(n-1)R is the scalar curvature of M, H is the mean curvature, and $S = \sum_{i,j} h_{ij}^2$ is the norm square of the second fundamental form of M^n .

We choose e_1, \dots, e_n such that $h_{ij} = \lambda_i \delta_{ij}$. From (2.6), we have,

(2.12)
$$\omega_{n+1,i} = \lambda_i \omega_i, \quad i = 1, 2, \cdots, n$$

Hence, we have from the structure equations of M^n ,

(2.13)
$$\begin{aligned} d\omega_{n+1,i} &= d\lambda_i \wedge \omega_i + \lambda_i d\omega_i \\ &= d\lambda_i \wedge \omega_i + \lambda_i \sum_j \omega_{ij} \wedge \omega_j. \end{aligned}$$

On the other hand, we have on the curvature forms of $M_1^{n+1}(c)$,

(2.14)
$$\Omega_{n+1,i} = -\frac{1}{2} \sum_{C,D} K_{n+1iCD} \omega_C \wedge \omega_D$$
$$= \frac{1}{2} \sum_{C,D} c(\delta_{n+1C} \delta_{iD} - \delta_{n+1D} \delta_{iC}) \omega_C \wedge \omega_D$$
$$= c \omega_{n+1} \wedge \omega_i = 0.$$

Therefore, from the structure equations of $M_1^{n+1}(c)$, we have,

(2.15)
$$d\omega_{n+1,i} = \sum_{j} \omega_{n+1j} \wedge \omega_{ji} - \omega_{n+1n+1} \wedge \omega_{n+1i} + \Omega_{n+1i}$$
$$= \sum_{j}^{j} \lambda_{j} \omega_{ij} \wedge \omega_{j}.$$

From (2.13) and (2.15), we obtain:

(2.16)
$$d\lambda_i \wedge \omega_i + \sum_j (\lambda_i - \lambda_j) \omega_{ij} \wedge \omega_j = 0.$$

Letting

(2.17)
$$\psi_{ij} = (\lambda_i - \lambda_j)\omega_{ij},$$

we have $\psi_{ij} = \psi_{ji}$. Equation (2.16) can be written as:

(2.18)
$$\sum_{j} (\psi_{ij} + \delta_{ij} d\lambda_j) \wedge \omega_j = 0.$$

By Cartan's Lemma, we get

(2.19)
$$\psi_{ij} + \delta_{ij} d\lambda_j = \sum_k Q_{ijk} \omega_k,$$

where the Q_{ijk} are uniquely determined functions such that

3. Proof of Main Theorem

We firstly state a proposition which can be proved by making use of the similar method due to Otsuki [11] for Riemannian space forms.

Proposition 3.1. Let M be a hypersurface in an (n+1) dimensional Lorentzian space form $M_1^{n+1}(c)$ such that the multiplicities of the principal curvatures are constant. Then, the distribution of the space of the principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of the principal vectors.

Let M^n be an *n* dimensional complete oriented space-like hypersurface with non-zero constant mean curvature and with two distinct principal curvatures, one of which is simple. We can choose an orientation for M^n such that H > 0. Without loss of generality, we may assume,

$$\lambda_1 = \lambda_2 = \dots = \lambda_{n-1} = \lambda, \quad \lambda_n = \mu,$$

where λ_i for $i = 1, 2, \dots, n$ are the principal curvatures of M^n . Therefore, we know that

(3.1)
$$(n-1)\lambda + \mu = nH, \quad S = (n-1)\lambda^2 + \mu^2.$$

We have,

(3.2)
$$\mu = nH - (n-1)\lambda.$$

From

$$\lambda - \mu = n(\lambda - H) \neq 0,$$

we get $\lambda - H \neq 0$.

