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LIE TRIPLE DERIVATION ALGEBRA OF VIRASORO-LIKE ALGEBRA

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Abstract. Let $\mathfrak{L}$ be the Virasoro-like algebra and $\mathfrak{g}$ its derived algebra, respectively. We investigate the structure of the Lie triple derivation algebra of $\mathfrak{L}$ and $\mathfrak{g}$. We prove that they are both isomorphic to $\mathfrak{L}$, which provides two examples of invariance under triple derivation.

1. **Introduction**

Let A be a Lie algebra over the complex number field $\mathbb{C}$. Recall that a $\mathbb{C}$-linear mapping $\phi : A \to A$ is called a Lie derivation if

$$\phi([a,b]) = [\phi(a), b] + [a, \phi(b)] \qquad (\forall\ a, b \in A).$$

If ϕ satisfies

$$\phi([[a,b],c]) = [[\phi(a),b],c] + [[a,\phi(b)],c] + [[a,b],\phi(c)] \tag{1.1}$$

for all $a, b, c \in A$, it is called a Lie triple derivation.

A derivation is a triple derivation. Like derivation algebra, the set of triple derivations is a Lie algebra under the usual bracket. It is denoted by $\mathrm{Der}^{(3)}A$. For some algebra, the derivation algebra $\mathrm{Der}A$ is a proper subalgebra of $\mathrm{Der}^{(3)}A$, see [7]. Lie triple derivation algebra of some kinds of operator algebras have been studied in [1, 4].

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Recently, Virasoro algebra, as a universal central extension of Witt algebra, has played an important role in mathematics and physics[9]. The q-analog Virasoro algebra was introduced by Kirkman et al as a derivation subalgebra of quantum torus (see [6]). On one hand, Virasoro-like algebra is the generalization of Virasoro algebra. On the other hand, it can be regarded as a special case of q-analog Virasoro algebra when $q = 1$. Lin and Tan [8] and Wang and Zhao[11] studied the nonzero level Harish-Chandra modules and Verma module of Virasoro-like algebra, respectively. Jiang and Meng[5] gave the structure of its derivation algebra and the automorphism group. Song *et al.* got the quantization of generalized Virsoro-like algebra in [10]. As early as in 1957, Herstein in [3] proved that any Jordan triple derivation is an ordinary derivation on a prime ring of characteristic different from 2. In this paper, we will prove that the Lie triple derivation of Virasoro-like algebra $\mathfrak{L}$ is isomorphic to $\mathfrak{L}$. Meanwhile we obtain that any Lie triple derivation of Virasoro-like algebra is a Lie derivation, i.e.,

$$\mathrm{Der}\mathfrak{L} = \mathrm{Der}^{(3)}\mathfrak{L}. \tag{1.2}$$

The paper is arranged as follows. In Section 2, we give some notations and recall some basic facts on $\mathfrak{L}$. In Section 3, we study the graded version of $\mathrm{Der}^{(3)}\mathfrak{L}$, which is a generalization of [2]. In Section 4, we give the main result of this paper.

2. **Preliminaries**

Throughout this paper we use $\mathbb{C}$, $\mathbb{Z}$ to denote the set of complex numbers, and integers, respectively. We assume that $\mathcal{U}$ is a 2-dimensional vector space over $\mathbb{C}$. We choose

$$e_1 = (1, 0), \quad e_2 = (0, 1)$$

as the basis of $\mathcal{U}$ and let $(\cdot, \cdot)$ be the standard bilinear form on $\mathcal{U}$, i.e.,

$$(e_i, e_j) = \delta_{ij}, \quad 1 \leq i, j \leq 2.$$

Let $A = \mathbb{C}[x_1^{\pm}, x_2^{\pm}]$ be a Laurent polynomial with 2 commuting variables x_1, x_2 over $\mathbb{C}$.

