Bulletin of the Iranian Mathematical Society Vol. 38 *No. 1* (*2012*), *pp 27-38.*

FUNCTION SPACES OF REES MATRIX SEMIGROUPS

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Communicated by Gholam hossein Esslamzadeh

Abstract. We characterize function spaces of Rees matrix semigroups. Then we study these spaces by using the topological tensor product technique.

1. **Introduction**

Let S be a semitopological semigroup. A semigroup compactification of S is a compact right topological semigroup containing a dense homomorphic image of S. The theory of semigroup compactifications has already received extensive treatment, including two major topics of existence of semigroup compactifications of a semigroup and the structure of its compactifications. Semigroup extensions are extremely useful tools in characterizing function spaces on topological semigroups and in particular those spaces associated with semigroup compactifications; See [1] for instance. A large class of semigroups which has been studied extensively from various points of view, is the class of completely 0-simple and completely simple semigroups. Following Munn [11], Esslamzadeh in [3, 4, 5] studied structure of regular semigroups with a finite number of idempotents and in particular completely 0-simple semigroups, by using their underlying groups. In this paper we study function spaces on these semigroups, by using the techniques of ℓ^1-Munn algebras. To our best

MSC(2010): Primary: 43A15 ; Secondary 43A60

Keywords: Semigroup compactification, completely 0-simple semigroup, topological tensor product.

Received: 30 April 2009, Accepted: 21 June 2010.

knowledge this approach is new. Topological tensor products of topological semigroups were introduced for the first time in the PhD thesis of the author [7]. See [8, 9] for the structure of topological tensor products of topological semigroups, their function spaces and compactifications.

This paper is organized as follows: In Section 2, we introduce our notations. In Sections 3 and 4, we use extension techniques to characterize spaces of functions on a completely 0-simple semigroup. In the last section we study these spaces by using the topological tensor product.

2. **Preliminaries**

Throughout we use the notations of [1]. For the terms which are not introduced here, the reader may be referred to one of [1, 4, 8, 9]. Suppose S is a semitopological semigroup and (ψ, X) is a semigroup compactification of S, that is, X is a compact Hausdorff right topological semigroup and let $\psi : S \to X$ is a continuous homomorphism such that $\overline{\psi(S)} = X$ and $\psi(S) \subseteq \Lambda(X)$ where

$$\Lambda(X) = \{t \in X : s \to ts : X \to X, \text{ is continuous}\}$$

is the topological center of X. We say that (ψ, X) has the left [right] joint continuity property if the mapping $(s, x) \to \psi(s)x$ $[(x, s) \to x\psi(s)]$ is continuous.

Let $\mathcal{B}(S)$ be the C^*-algebra of all bounded complex valued functions on S, $\mathcal{F}$ a unital C^*-subalgebra of $\mathcal{B}(S)$, $S^{\mathcal{F}}$ the set of all multiplicative means on $\mathcal{F}$ and $\varepsilon : S \to S^{\mathcal{F}}$ be the evaluation mapping. We say that $\mathcal{F}$ is m-admissible if $T_\mu(\mathcal{F}) \subseteq \mathcal{F}$ for all $\mu \in S^{\mathcal{F}}$, where $T_\mu(f)(s) = \mu(L_s(f))$, $s \in S$, $f \in \mathcal{F}$. If we equip $S^{\mathcal{F}}$ with the Gelfand topology then $S^{\mathcal{F}}$ with multiplication $\mu\nu(f) = \mu(T_\nu(f))$, $\mu, \nu \in S^{\mathcal{F}}$ is a compact Hausdorff right topological semigroup. Moreover, the evaluation mapping is a continuous homomorphism into a dense subsemigroup of $S^{\mathcal{F}}$ which is contained in the topological center of $S^{\mathcal{F}}$. Now if (ψ, X) is a compactification of S, then $\psi^*(C(X))$ is an m-admissible subalgebra of $C(S)$. Conversely, if $\mathcal{F}$ is an m-admissible subalgebra of $C(S)$, then there exists a unique (up to isomorphism) compactification (ψ, X) of S such that $\psi^*(C(X)) = \mathcal{F}$. In other words, the compactification corresponding to the m-admissible subalgebra $\mathcal{F}$ is $(\varepsilon, S^{\mathcal{F}})$. Moreover, $\varepsilon^*(C(S^{\mathcal{F}})) = \mathcal{F}$.