Let $\overline{\omega} = |\lambda - H|^{-\frac{1}{n}}$. We denote the integral submanifold through $x \in M^n$ corresponding to λ by $M_1^{n-1}(x)$. Let

(3.3)
$$d\lambda = \sum_{k=1}^{n} \lambda_{k} \,\omega_{k}, \quad d\mu = \sum_{k=1}^{n} \mu_{k} \,\omega_{k}.$$

From Proposition 3.1, we have,

(3.4)
$$\lambda_{1} = \lambda_{2} = \dots = \lambda_{n-1} = 0 \text{ on } M_{1}^{n-1}(x).$$

From (3.2), we have,

(3.5)
$$d\mu = -(n-1)d\lambda.$$

Hence, we also have,

(3.6)
$$\mu_{,1} = \mu_{,2} = \dots = \mu_{,n-1} = 0 \text{ on } M_1^{n-1}(x).$$

In this case, we may locally consider λ to be a function of the arc length s of the integral curve of the principal vector field e_n corresponding to the principal curvature μ . From (2.19) and (3.4), we have, for $1 \leq j \leq n-1$,

(3.7)
$$d\lambda = d\lambda_j = \sum_{k=1}^n Q_{jjk}\omega_k$$
$$= \sum_{k=1}^{n-1} Q_{jjk}\omega_k + Q_{jjn}\omega_n = \lambda_{,n}\,\omega_n.$$

Therefore, we have,

(3.8)
$$Q_{jjk} = 0, \ 1 \le k \le n-1, \ \text{and} \ Q_{jjn} = \lambda_{,n}.$$

By (2.19) and (3.6), we have,

(3.9)
$$d\mu = d\lambda_n = \sum_{k=1}^n Q_{nnk}\omega_k$$
$$= \sum_{k=1}^{n-1} Q_{nnk}\omega_k + Q_{nnn}\omega_n = \sum_{i=1}^n \mu_{,i}\,\omega_i = \mu_{,n}\,\omega_n.$$

Hence, we obtain:

(3.10)
$$Q_{nnk} = 0, \quad 1 \le k \le n-1, \text{ and } Q_{nnn} = \mu_{,n}.$$

From (3.5), we get

(3.11)
$$Q_{nnn} = \mu_{,n} = -(n-1)\lambda_{,n} \,.$$

From the definition of ψ_{ij} , if $i \neq j$, we have $\psi_{ij} = 0$, for $1 \leq i \leq n-1$, and $1 \leq j \leq n-1$. Therefore, from (2.19), if $i \neq j$, $1 \leq i \leq n-1$ and $1 \leq j \leq n-1$, we have,

By (2.19),(3.8),(3.10),(3.11) and (3.12), we get

(3.13)
$$\psi_{jn} = \sum_{k=1}^{n} Q_{jnk} \omega_k$$
$$= Q_{jjn} \omega_j + Q_{jnn} \omega_n = \lambda_{,n} \omega_j.$$

From (2.19), (3.2) and (3.13), we have,

(3.14)
$$\omega_{jn} = \frac{\psi_{jn}}{\lambda - \mu} = \frac{\lambda_{,n}}{\lambda - \mu} \omega_j = \frac{\lambda_{,n}}{n(\lambda - H)} \omega_j.$$

Therefore, from the structure equations of M^n , we have,

$$d\omega_n = \sum_{k=1}^{n-1} \omega_k \wedge \omega_{kn} + \omega_{nn} \wedge \omega_n = 0.$$

Therefore, we may put $\omega_n = ds$. By (3.7) and (3.9), we get

$$d\lambda = \lambda_{,n} \, ds, \quad \lambda_{,n} = \frac{d\lambda}{ds},$$

and

$$d\mu = \mu_{,n} \, ds, \quad \mu_{,n} = \frac{d\mu}{ds}$$

Then, we have,

(3.15)
$$\omega_{jn} = \frac{\frac{d\lambda}{ds}}{n(\lambda - H)}\omega_j = \frac{d\{\log|\lambda - H|^{\frac{1}{n}}\}}{ds}\omega_j.$$

Equation (3.15) shows that the integral submanifold $M_1^{n-1}(x)$ corresponding to λ and s is umbilical in M^n and $M_1^{n+1}(c)$. From (3.15) and the structure equations of $M_1^{n+1}(c)$, we have,

$$d\omega_{jn} = \sum_{k=1}^{n-1} \omega_{jk} \wedge \omega_{kn} + \omega_{jn} \wedge \omega_{nn} - \omega_{jn+1} \wedge \omega_{n+1n} + \Omega_{jn}$$
$$= \sum_{k=1}^{n-1} \omega_{jk} \wedge \omega_{kn} - \omega_{jn+1} \wedge \omega_{n+1n} - c\omega_{j} \wedge \omega_{n}$$
$$= \frac{d\{\log|\lambda - H|^{\frac{1}{n}}\}}{ds} \sum_{k=1}^{n-1} \omega_{jk} \wedge \omega_{k} - (c - \lambda\mu)\omega_{j} \wedge ds.$$