The elements of A are denoted by $x^r = x_1^{r_1} x_2^{r_2}$ for $r = (r_1, r_2)$. Set $D_i(r) = x^r x_i(\partial/\partial x_1)$ for $r \in \mathbb{Z}^2$. For all $u = (u_1, u_2) \in \mathcal{U}$, we define

$$D(u, r) = u_1 D_1(r) + u_2 D_2(r).$$

The space Der$A$ is a Lie algebra under the following bracket:

$$[D(u,r), D(u',r')] = D(u'', r+r'),$$

where $u'' = (u,r')u' - (u',r)u$.
A subalgebra $\mathfrak{L}$ of Der$A$ is called (centerless) Virasoro-like algebra if it satisfies

$$\mathfrak{L} = \bigoplus_{r\in\mathbb{Z}^2} \mathfrak{L}_r,$$

where $\mathfrak{L}_r = \{D(u,r)|(u,r)=0\}$. Let $\mathfrak{g} = [\mathfrak{L},\mathfrak{L}]$ be the derived algebra of $\mathfrak{L}$, then $\mathfrak{L} = \mathfrak{g} \bigoplus \mathfrak{h}$, where

$$\mathfrak{g}_r = \begin{cases} \mathfrak{L}_r, r \neq 0 \\ 0 \quad\ \ r = 0 \end{cases}$$

and $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{L}$.

In this paper we study the centerless Virasoro-like algebra.

3. Gradation of $\mathfrak{L}$ and $\mathbf{Der}^{(3)}\mathfrak{L}$

In this section, we prove that $\mathbf{Der}^{(3)}\mathfrak{L}$ and $\mathbf{Der}^{(3)}\mathfrak{g}$ are both graded algebras.

Let G be an abelian group, $L = \bigoplus_{g\in G} L_g$ a G-graded Lie algebra over an algebraically closed field $\mathbb{F}$. Let V be a G-graded L-module. A linear mapping $\varphi : L \longrightarrow V$ is called a triple derivation if

(3.1) $\varphi([x,[y,z]]) = z.y.\varphi(x) - z.x.\varphi(y) + x.y.\varphi(z) - y.x.\varphi(z)$

for all $x, y, z \in L$. We say that a triple derivation φ has degree g if $\varphi(L_h) \subseteq V_{g+h}$.

This definition generalizes (1.1): Take the regular module $V = L$, then (3.1) will be (1.1).

Theorem 3.1. *Let L be a finitely generated Lie algebra and V a G-graded L-module. Then*

$$\mathrm{Der}^{(3)}_{\mathbb{F}}(L,V) = \bigoplus_{g\in G} \mathrm{Der}^{(3)}_{\mathbb{F}}(L,V)_g.$$

This theorem can be proved using a similar argument as in [2].

Proof. Let S be a finite generating set of L. For each element $g \in G$ we define two canonical projections $p_g : L \longrightarrow L_g$ and $\pi_g : V \longrightarrow V_g$. Suppose $\varphi : L \longrightarrow V$ is a triple derivation and there are finite subsets $Q, R \subset G$ such that

$$S \subset \sum_{g \in Q} L_g \quad \text{and} \quad \varphi(S) \subset \sum_{g \in R} V_g.$$

For $g \in G$ let $\varphi_g := \sum_{h \in G} \pi_{g+h} \circ \varphi \circ p_h$. Now for $x_h \in L_h,\ x_k \in L_k$ and $x_l \in L_l$, we have

$$\begin{aligned} & \varphi_g([[x_h, x_k], x_l]) \\ = \;& \pi_{g+h+k+l} \circ \varphi([[x_h, x_k], x_l]) \\ = \;& \pi_{g+h+k+l}(x_l.x_k.\varphi(x_h) - x_l.x_h.\varphi(x_h) + x_h.x_k.\varphi(x_l) - x_k.x_h.\varphi(x_l)) \\ = \;& x_l.x_k.\pi_{g+h}(\varphi(x_h)) - x_l.x_h.\pi_{g+h}(\varphi(x_h)) \\ & \qquad + x_h.x_k.\pi_{g+l}(\varphi(x_l)) - x_k.x_h.\pi_{g+l}(\varphi(x_l)) \\ = \;& x_l.x_k.(\varphi_g(x_h)) - x_l.x_h.(\varphi_g(x_h)) + x_h.x_k.(\varphi_g(x_l)) - x_k.x_h.(\varphi_g(x_l)), \end{aligned}$$

which follows that φ_g is contained in $\mathrm{Der}_{\mathbb{F}}^{(3)}(L, V)_g$.