Let S and T be semitopological semigroups with semigroup compactifications S' and T'. A continuous function $\varphi' : S' \to T'$ is an extension of the continuous function $\varphi : S \to T$ if $\varphi' \circ \varepsilon_S = \varepsilon_T \circ \varphi$ and φ' is uniquely determined by φ. Such an extension exists if and only if $\varphi^*(B) \subseteq A$, where A and B are the associated function spaces of the compactifications. Let S' and S'' be compactifications of S. Then S' is a factor of S'' if the identity map on S has an extension $\varphi : S'' \to S'$. A compactification with a given property $\mathcal{P}$ is called a $\mathcal{P}$-compactification. A universal $\mathcal{P}$-compactification of S is a $\mathcal{P}$-compactification of which, every $\mathcal{P}$-compactification of S is a factor. Universal $\mathcal{P}$-compactifications, if they exist, are unique (up to isomorphism). We denote the universal $\mathcal{P}$-compactification of S by $S^{\mathcal{P}}$.

Let $G^0 = G \cup \{0\}$ [respectively G] be a group with zero [respectively group], I and J be arbitrary nonempty sets. Let P be a $J \times I$ matrix over G^0 [respectively G]. The set $S = G \times I \times J \cup \{0\}$ [respectively $S = G \times I \times J$] is a semigroup under the composition $(a, i, j) \circ (b, l, k) = (aP_{jl}b, i, k)$ if $P_{jl} \neq 0$ and zero otherwise, [respectively $(a, i, j) \circ (b, l, k) = (aP_{jl}b, i, k)$] that we denote by $\mathcal{M}^0(G, P)$ [respectively $\mathcal{M}(G, P)$] and we call it the Rees $I \times J$ matrix semigroup over G^0 [respectively G] with the sandwich matrix P.

As it was observed in [4] the reduced semigroup algebra of a completely 0 -simple semigroup S is isometrically isomorphic to an appropriate ℓ^1-Munn algebra over the group algebra of the underlying group of S. This fact is one of the authors motivations to study function spaces of completely 0-simple semigroups with a new approach.

3. **Compactifications of a completely 0-simple semigroup**

In this section first we introduce an extension of a completely 0-simple semigroup $S = \mathcal{M}^0(G, P)$. Then we characterize compactifications of a completely 0-simple semigroup.

Following Clifford and Preston [2], let S and T be disjoint semigroups, and let T have a zero element 0. A semigroup Ω is called an extension of S by T if it contains S as an ideal and if the Rees factor semigroup $\frac{\Omega}{S}$ is isomorphic to T. A mapping $A \mapsto \bar{A}$ of $T^* = T \setminus \{0\}$ into S is a partial homomorphism if and only if $\overline{AB} = \overline{A}\,\overline{B}$, whenever $AB \neq 0$. On the other hand by [2, 4.19], a partial homomorphism $A \to \overline{A}$ of the partial groupoid T^* into S determines an extension Ω of S by T as follows: For $A, B \in T$ and $s, t \in S$,

$$(P1) \quad A \circ B = \begin{cases} AB & if AB \neq 0 \\ \overline{A}\,\overline{B} & if AB = 0 \end{cases}$$

$$(P2)\quad A\circ s=\overline{A}s,\qquad (P3)\quad s\circ A=s\overline{A},\qquad (P4)\quad s\circ t=st.$$

Let G a be a topological group, I and J arbitrary nonempty sets and $P=(p_{ji})$ a $J\times I$ matrix with entries in $G^0=G\cup\{0\}$. The Rees matrix semigroup $S=\mathcal{M}^0(G,P)$ where $G\times I\times J$ is equipped with the product topology and $\{0\}$, considered as an isolated point of S, is a topological semigroup. Suppose $i\to u_i$ and $j\to v_j$ are mappings of I and J into G, respectively. Let W be a continuous homomorphism of G into G such that $W(p_{ji})=v_ju_i$ whenever $p_{ji}\neq 0$. Then the map $\theta:S\to G$ defined by $\theta(g,i,j)=u_i(W(g))v_j$ is a continuous partial homomorphism of S into G [2, Theorem 4.22]. Hence we have an extension Ω of G by S. In this case we define the relation ρ on Ω by $\sigma_1\rho\sigma_2$ if and only if $\sigma_1=g\sigma_2$ for some $g\in G$. We fix these notations for the rest of this paper.