From (3.15), we have,

$$d\omega_{jn} = \frac{d^2 \{\log|\lambda - H|^{\frac{1}{n}}\}}{ds^2} ds \wedge \omega_j + \frac{d \{\log|\lambda - H|^{\frac{1}{n}}\}}{ds} d\omega_j$$

$$= \frac{d^2 \{\log|\lambda - H|^{\frac{1}{n}}\}}{ds^2} ds \wedge \omega_j + \frac{d \{\log|\lambda - H|^{\frac{1}{n}}\}}{ds} \sum_{k=1}^n \omega_{jk} \wedge \omega_k$$

$$= \{-\frac{d^2 \{\log|\lambda - H|^{\frac{1}{n}}\}}{ds^2} + [\frac{d \{\log|\lambda - H|^{\frac{1}{n}}\}}{ds}]^2\} \omega_j \wedge ds$$

$$+ \frac{d \{\log|\lambda - H|^{\frac{1}{n}}\}}{ds} \sum_{k=1}^{n-1} \omega_{jk} \wedge \omega_k.$$

From the above two equalities, we have,

(3.16)
$$\frac{d^2 \{ \log |\lambda - H|^{\frac{1}{n}} \}}{ds^2} - \{ \frac{d \{ \log |\lambda - H|^{\frac{1}{n}} \}}{ds} \}^2 - (c - \lambda \mu) = 0.$$

From (3.2), we get (3.17) $\frac{d^2 \{ \log |\lambda - H|^{\frac{1}{n}} \}}{ds^2} - \{ \frac{d \{ \log |\lambda - H|^{\frac{1}{n}} \}}{ds} \}^2 - \{ (n-1)\lambda^2 - nH\lambda + c \} = 0.$

Since we define $\varpi = |\lambda - H|^{-\frac{1}{n}}$, then we obtain from the above equation,

(3.18)
$$\frac{d^2\varpi}{ds^2} + \varpi\{(n-1)\lambda^2 - nH\lambda + c\} = 0.$$

We can now prove the following lemmas.

Lemma 3.2. Let

$$P_H(t) = (n-1)t^2 - nHt + c.$$

Then, for $c \leq 0$ or c > 0 and $H^2 \geq c$, $P_H(t)$ has two real roots t_1 and t_2 and

(i) if $t \ge H$, then $t \ge t_2$ holds if and only if $P_H(t) \ge 0$ and $t \le t_2$ holds if and only if $P_H(t) \le 0$.

(2) If $t \leq H$, then $t \geq t_1$ holds if and only if $P_H(t) \leq 0$ and $t \leq t_1$ holds if and only if $P_H(t) \geq 0$, where for c = 0, $t_1 = 0$ and $t_2 > 0$, for c < 0, $t_1 < 0$ and $t_2 > 0$, for c > 0 and $H^2 \geq c$, $t_1 > 0$ and $t_2 > 0$.

Proof. We have,

$$\frac{dP_H(t)}{dt} = 2(n-1)t - nH.$$

It follows that the solution of $\frac{dP_H(t)}{dt} = 0$ is $t_0 = \frac{nH}{2(n-1)} > 0$. Therefore, we know that $t \le t_0$ if and only if $P_H(t)$ is a decreasing function, $t \ge t_0$ if and only if $P_H(t)$ is an increasing function and $P_H(t)$ obtain its minimum at $t = t_0$.

Since $P_H(t)$ is continuous and $P_H(t_0) = c - \frac{n^2 H^2}{4(n-1)} < 0$, then we infer that $P_H(t)$ has two distinct real roots t_1 and t_2 with $t_1 < t_0 < t_2$. From $P_H(0) = c$, we infer that for c = 0, $t_1 = 0$ and $t_2 > 0$, for c < 0, $t_1 < 0$ and $t_2 > 0$, for c < 0, $t_1 < 0$ and $t_2 > 0$, for c > 0 and $H^2 \ge c$, $t_1 > 0$ and $t_2 > 0$.