Let $T = \{g - h | g \in R, h \in Q\}$. For any $y \in S$ we have

$$\begin{aligned} \varphi(y) &= \sum_{g \in R} \pi_g \circ \varphi(y) = \sum_{g \in R} \sum_{h \in Q} \pi_g \circ \varphi \circ p_h(y) \\ &= \sum_{g \in R} \sum_{h \in Q} \pi_{g-h}(y) \\ &= \sum_{q \in T} \varphi_q(y). \end{aligned}$$

This shows that $\varphi = \sum_{q \in T} \varphi_q$ on S and subsequently L. $\square$

Lemma 3.2. *The Lie algebra* $\mathfrak{g}$ *is generated by*

$$D(e_1, \pm e_2), D(e_2, \pm e_1),$$

where e_1, e_2 *are defined above.*

Note that $\mathfrak{L}$ is generated by $\mathfrak{g}$ and $D(e_1, 0), D(e_2, 0)$, so $\mathfrak{L}$ is also finitely generated with the generators

$$D(e_1, \pm e_2), D(e_2, \pm e_1), D(e_1, 0), D(e_2, 0).$$

Theorem 3.1 and Lemma 3.2 imply that $\mathrm{Der}^{(3)}(\mathfrak{L})$ and $\mathrm{Der}^{(3)}(\mathfrak{g})$ are $\mathbb{Z}^2$-graded Lie algebras.

4. Structure of $\mathbf{Der}^{(3)}\mathfrak{g}$ and $\mathbf{Der}^{(3)}\mathfrak{L}$

4.1. **The Structure of $\mathbf{Der}^{(3)}\mathfrak{g}$.** In this section we will characterize the structure $\mathbf{Der}^{(3)}\mathfrak{g}$ and we will show that $\mathbf{Der}^{(3)}\mathfrak{g} \cong \mathfrak{L}$.

Lemma 4.1. *For any nonzero $r \in \mathbb{Z}^2$, $(\mathbf{Der}^{(3)}\mathfrak{g})_r = ad\mathfrak{g}_r$.*

Proof. We only need to show that there exists an element $y \in \mathfrak{g}_r$ such that $ady = \varphi$ for any $\varphi \in (\mathbf{Der}^{(3)}\mathfrak{g})_r$ and $r \in \mathbb{Z}^2$.

By Lemma 3.2, it is enough to prove $ady = \varphi$ holds for the generators of $\mathfrak{g}$.

Suppose $\varphi(D(e_1, e_2)) = D(w, e_2 + r)$, where $w = (w_1, w_2) \in \mathcal{U}$.

Case 1: $r_1 \neq 0$. By choosing

$$u = (-\frac{w_2}{r_1}, -\frac{r_1 w_1 + w_2}{r_1^2}),$$

we get

$$\varphi(D(e_1, e_2)) = adD(u, r)(D(e_1, e_2)).$$

It is convenient to replace φ by $\varphi - adD(u, r)$, then $\varphi(D(e_1, e_2)) = 0$.

We assume $\varphi(D(e_1, -e_2)) = D(v, -e_2 + r)$ with $v = (v_1, v_2) \in \mathcal{U}$.

Observe that $[D(e_1, e_2), D(e_1, -e_2)] = 0$, so

$$\varphi([[D(e_1, e_2), D(e_1, -e_2)], D(e_1, e_2)]) = 0.$$

However, φ is a triple derivation, so we have

$$\begin{aligned} &\varphi([[D(e_1, e_2), D(e_1, -e_2)], D(e_1, e_2)]) \\ &= [[D(e_1, e_2), D(v, -e_2 + r)], D(e_1, e_2)] \\ &= D(v', e_2 + r) \end{aligned} \tag{4.1}$$

for some $v' \in \mathbb{C}^2$. Subsequently, $\varphi([[D(e_1, e_2), D(e_1, -e_2)], D(e_1, e_2)]) = D(v', e_2 + r) = 0$, which forces $v' = 0$.