Lemma 3.1. *The relation ρ is an equivalence relation on Ω.*

Proof. Straightforward. □

Lemma 3.2. *For every $\sigma\in\Omega$ we have $G\sigma=\sigma G$.*

Proof. Let $\sigma\in\Omega$ and $g\in G$. Then $g\sigma\in G\sigma$. If $\sigma=s\in G$, then

$$g\sigma=gs=s(s^{-1}gs)=sg'=\sigma g'\in\sigma G$$

where $g'=s^{-1}gs\in G$. If $\sigma=A\in S$, then

$$g\sigma=gA=g(\theta(A))=\theta(A)(\theta(A)^{-1}g\theta(A))\in\sigma G.$$

Thus $G\sigma\subseteq\sigma G$. Similarly, $\sigma G\subseteq G\sigma$. □

Corollary 3.3. *The relation ρ is a congruence on Ω.*

Remark 3.4. *The identity $[e]=\{\sigma\in\Omega\ \mid\ \exists g\in G,\sigma=ge\}=G$ implies that $\frac{\Omega}{G}=\frac{\Omega}{\rho}\simeq S$.*

Now assume that G is a compact topological group, $S=\mathcal{M}^0(G,P)$ as above is considered with product topology and Ω is a topological groupoid. The notions of topological groupoid and topologial groupoid compactifications are defined similar to their semigroup analogs.

Lemma 3.5. *Let (ψ,X) be a topological groupoid compactification of topological groupoid Ω, $\psi(G)=\hat{G}$, and $\hat{\rho}=\{(x_1,x_2)\in X\times X\mid \exists\hat{g}\in\hat{G},\ x_1=\hat{g}_1x_2\}$. Then $\hat{\rho}$ is a closed congruence on X.*

Proof. Clearly $\hat{\rho}$ is an equivalence relation on X. Let $x\in X$, $\psi(\sigma_\alpha)\to x$ and $\psi(g)=\hat{g}$. By lemma 2.2, $\sigma_\alpha g=g'\sigma_\alpha$ for some $g'\in G$. Thus, $\lim_\alpha\psi(\sigma_\alpha)\psi(g)=\lim_\alpha\psi(g')\psi(\sigma_\alpha)$. So $x\hat{g}=\hat{g}'x$ and hence $x\hat{G}\subseteq\hat{G}x$.

Similarly $\hat{G}x \subseteq x\hat{G}$. Thus $\hat{G}x = x\hat{G},\ x \in X$. This implies that $\hat{\rho}$ is a congruence on X. To show that $\hat{\rho}$ is closed, let $\{\hat{x_\alpha}\}$ and $\{\hat{y_\alpha}\}$ be nets in X such that $\hat{x_\alpha} \to \hat{x}$, $\hat{y_\alpha} \to \hat{y}$ and $\hat{x_\alpha}\hat{\rho}\hat{y_\alpha}$. There exists $\hat{g_\alpha} \in \hat{G}$ such that $\hat{x_\alpha} = \hat{g_\alpha}\hat{y_\alpha}$. We may choose $g_\alpha \in G$ such that $\psi(g_\alpha) = \hat{g_\alpha}$. Compactness of G allows us to assume that g_α converges to some $g \in G$. So $\psi(g_\alpha) \to \psi(g)$, and by joint continuity of product of X, $\hat{g_\alpha}y_\alpha \to \hat{g}\hat{y}$. Therefore $\hat{x} = \hat{g}\hat{y}$, that is, $\hat{x}\hat{\rho}\hat{y}$. □

Theorem 3.6. *Let (ψ, X) be a topological groupoid compactification of topological groupoid Ω, where Ω is an extension of G by S. Then $\frac{X}{\hat{\rho}}$ is a topological groupoid compactification of S.*

Proof. First note that if $\sigma_1\rho\sigma_2$ ($\sigma_1, \sigma_2 \in \Omega$), then $\psi(\sigma_1)\hat{\rho}\psi(\sigma_2)$. Thus ψ preserves congruence and hence there exists a continuous homomorphism $\hat{\psi} : \frac{\Omega}{\rho} \to \frac{X}{\hat{\rho}}$ such that the following diagram commutes.