Since $t_0 \leq H$ and $P_H(H) = c - H^2 \leq 0$, then we know that $H \leq t_2$. In fact, if $H > t_2$, then from the increasing property of $P_H(t)$, we have $P_H(H) > P_H(t_2) = 0$, which is a contraction.

Now, we prove the second part of Lemma 3.2. If $t \ge H$, then from the increasing property of $P_H(t)$, we obtain that $t \ge t_2$ holds if and only if $P_H(t) \ge P_H(t_2) = 0$ and $t \le t_2$ holds if and only if $P_H(t) \le P_H(t_2) = 0$.

If $t \leq H$, then from the decreasing property of $P_H(t)$, we directly obtain that $t \leq t_1$ holds if and only if $P_H(t) \geq P_H(t_1) = 0$.

Now, we consider the case $t \leq H$ and $t \geq t_1$. From $t \geq t_1$, we have $t \in [t_1, t_0]$ or $t \in [t_0, H]$. If $t \in [t_1, t_0]$, then from the decreasing property of $P_H(t)$, we infer that $P_H(t) \leq P_H(t_1) = 0$; if $t \in [t_0, H]$, then from

the increasing property of $P_H(t)$, we infer that $P_H(t) \leq P_H(H) \leq 0$. Hence, if $t \geq t_1$, then $P_H(t) \leq 0$. On the other hand, if $P_H(t) \leq 0$, then by $t \leq H$, we can prove $t \geq t_1$. In fact, if $t < t_1$, then from the decreasing property of $P_H(t)$, we infer that $P_H(t) > P_H(t_1) = 0$, which is a contradiction to having $P_H(t) \leq 0$. Therefore, if $t \leq H$, then $t \geq t_1$ holds if and only if $P_H(t) \leq 0$. The proof of the lemma is now complete.

Lemma 3.3. Let

$$S(t) = (n-1)t^2 + [nH - (n-1)t]^2.$$

(i) If $t \ge H$, then $t \ge t_2$ holds if and only if $S(t) \ge S(t_2)$ and $t \le t_2$ holds if and only if $S(t) \le S(t_2)$.

(ii) If $t \leq H$, then $t \geq t_1$ holds if and only if $S(t) \leq S(t_1)$ and $t \leq t_1$ holds if and only if $S(t) \geq S(t_1)$, where t_1 and t_2 are the two distinct real roots of $P_H(t)$ and $t_1 < t_2$.

Proof. We have,

$$\frac{dS(t)}{dt} = 2n(n-1)(t-H).$$

It follows that the solution of $\frac{dS(t)}{dt} = 0$ is t = H. Therefore, $t \leq H$ if and only if S(t) is a decreasing function, $t \geq H$ if and only if S(t) is an increasing function and S(t) obtain its minimum at t = H.

From the proof of Lemma 3.2, we know that $t_1 < H \leq t_2$. Since $t \geq H$ if and only if S(t) is an increasing function, then we infer that if $t \geq H$, then $t \geq t_2$ holds if and only if $S(t) \geq S(t_2)$ and $t \leq t_2$ holds if and only if $S(t) \leq S(t_2)$.

If $t \leq H$, then from the decreasing property of S(t), we directly have $t \geq t_1$ holds if and only if $S(t) \leq S(t_1)$ and $t \leq t_1$ holds if and only if $S(t) \geq S(t_1)$. The proof of the lemma is now complete.

Proof of Main Theorem. Putting $t = \lambda$, from (3.18), we have,

(3.19)
$$\frac{d^2\varpi}{ds^2} + \varpi P_H(t) = 0.$$

Since

$$\lambda - \mu = n(t - H) \neq 0,$$

then we have, $t - H \neq 0$.