On the other hand, direction computation of v' gives that

$$\begin{cases} 2r_1 v_2 - r_1^2 v_1 = 0 \\ r_1^2 v_2 = 0, \end{cases}$$

which implies that $\varphi(D(e_1, -e_2)) = 0$.

We prove $\varphi(D(e_2, e_1)) = \varphi(D(e_2, -e_1)) = 0$ in several steps.

We first assume that $\varphi(D(e_2,e_1)) = D(\overline{v}, e_1+r)$ with $\overline{v} = (\overline{v}_1, \overline{v}_2)$.

If $r \neq (-2, r_2)$, by considering the image of φ on the following equality

$$[[D(e_1,e_2), D(e_2,e_1)], D(e_1,-e_2)] = D(e_2,e_1),$$

we know $(r_1+1)^2\overline{v} = \overline{v}$. Hence $\overline{v} = 0$.

Similarly if $r \neq (2, r_2)$, by applying φ on

$$[[D(e_1,e_2), D(e_2,-e_1)], D(e_1,-e_2)] = -D(e_2,-e_1),$$

we get $\varphi(D(e_2,-e_1)) = 0$.

So we can conclude that if:
$r = (-2, r_2)$,

$$\varphi(D(e_1,\pm e_2)) = \varphi(D(e_2,-e_1)) = 0;$$

if $r = (2, r_2)$,

$$\varphi(D(e_1,\pm e_2)) = \varphi(D(e_2,e_1)) = 0.$$

Next we consider the case $r = (-2, r_2)$.

If $r_2 \neq 0$,

$$\begin{aligned} & \varphi([[D(e_2,e_1), D(e_2,-e_1)], D(e_1,e_2)]) \\ = \ & [[D(\overline{v}, e_1+r), D(e_2,-e_1)], D(e_1,e_2)] \\ = \ & D(\overline{v''}, e_2+r) \end{aligned}$$

where $\overline{v''} = -((2r_2+1)\overline{v}_1 + r_2\overline{v}_2, 2\overline{v}_1 + 2r_2\overline{v}_2)$, which tells us $\overline{v} = 0$. i.e. $\varphi(D(e_2,e_1)) = 0$.

If $r = (-2, 0)$, then

$$[[D(e_1,e_2), D(e_2,e_1)], D(e_2,-e_1)] = D(e_1,e_2)$$

also implies $\varphi(D(e_2,e_1)) = 0$.

Similar discussion will lead to $\varphi(D(e_2,-e_1)) = 0$ for $r = (-2, r_2)$ with $r_2 \in \mathbb{Z}$.

Case 2: $r_1 = 0$. Suppose $\varphi(D(e_2,e_1)) = D(w, e_1+r)$, we choose an element $D(u', r) \in \mathfrak{g}_r$ with

$$u' = (-\frac{w_1}{r_2}, -\frac{w_1 + r_2 w_2}{r_2^2}).$$

Then we get

$$(\varphi - adD(u', r))(D(e_2,e_1)) = 0.$$

Now we assume $\varphi(D(e_2,-e_1)) = D(\check{w}, -e_1+r)$ with $\check{w} = (\check{w}_1\check{w}_2)$.

Similarly as before (see (4.1)),

$$[[D(e_2, e_1)D(e_2, -e_1)]D(e_2, e_1)] = 0$$

implies that we also have

$$\left\{\begin{array}{l} -r_2^2\check{w}_1 = 0 \\ -r_2^2\check{w}_2 + 2r_2\check{w}_1 = 0. \end{array}\right. \tag{4.2}$$

Thus $\varphi(D(e_2, -e_1)) = 0$.