$$\begin{tikzcd}
\Omega \arrow[r, "\psi"] \arrow[d, "\pi"'] & X \arrow[d, "\hat{\pi}"] \\
\frac{\Omega}{\rho} \arrow[r, "\hat{\psi}"'] & \frac{X}{\hat{\rho}}
\end{tikzcd}$$

Clearly $\frac{X}{\hat{\rho}}$ is a compact Hausdorff topological groupoid [1. Proposition 1.3.8]. We have

$$\overline{\hat{\psi}(\frac{\Omega}{\rho})} = \overline{\hat{\psi} o \pi(\Omega)} = \overline{\hat{\pi} o \psi(\Omega)} \supseteq \hat{\pi}(\overline{\psi(\Omega)}) = \hat{\pi}(X) = \frac{X}{\hat{\rho}}.$$

Also

$$\hat{\psi}(\frac{\Omega}{\rho}) = \hat{\psi} o \pi(\Omega) = \hat{\pi} o \psi(\Omega) \subseteq \hat{\pi}(\Lambda(X)) = \Lambda(\hat{\pi}(X)) = \Lambda(\frac{X}{\hat{\rho}}).$$

Therefore $\frac{X}{\hat{\rho}}$ is a compactification of $\frac{\Omega}{\rho}$ and hence is a compactification of S. □

4. **Function spaces of** $S = \mathcal{M}^0(G, P)$

Theorem 4.1. *Let G be a compact group, $S = \mathcal{M}^0(G, P)$ and Ω an extension of G by S. Let (ε_S, S^{ap}) and $(\varepsilon_\Omega, \Omega^{ap})$ be the almost periodic compactifications of S and Ω, respectively. Then $S^{ap} \simeq \frac{\Omega^{ap}}{\hat{G}}$, where, $\hat{G} = \varepsilon_\Omega(G)$.*

Proof. By Theorem 3.6, $(\hat{\varepsilon_\Omega}, \frac{\Omega^{ap}}{\hat{G}})$ is a compactification of $\frac{\Omega}{G} \simeq S$, where $\hat{G} = \varepsilon_\Omega(G)$. The universal property of the *ap*-compactification (ε_S, S^{ap}) of S [1, Theorem 1.4.10] provides a continuous homomorphism ϕ_1 : $S^{ap} \longrightarrow \frac{\Omega^{ap}}{\hat{G}}$ such that the following diagram commutes.

$$\begin{tikzcd} S \arrow[r, "\varepsilon_S"] \arrow[d, "\hat{\varepsilon_\Omega}"'] & (\frac{\Omega}{G})^{ap} = S^{ap} \arrow[dl, "\varphi_1"] \\ \frac{\Omega^{ap}}{\hat{G}} & \end{tikzcd}$$

On the other hand, the homomorphism

$$\eta : \Omega \xrightarrow{\pi} \frac{\Omega}{G} \simeq S \xrightarrow{\varepsilon_S} S^{ap}$$

provides a continuous homomorphism $\varphi_2 : \Omega^{ap} \longrightarrow S^{ap}$ such that the following diagram commutes.

$$\begin{tikzcd} \Omega \arrow[r, "\varepsilon_\Omega"] \arrow[d, "\eta"'] & \Omega^{ap} \arrow[dl, "\varphi_2"] \\ S^{ap} & \end{tikzcd}$$

Now let $\hat{\sigma_1}\hat{\rho}\hat{\sigma_2}$ ($\hat{\sigma_1}, \hat{\sigma_2} \in \Omega^{ap}$) and choose nets $\{u_\alpha\}$, $\{v_\alpha\}$ in Ω such that $\lim_\alpha \varepsilon_\Omega(u_\alpha) = \hat{\sigma_1}$, and $\lim_\alpha \varepsilon_\Omega(v_\alpha) = \hat{\sigma_2}$. We have $\hat{\sigma_1} = \hat{g}\hat{\sigma_2}$, where $\hat{g} = \varepsilon_\Omega(g)$ for some $g \in G$. Thus

$$\begin{aligned} \varphi_2(\hat{\sigma_1}) = \varphi_2(\hat{g}\hat{\sigma_2}) &= \varphi_2(\varepsilon_\Omega(g) \lim_\alpha \varepsilon_\Omega(v_\alpha)) \\ &= \lim_\alpha \varphi_2 o \varepsilon_\Omega(g v_\alpha) = \lim_\alpha \eta(g v_\alpha) \\ &= \lim_\alpha \eta(g)\eta(v_\alpha) = \lim_\alpha \varphi_2 o \varepsilon_\Omega(v_\alpha) \\ &= \varphi_2(\hat{\sigma_2}). \end{aligned}$$