(1) For c = 0, (i) if $S \le (n-1)t_2^2$, then we consider two cases t > H and t < H:

Case t > H: Since $S(t_2) = (n-1)t_2^2$, then from Lemma 3.2, Lemma 3.3 and (3.19), we have $S(t) \leq (n-1)t_2^2 = S(t_2)$ holds if and only if $t \leq t_2$ if and only if $P_H(t) \leq 0$ and if and only if $\frac{d^2\varpi}{ds^2} \geq 0$. Thus, $\frac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. Therefore, by the similar assertion in Wei [14], we have that $\varpi(s)$ must is monotonic when s tends to infinity. From the definition of $\varpi(s)$ and $\lim_{s\to\infty} \lambda \neq H$, we infer that the positive function $\varpi(s)$ is bounded. Since $\varpi(s)$ is bounded and monotonic, when s tends to infinity, we know that both $\lim_{s\to-\infty} \varpi(s)$ and $\lim_{s\to+\infty} \varpi(s)$ exist and then we get

$$\lim_{s \to -\infty} \frac{d\varpi(s)}{ds} = \lim_{s \to +\infty} \frac{d\varpi(s)}{ds} = 0.$$

From the monotonicity of $\frac{d\varpi(s)}{ds}$, we have $\frac{d\varpi(s)}{ds} \equiv 0$ and $\varpi(s) = \text{constant}$. From $\varpi = |\lambda - H|^{-\frac{1}{n}}$ and (3.1), we have λ and μ are constants; that is, M^n is isoparametric. Therefore, by the congruence Theorem of Abe et al. [1], M^n is isometric to the Euclidean cylinder $H^{n-1}(c_1) \times R^1$, where $c_1 < 0$.

Case t < H: Since $S(t_2) = (n-1)t_2^2 = \frac{n^2H^2}{n-1} \le n^2H^2 = S(t_1)$, then we have $S \le S(t_1)$. From Lemma 3.2, Lemma 3.3 and (3.19), we have $S(t) \le S(t_1)$ holds if and only if $t \ge t_1$ if and only if $P_H(t) \le 0$ and if and only if $\frac{d^2\omega}{ds^2} \ge 0$. Thus, $\frac{d\omega}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. By the same assertion as above, we know that M^n is isometric to the Euclidean cylinder $H^{n-1}(c_1) \times R^1$, where $c_1 < 0$.

(ii) If $(n-1)t_2^2 \leq S \leq n^2 H^2$, we also consider two cases t > H and t < H.

Case t > H: Since $S \ge (n-1)t_2^2 = S(t_2)$, then from Lemma 3.2, Lemma 3.3 and (3.19), we have $S(t) \ge S(t_2)$ holds if and only if $t \ge t_2$ if and only if $P_H(t) \ge 0$ and if and only if $\frac{d^2\omega}{ds^2} \le 0$. Thus, $\frac{d\omega}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. Combining $\frac{d^2\omega}{ds^2} \le 0$ with the boundedness of $\varpi(s)$, similar to the proof of (i), we know that $\varpi(s)$, λ and μ are constants, that is, M^n is isoparametric. By the congruence Theorem of Abe et al. [1], we know that M^n is isometric to the hyperbolic cylinder $H^1(c_1) \times R^{n-1}$ or the Euclidean cylinder $H^{n-1}(c_1) \times R^1$, where $c_1 < 0$.

Case t < H: Since $S \le n^2 H^2 = S(t_1)$, from Lemma 3.2, Lemma 3.3 and (3.19), we have $S(t) \le S(t_1)$ holds if and only if $t \ge t_1$ if and only if $P_H(t) \le 0$ and if and only if $\frac{d^2 \omega}{ds^2} \ge 0$. Thus, $\frac{d\omega}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. By the same assertion as above, we know that M^n is isometric to the hyperbolic cylinder $H^1(c_1) \times R^{n-1}$ or the Euclidean cylinder $H^{n-1}(c_1) \times R^1$, where $c_1 < 0$.

(2) For c < 0, (i) if $S \le (n-1)t_2^2 + c^2t_2^{-2}$, we consider two cases t > H and t < H.