If $r_2 \neq -2$, $\varphi(D(e_1, e_2)) = 0$ due to the fact that

$$\varphi([[D(e_2, e_1)D(e_1, e_2)]D(e_2, -e_1)]) = \varphi(-D(e_1, e_2)).$$

If $r_2 = -2$, i.e. $r = (0, -2)$, we also have

$$\varphi([[D(e_2, e_1)D(e_1, e_2)]D(e_1, -e_2)]) = \varphi(D(e_2, e_1)) = 0.$$

Similar discussion on $r_2 = 2$ or $r_2 \neq 2$, gives $\varphi(D(e_1, -e_2)) = 0$. Therefore $\varphi(D(e_i, e_j)) = 0,\ 1 \leq i \neq j \leq 2$. By Lemma 3.2 we also have $\varphi(\mathfrak{g}) = 0$ for all nonzero $r \in \mathbb{Z}^2$. This completes the proof. $\square$

Theorem 4.2. $\mathbf{Der}^{(3)}\mathfrak{g} \cong \mathfrak{L}$*.*

Proof. We first define the action of $D(u, 0) \in \mathfrak{h}$ on $\mathfrak{g}$ as follows:

$$D(u, 0).D(v, s) = [D(u, 0), D(v, s)] = (u, s)D(v, s),\ \forall s \in \mathbb{Z}^2 \setminus \{0\},$$

which is a 0-homogeneous outer derivation of $\mathfrak{g}$.

We note that $\varphi(D(e_i, \pm e_j)) = c_{ij}^{\pm} D(e_i, \pm e_j)$ and

$$[[D(e_i, e_j)D(e_j, e_i)]D(e_j, -e_i)]) = D(e_i, e_j),\ 1 \leq i \neq j \leq 2.$$

By applying φ to both sides it follows that $c_{ij}^{+} = -c_{ij}^{-}$.

We then define

$$h = c_{21}^{+} D(e_1, 0) + c_{12}^{+} D(e_2, 0).$$

Then we can check $(\varphi - h)(\mathfrak{g}) = 0$ when $r = 0$.

Therefore by Section 3 and Lemma 4.1 we have

$$\begin{aligned}
\mathbf{Der}^{(3)}\mathfrak{g} &= \oplus_{r\in\mathbb{Z}^2}\mathbf{Der}^{(3)}\mathfrak{g}_r \\
&\cong \oplus_{r\in\mathbb{Z}^2\setminus\{0\}} ad\mathfrak{g}_r \oplus \mathfrak{h} \\
&= ad\mathfrak{g} \oplus \mathfrak{h} \\
&= ad\mathfrak{L} \\
&= \mathfrak{L}.
\end{aligned}$$

□

4.2. **The Structure of $\mathbf{Der}^{(3)}\mathfrak{L}$.** In this section we will prove $\mathbf{Der}^{(3)}\mathfrak{L} \cong \mathfrak{L}$ by introducing the following lemma.

Lemma 4.3. *Suppose $\varphi \in \mathbf{Der}^{(3)}\mathfrak{L}$. If $\varphi(\mathfrak{g}) = 0$, then $\varphi(\mathfrak{h}) = 0$.*

Proof. We only consider $D(e_1, 0)$.

Assume that $\varphi(D(e_1, 0)) = D(\tilde{u}, r)$, where $\tilde{u} = (\tilde{u_1}, \tilde{u_2})$.

Case 1: $r \neq 0$.

If $r_1 \neq 0$, applying φ on both sides of

$$[[D(e_1, e_2), D(e_1, 0)], D(e_1, e_2)] = 0,$$

we have

$$\begin{cases} 2r_1\tilde{u}_2 - r_1^2\tilde{u}_1 = 0 \\ r_1^2\tilde{u}_2 = 0, \end{cases} \tag{4.3}$$

which follows that $\tilde{u} = 0$ and $\varphi(D(e_1, 0)) = 0$.

If $r_2 \neq 0$, it follows from

$$[[D(e_2, e_1), D(e_1, 0)], D(e_2, -e_1)] = 0.$$

Case 2: $r = 0$.