So φ_2 preserves congruence. Thus there exists continuous homomorphism $\psi : \frac{\Omega^{ap}}{\hat{G}} \longrightarrow S^{ap}$ such that the following diagram commutes.

$$\begin{tikzcd} \Omega^{ap} \arrow[r, "\varphi_2"] \arrow[d, "\hat{\pi}'"'] & S^{ap} \\ \frac{\Omega^{ap}}{\hat{G}} \arrow[ur, "\psi"'] & \end{tikzcd}$$

Now, we show that $\varphi_1 o \psi = id_{\frac{\Omega^{ap}}{\hat{G}}}$. If $(\hat{\pi})'(t) \in \frac{\Omega^{ap}}{\hat{G}}$, then we can find a net $\{\sigma_\alpha\}$ in Ω such that $\lim_\alpha \varepsilon_\Omega(\sigma_\alpha) = t$. Now

$$\begin{aligned} \varphi_1 o \psi(\hat{\pi}'(t)) &= \varphi_1 o \varphi_2(t) = \lim_\alpha \varphi_1 o \varphi_2(\varepsilon_\Omega(\sigma_\alpha)) \\ &= \lim_\alpha \varphi_1 o \eta(\sigma_\alpha) = \lim_\alpha \varphi_1 o \varepsilon_S o \pi(\sigma_\alpha) \\ &= \lim_\alpha \hat{\varepsilon_\Omega} o \pi(\sigma_\alpha) \lim_\alpha \hat{\pi}'(\varepsilon_\Omega(\sigma_\alpha)) \\ &= \hat{\pi}'(\lim_\alpha \varepsilon_\Omega(\sigma_\alpha)) = \hat{\pi}'(t). \end{aligned}$$

Therefore $S^{ap} \simeq \frac{\Omega^{ap}}{\hat{G}}$. □

Theorem 4.2. *Let G be a compact group, $S = \mathcal{M}^0(P, G)$ and Ω an extension of G by S. Let (ε_s, S^{sap}) and $(\varepsilon_\Omega \Omega^{sap})$ be the strongly almost periodic compactifications of S and Ω, respectively. Then $S^{sap} \simeq \frac{\Omega^{sap}}{\hat{G}}$, where, $\hat{G} = \varepsilon_\Omega(G)$.*

Proof. Since (ε_s, S^{sap}) is the universal topological group compactification of S, an argument similar to that of Theorem 4.1 shows that $S^{sap} \simeq \frac{\Omega^{sap}}{\hat{G}}$. □

Note that the above results are true for similar spaces of functions. In fact:

Theorem 4.3. *With the assumptions of the preceding theorem, let $(\varepsilon_s, S^{\mathcal{P}})$ and $(\varepsilon_\Omega \Omega^{\mathcal{P}})$ be the universal $\mathcal{P}$-compactifications of S and Ω, respectively. Then $S^{\mathcal{P}} \simeq \frac{\Omega^{\mathcal{P}}}{\hat{G}}$ where $\hat{G} = \varepsilon_\Omega(G)$, provided that $\mathcal{P}$ has joint continuity property.*

5. Function spaces of $S = \mathcal{M}^0(G, P)$ and topological tensor product

Let S and T be two topological semigroups with identity and let X be a non-empty topological space. Then X is called a topological left

S-system if there is an action $(s,x) \longrightarrow sx$ of $S \times X$ into X which is jointly continuous and $s_1(s_2x) = (s_1s_2)x$, $1_sx = x$ $(s_1, s_2 \in S, x \in X)$. Similarly a topological right S-system is defined. A topological left S-system and a topological right T-system is called a topological (S,T)-bisystem if $(sx)t = s(xt)$, $(s \in S, t \in T, x \in X)$.

Let X, Y be two topological left S-systems and let $\varphi : X \longrightarrow Y$ be a continuous map. We say that φ is a topological left S-map if $\varphi(sx) = s\varphi(x)$, $(x \in X, s \in S)$. Similarly we can define a topological right T-map.