Case t > H: Since

$$S(t_2) = (n-1)t_2^2 + [nH - (n-1)t_2]^2$$

= $(n-1)t_2^2 + [nH - (n-1)t_2 - \frac{c}{t_2} + \frac{c}{t_2}]^2$
= $(n-1)t_2^2 + \{\frac{-1}{t_2}[(n-1)t_2^2 - nHt_2 + c] + \frac{c}{t_2}\}^2$
= $(n-1)t_2^2 + \{\frac{-1}{t_2}P_H(t_2) + \frac{c}{t_2}\}^2$
= $(n-1)t_2^2 + c^2t_2^{-2}$,

Then from Lemma 3.2, Lemma 3.3 and (3.19), we have $S(t) \leq S(t_2)$ holds if and only if $t \leq t_2$ if and only if $P_H(t) \leq 0$ and if and only if $\frac{d^2 \varpi}{ds^2} \geq 0$. Thus, $\frac{d \varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. By the same assertion in the proof of (1), we know that $\varpi(s)$, λ and μ are constants; that is, M^n is isoparametric. By the congruence Theorem of Abe et al. [1], we know that M^n is isometric to the hyperbolic cylinder $H^1(c_1) \times H^{n-1}(c_2)$, where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 < 0$.

Case t < H: By a direct calculation, we have $S(t_1) = (n-1)t_1^2 + c^2t_1^{-2}$. From (1.8), we have $S(t_2) \leq S(t_1)$. Hence, we obtain that $S \leq S(t_1)$. From Lemma 3.2, Lemma 3.3 and (3.19), we have $S(t) \leq S(t_1)$ holds if and only if $t \geq t_1$ if and only if $P_H(t) \leq 0$ and if and only if $\frac{d^2\omega}{ds^2} \geq 0$. Thus, $\frac{d\omega}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. By the same assertion as above, we know that M^n is isometric to the hyperbolic cylinder $H^1(c_1) \times H^{n-1}(c_2)$, where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 < 0$.

cylinder $H^1(c_1) \times H^{n-1}(c_2)$, where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 < 0$. (ii) If $(n-1)t_2^2 + c^2t_2^{-2} \le S \le (n-1)t_1^2 + c^2t_1^{-2}$, then we consider two cases t > H and t < H.

Case t > H: Since $S \ge (n-1)t_2^2 + c^2t_2^{-2} = S(t_2)$, then from Lemma 3.2, Lemma 3.3 and (3.19), we have $S(t) \ge S(t_2)$ holds if and only if $t \ge t_2$ if and only if $P_H(t) \ge 0$ and if and only if $\frac{d^2\omega}{ds^2} \le 0$. Thus, $\frac{d\omega}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. Similar to the proof of (1), we know that $\varpi(s)$, λ and μ are constants; that is, M^n is isoparametric. By the congruence Theorem of Abe et al. [1], we know that M^n is isometric to the hyperbolic cylinder $H^1(c_1) \times H^{n-1}(c_2)$, where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 < 0$.

Case t < H: Since $S \leq (n-1)t_1^2 + c^2t_1^{-2} = S(t_1)$, then from Lemma 3.2, Lemma 3.3 and (3.19), we have $S(t) \leq S(t_1)$ holds if and only if $t \geq t_1$ if and only if $P_H(t) \leq 0$ and if and only if $\frac{d^2\varpi}{ds^2} \geq 0$. Thus, $\frac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. By the same assertion as above, we know that M^n is isometric to the hyperbolic cylinder $H^1(c_1) \times$ $H^{n-1}(c_2)$, where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 < 0$. (3) For c > 0 and $H^2 \geq c$, if (i) $S \leq (n-1)t_2^2 + c^2t_2^{-2}$ or (ii) $(n-1)t_2^2 + c^2t_2^{-2} \leq S \leq (n-1)t_1^2 + c^2t_1^{-2}$, then by the same assertion in

(3) For c > 0 and $H^2 \ge c$, if (i) $S \le (n-1)t_2^2 + c^2t_2^{-2}$ or (ii) $(n-1)t_2^2 + c^2t_2^{-2} \le S \le (n-1)t_1^2 + c^2t_1^{-2}$, then by the same assertion in the proof of (2), we can also prove that $\varpi(s)$, λ and μ are constants; that is, M^n is isoparametric. By the congruence Theorem of Abe et al. [1], we know that M^n is isometric to the hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$ or the spherical cylinder $H^{n-1}(c_1) \times S^1(c_2)$, where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 > 0$. This completes the proof of the Main Theorem.

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