Note that

$$[[D(e_1, e_2), D(e_1, 0)], D(e_2, e_1)] = 0.$$

According to our assumption, the operation of φ on the both sides is

$$[[D(e_1, e_2), \tilde{u}_1 D(e_1, 0) + \tilde{u}_2 D(e_2, 0)], D(e_j, e_i)] = 0.$$

By a simple calculation we can get $\tilde{u}_2 = 0$.

The same action on

$$[[D(e_2, e_1), D(e_1, 0)], D(e_1, 0)] = D(e_2, e_1)$$

tells us $\tilde{u}_1 = 0$. This completes the proof. □

We therefore have:

Theorem 4.4. $\mathbf{Der}^{(3)}\mathfrak{L} \cong \mathfrak{L}$.

Proof. Let ϕ be a triple derivation of $\mathfrak{L}$. By Theorem 3.1 we have

$$\phi = \sum_{r\in\mathbb{Z}^2} \phi_r.$$

Repeat the process of Section 4.1, we deduce that there exists an element $x_r \in \mathfrak{L}_r$ such that $(\phi_r - adx_r)(\mathfrak{g}) = 0$, for all $r \in \mathbb{Z}^2$. By Lemma 4.3, we obtain $(\phi_r - adx_r)(\mathfrak{L}) = 0$. So

$$\begin{aligned} \mathbf{Der}^{(3)}\mathfrak{L} &= \oplus_{r\in\mathbb{Z}^2}\mathbf{Der}^{(3)}\mathfrak{L}_r \\ &= \oplus_{r\in\mathbb{Z}^2} ad\mathfrak{L}_r \\ &= ad\mathfrak{L} \\ &\cong \mathfrak{L}. \end{aligned}$$

□

Corollary 4.5. *Suppose* $\mathfrak{g}$ *and* $\mathfrak{L}$ *are defined as above, Then*

$$\mathbf{Der}\mathfrak{g} = \mathbf{Der}^{(3)}\mathfrak{g} = \mathbf{Der}\mathfrak{g} = \mathbf{Der}^{(3)}\mathfrak{L} \cong \mathfrak{L}.$$

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References

[1] J.-H. Zhang, B.-W. Wu and H.-X. Cao, Lie triple derivations of nest algebras, *Linear Algebra Appl.* **416** (2006), no. 2-3, 559–567.

[2] R. Farnsteiner, Derivations and central extensions of finitely generated graded Lie algebras, *J. Algebra* **118**, (1988), no. 1, 33–45.

[3] I. Herstein, Jordan derivation of primes, *Proc. Amer. Math. Soc.* **8** (1957) 1104–1110.

[4] P. Ji and L. Wang, Lie triple derivations of TUHF algebras, *Linear Algebra Appl.* **403** (2005) 399–408.

[5] C.-B. Jiang, D.-J. Meng, The automorphism group of the derivation algebra of the Virasoro-like algebra, *Adv. Math.* (China) **27** (1998), no. 2, 175–183.

[6] E. Kirkman, C. Procesi and L. Small, A q-analog for the Virasoro algebra. *Comm. Algebra* **22** (1994), no. 10, 3755–3774.
[7] H. Wang, Q. Li, Lie triple derivation of the Lie algebra of strictly upper triangular matrix over a commutative ring, *Linear Algebra Appl.* **430** (2009), no. 1, 66–77.
[8] W. Lin, S. Tan, Nonzero level Harish-Chandra modules over the Virasoro-like algebra, *J. Pure Appl. Algebra* **204** (2006), no. 1, 90–105.
[9] O. Mathieu, Classification of Harish-Chandra modules over the Virasoro algebra, *Invent. Math.* **107** (1992), 225–234.
[10] S.-A. Song, Y.-C. Su and Y.-Z. Wu, Quantization of generalized Virasoro-like algebras, *Linear Algebra Appl.* **428** (2008), no. 11, 2888–2899.
[11] X.-D. Wang, K.-M. Zhao, Verma modules over the Virasoro-like algebra, *J. Aust. Math. Soc.* **80** (2006), 179–191.

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