Now let X be a topological left S-system and Y be a topological right T-system. Then X×Y equipped with the product topology, is a topological (S,T)-bisystem (that is, $s_1s_2(x,y) = s_1(s_2x,y)$, $1_s(x,y) = (x,y)$, $(x,y)t_1t_2 = (x,yt_1)t_2$, $(x,y)1_T = (x,y)$, for all $s_1, s_2 \in S$ and $t_1, t_2 \in T$). Let Z be a topological (S,T)-bisystem. We say that $\beta : X \times Y \longrightarrow Z$ is a topological (S,T)-map if β is a topological left S-map and a topological right T-map.

Let S and T be two topological semigroups with identities 1_S and 1_T respectively. Then S can be regarded as a topological (S,S)-bisystem where the action of S on S is just its multiplication. Let $\sigma : S \longrightarrow T$ be a continuous homomorphism. Then T can be regarded as a topological (S,T)-bisystem by $st = \sigma(s)t$, $(s \in S, t \in T)$. Let C be a topological (S,T)-bisystem which is also a semigroup with identity, and let $\beta : S \times T \longrightarrow C$ be a topological (S,T)-map. We say that β is a topological σ-bimap if $\beta(ss', t) = \beta(s, \sigma(s')t)$, $(s, s' \in S, t \in T)$.
By a topological tensor product of S and T we mean a pair (P,ψ) where P is a topological (S,T)-bisystem and $\psi : S \times T \longrightarrow P$ is a topological σ-bimap such that for every topological (S,T)-bisystem C and every topological σ-bimap $\beta : S \times T \longrightarrow C$, there exists a unique topological (S,T)-map $\overline{\beta} : P \longrightarrow C$ such that the diagram

$$\begin{tikzcd} S \times T \arrow[r, "\psi"] \arrow[d, "\beta"'] & P \arrow[dl, "\overline{\beta}"] \\ C & \end{tikzcd}$$

commutes. In [9] Medghalchi and the author proved the existence of topological tensor product of S and T with respect to σ which is denoted by $S \otimes_\sigma T$. Moreover in [9, 10] a characterization of the space of functions of them was proved and it was shown that $S \times T$ can be considered as an extension of the topological tensor product of S and T .

We recall the following results from [9] and [10].

Theorem 5.1. [*9, Theorem 3.3*] *Let S and T be two topological semigroups with identities, and let $\sigma : S \longrightarrow T$ be a continuous homomorphism. Then there is a unique topological tensor product of S and T.*

Theorem 5.2. [*9, Theorem 4.2*] *Let S and T be two topological semigroups with identities, and let $\sigma : S \longrightarrow T$ be a continuous homomorphism. Then $(S \otimes_\sigma T)^{sap} \simeq S^{sap} \otimes_\eta T^{sap}$ where η is an appropriate homomorphism from S^{sap} into T^{sap}*

Lemma 5.3. [*10, Lemma 5,2*] *Let G_1 and G_2 be two topological groups and let $\sigma : G_1 \longrightarrow G_2$ be a continuous homomorphism. Then $N = \{(m, n) \in G_1 \times G_2 : (m, n)\rho(1_{G_1}, 1_{G_2})\}$ is a closed normal subgroup of $G_1 \times G_2$.*

Theorem 5.4. [*10, Theorem 5.3*] *Let G_1 and G_2 be two topological groups and let $\sigma : G_1 \longrightarrow G_2$ be a continuous homomorphism. Then $G_1 \otimes_\sigma G_2 = (G_1 \times G_2)/N$, where $N = \{(m, n) \in G_1 \times G_2 : (m, n)\rho(1_{G_1}, 1_{G_2})\}$.*

Theorem 5.5. *Let G be a compact group and $S = \mathcal{M}^0(G, P)$. Then $(S \otimes_\theta G)^{sap} \simeq \frac{S^{sap} \times G}{N}$, for some closed normal subgroup N of $S^{sap} \times G$.*

Proof. Define $\theta : S \to G$ by $\theta(g, i, j) = u_i(W(g))v_j$. Observe that θ is a continuous partial homomorphism of S into G. So by Theorem 5.1, $S \otimes_\theta G$ exists. On the other hand by Theorem 5.2 $(S \otimes_\theta G)^{sap} \simeq S^{sap} \otimes_\eta G^{sap} \simeq s^{sap} \otimes_\eta G$. Finally, by Lemma 5.2 and Theorem 5.3, $(S \otimes_\theta G)^{sap} \simeq \frac{S^{sap} \times G}{N}$, where

$$N = \{(m, n) \in S^{sap} \times G \,|\, m \otimes_\eta n = 1_{S^{sap}} \otimes_\eta 1_G\}$$

is a closed normal subgroup of $S^{sap} \times G$. □

Theorem 5.6. *Let G be a compact group and $S = \mathcal{M}^0(G, P)$. Let $(S \otimes_\theta G)^{\mathcal{P}}$ and $S^{\mathcal{P}}$ be the universal topological group compactifications of $S \otimes_\theta G$ and S, respectively. Then $(S \otimes_\theta G)^{\mathcal{P}} \simeq S^{\mathcal{P}} \otimes_\eta G$ if $\mathcal{P}$ has joint continuity property.*

Proof. Let $(\varepsilon_{S\otimes_\theta G}, (S \otimes_\theta G)^{\mathcal{P}}), (\varepsilon_S, S^{\mathcal{P}})$ and $(\varepsilon_G, G^{\mathcal{P}})$ be the universal topological group $\mathcal{P}$-compactifications of $S \otimes_\theta G, S$ and G, respectively. By [9, 3.7], $(\delta_{S\otimes_\theta G}, S^{\mathcal{P}} \otimes_\eta G^{\mathcal{P}})$ is a topological group compactification of $S \otimes_\theta G$. The universal property of $\mathcal{P}$-compactification $(\varepsilon_{S\otimes_\theta G}, (S \otimes_\theta G)^{\mathcal{P}})$

gives a continuous homomorphism $\phi : (S \otimes_\theta G)^{\mathcal{P}} \longrightarrow S^{\mathcal{P}} \otimes_\eta G^{\mathcal{P}}$ such that the following diagram commutes.

$$\begin{array}{lcl} S \otimes_\theta G & \xrightarrow{\varepsilon_{S\otimes_\theta G}} & (S \otimes_\theta G)^{\mathcal{P}} \\ \delta_{S\otimes_\theta G} \downarrow & \swarrow \phi & \\ S^{\mathcal{P}} \otimes_\eta G^{\mathcal{P}} & & \end{array}$$

Also, since $(\varepsilon_S \times \varepsilon_G, (S \times G)^{\mathcal{P}})$ is a universal topological group compactification of $S \times G$, via the homomorphism

$$\zeta : S \times G \xrightarrow{\pi_1} S \otimes_\theta G \xrightarrow{\varepsilon_{S\otimes_\theta G}} (S \otimes_\theta G)^{\mathcal{P}},$$

then there is a continuous homomorphism $\phi_1 : (S \times G)^{\mathcal{P}} \longrightarrow (S \otimes_\theta G)^{\mathcal{P}}$ such that the following diagram commutes.

$$\begin{array}{lcl} S \times G & \xrightarrow{\zeta} & (S \otimes_\theta G)^{\mathcal{P}} \\ \varepsilon_S \times \varepsilon_G \downarrow & \nearrow \phi_1 & \\ (S \times G)^{\mathcal{P}} & & \end{array}$$

By [1, 3.3.4] $(S \times G)^{\mathcal{P}} = S^{\mathcal{P}} \times G^{\mathcal{P}}$. Thus we can assume that ϕ_1 is a map from $S^{\mathcal{P}} \times G^{\mathcal{P}}$ into $(S \otimes_\theta G)^{\mathcal{P}}$. Observe that ϕ_1 preserves congruence, for, if $vv' \otimes_\eta \mu = v \otimes_\eta \eta(v')\mu$, where $v, v' \in S^{\mathcal{P}}$, $\mu \in G^{\mathcal{P}}$, we can get the nets $\{s_\alpha\}, \{s'_\beta\}$ in S and $\{t_\gamma\}$ in G such that $\lim_\alpha \varepsilon_S(s_\alpha) = v$, $\lim_\beta \varepsilon_S(s'_\beta) = v'$ and $\lim_\gamma \varepsilon_G(t_\gamma) = \mu$. Therefore,

$$\begin{aligned} \phi_1(vv' \otimes_\eta \mu) &= \phi_1(\lim_{\alpha,\beta,\gamma} \varepsilon_S \times \varepsilon_G(s_\alpha s'_\beta, t_\gamma)) \\ &= \lim_{\alpha,\beta,\gamma} \phi_1(\varepsilon_S \times \varepsilon_G(s_\alpha s_\beta, t_\gamma)) \\ &= \lim_{\alpha,\beta,\gamma} \varepsilon_{S\otimes_\theta G}(\pi_1(s_\alpha s_\beta, t_\gamma)) \\ &= \lim_{\alpha,\beta,\gamma} \varepsilon_{S\otimes_\theta G}(\pi_1(s_\alpha, \theta(s'_\beta)t_\gamma)). \end{aligned}$$

On the other hand we have

$$\begin{aligned} \phi_1(v \otimes_\eta \eta(v')\mu) &= \phi_1(\lim_{\alpha,\beta,\gamma} \varepsilon_S \times \varepsilon_G(s_\alpha, \theta(s'_\beta)t_\gamma) \\ &= \dots \\ &= \lim_{\alpha,\beta,\gamma} \varepsilon_{S\otimes_\theta G}(\pi_1(s_\alpha, \theta(s'_\beta)t_\gamma)). \end{aligned}$$

Now, by the structure of topological tensor product [9, 3.3], ϕ_1 preserves congruence. Thus there exists a continuous homomorphism $\phi_2 : S^P \otimes_\eta$

$G^{\mathcal{P}} \longrightarrow (S \otimes_\theta G)^{\mathcal{P}}$ such that the following diagram commutes.

$$\begin{array}{ccc} S^{\mathcal{P}} \times G^{\mathcal{P}} & \xrightarrow{\phi_1} & (S \otimes_\theta G)^{\mathcal{P}} \\ \pi_2 \downarrow & \nearrow \phi_2 & \\ S^{\mathcal{P}} \otimes_\eta G^{\mathcal{P}} & & \end{array}$$

Now, $\phi \circ \phi_2$ is identity on $S^{\mathcal{P}} \otimes_\eta G^{\mathcal{P}}$ for, if $v \otimes_\eta \mu \in S^{\mathcal{P}} \otimes_\eta G^{\mathcal{P}}$, then we can find a net $\{s_\alpha\}$ in S and a net $\{t_\beta\}$ in G such that $\varepsilon_S(s_\alpha) \longrightarrow v$ and $\varepsilon_G(t_\beta) \longrightarrow \mu$. Now

$$\begin{aligned} \phi \circ \phi_2(v \otimes_\eta \mu) &= \phi \circ \phi_2(\pi_2(v, \mu)) = \phi(\phi_1(v, \mu)) \\ &= \lim_{\alpha,\beta} \phi(\phi_1(\varepsilon_S \times \varepsilon_G(s_\alpha, t_\beta)) \\ &= \lim_{\alpha,\beta} \phi(\zeta(s_\alpha, t_\beta)) = \lim_{\alpha,\beta} \phi(\varepsilon_{S\otimes_\theta G}(\pi_1(s_\alpha, t_\beta))) \\ &= \lim_{\alpha,\beta} \delta_{S\otimes_\theta G}(s_\alpha \otimes_\theta t_\beta) = v \otimes_\eta \mu \end{aligned}$$

thus, $(S \otimes_\theta T)^{\mathcal{P}} \simeq S^{\mathcal{P}} \otimes_\eta G^{\mathcal{P}}$. Finally, since G is compact then $(S \otimes_\theta G)^{\mathcal{P}} \simeq S^{\mathcal{P}} \otimes_\eta G$. □

Corollary 5.7. *Let G be a compact group and $S = \mathcal{M}^0(G, P)$. Let $(S \otimes_\theta G)^{\mathcal{P}}$ and $S^{\mathcal{P}}$ be universal topological group compactifications of $S \otimes_\theta G$ and S, respectively. Then $(S \otimes_\theta G)^{\mathcal{P}} \simeq \frac{S^{\mathcal{P}} \times G}{N}$, for some closed normal subgroup N of $S^{\mathcal{P}} \times G$, provided that $\mathcal{P}$ has joint continuity property.*

Acknowledgments

The author would like to sincerely thank the referee for his/her valuable comments and useful suggestions

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