TUTTE POLYNOMIALS OF FLOWER GRAPHS

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ABSTRACT. We give explicit expressions of the Tutte polynomial of a complete flower graph and a flower graph with some petals missing.

1. Introduction

There are several polynomials associated with a graph G. Polynomials play an important role in the study of graphs as they encode a variety of information about a graph. Chromatic polynomials of graphs are sometimes easy to compute. However, Tutte polynomials of graphs seem harder to find, and if known are complicated. For example, the chromatic polynomial of K_n is $\lambda \prod_{i=1}^{n-1}(\lambda - i)$, but the Tutte polynomial of the same structure as described by Tutte [7] and Welsh [9] has a more complicated form.

Finding explicit expressions of the Tutte polynomial for different classes of matroids and graphs is an active area of research; for example, see [1, 4]. Determining the Tutte polynomial of a graph has important implications beyond graph theory, including knot theory, theoretical physics and biology.

Flower graphs form a class of highly symmetric graphs. Like complete graphs, flower graphs have an attractive simple formula for the chromatic polynomial; see [5].

In Section 4, we give an explicit expression of the Tutte polynomial of an incomplete flower graph. In addition, we give a class of non-isomorphic graphs with the property that all the graphs in the class have the same Tutte polynomial. Finally, in Section 5, we give an explicit expression of the Tutte polynomial of a complete flower graph.

2. Flower graphs

Here, we define the tensor product of two graphs and use it to define a complete flower graph and an incomplete flower graph. Finally, we define a deformed flower graph.

Let G_1 and G_2 be two graphs where G_2 has a distinguished edge labelled d. The tensor product $G_1 \otimes G_2$ of two graphs G_1 and G_2 is defined to be the graph obtained by taking a 2-sum of G_1 with G_2 at each edge of G_1 and the distinguished edge d of G_2. A 2-sum between two graphs identifies an edge in one with an edge in the other, and then deletes the identified edge. A graph G is called an $(n \times m)$-*complete flower graph* if it is the tensor product $C_n \otimes C_m$ for $m \geq 2$ and $n \geq 2$. An $(n \times m)$-complete flower graph is denoted by F_{n_m}. It is clear that F_{n_m} has $n(m-1)$ vertices and nm edges. The m-cycles are called the *petals* and the n-cycle is called the center of F_{n_m}. A graph G is called an $(n \times m)$-*incomplete flower graph with i petals* if it is a graph obtained by the incomplete operation of forming the tensor product $C_n \otimes C_m$ such that only i edges of C_n are replaced by copies of C_m. An $(n \times m)$-incomplete flower graph with i petals is denoted by $F_{n_m}^i$. We remark that the positions of petals are irrelevant. Furthermore, $F_{n_m} = F_{n_m}^n$. It is clear by definition that we have several non-isomorphic graphs represented by $F_{n_m}^i$. We denote a flower graph with center C_n and $i = l + j$ petals such that l petals are of size m and j petals are of size k, by $F_{n_{m_l,k_j}}^i$. Such a flower graph is known as a *deformed flower graph*.

3. Tutte polynomial

Here, we review some basic needed properties of the Tutte polynomial.

The Tutte polynomial $T(G; x, y)$ of a graph G is a polynomial in two independent variables x and y and is defined by:

$$T(G; x, y) = \sum_{X \subseteq E} (x-1)^{r(E)-r(X)} (y-1)^{|X|-r(X)}.$$

The Tutte polynomial can also be computed using the following deletion and contraction formula:

T1. $T(I; x, y) = x$ and $T(L; x, y) = y$ where I is an isthmus and L is a loop.

T2. If e is an edge of the graph G and e is neither a loop nor an isthmus, then,

$$T(G; x, y) = T(G \backslash e; x, y) + T(G/e; x, y).$$

T3. If e is a loop or an isthmus of the graph G, then,

$$T(G; x, y) = T(e; x, y) T(G/e; x, y).$$

The following theorem follows from the deletion and contraction formula of the Tutte polynomials.

Theorem 3.1. *Let t_n be a tree on n vertices and let C_n be an n-cycle. Then,*

(i) $T(t_n; x, y) = x^{n-1}$.
(ii) $T(C_n; x, y) = y + \sum_{i=1}^{n-1} x^i$.

The Tutte polynomial under the tensor product $G_1 \otimes G_2$ of two graphs G_1 and G_2 is calculated in [3] as follows:

$$T(G_1 \otimes G_2; x, y) = T_C(G_2; x, y)^{|G_1| - \rho(G_1)} T_L(G_2; x, y)^{\rho(G_1)} T(G_1; X, Y),$$

where

$$X = \frac{(x-1)T_C(G_2; x, y) + T_L(G_2; x, y)}{T_L(G_2; x, y)}$$

$$Y = \frac{T_C(G_2; x, y) + (y-1)T_L(G_2; x, y)}{T_C(G_2; x, y)}$$

and

$$(x-1)T_C(G_2; x, y) + T_L(G_2; x, y) = T(G_2 \backslash d; x, y)$$
$$T_C(G_2; x, y) + (y-1)T_L(G_2; x, y) = T(G_2/d; xy).$$

4. **Tutte polynomial of incomplete flower graph**

Here, we give an explicit expression of the Tutte polynomial of an incomplete flower graph. Furthermore, we give a class of non-isomorphic graphs such that all graphs in this class have the same Tutte polynomial.

For simplicity in notation, if G_1 and G_2 are disjoint graphs, then the graph obtained by merging one vertex of G_1 and one vertex of G_2 will be denoted by $G_1 \cdot G_2$. Furthermore, for any graph $F_{n_m}^i$, the edges of a petal will be labelled as the following sequence $e_1, e_2, \cdots, e_m$, where e_1 is the edge in both the center and the petal. In particular, e_1 is adjacent to e_2 and e_m in the petal. It is clear that $F_{n_m}^i \backslash e_2 = F_{n_m}^{i-1} \cdot P_{m-2}$, where P_{m-2} is a path with $m - 2$ edges. Throughout this paper, we let the edge e be the petal edge e_2 unless otherwise stated.

Lemma 4.1. *Let $F_{n_m}^1$, for $m, n \geq 2$, be a flower graph with one petal. Then,*

$$T(F_{n_m}^1; x, y) = \left(\sum_{i=0}^{m-2} x^i \right) T(C_n; x, y) + yT(C_{n-1}; x, y).$$

Proof. We use induction on m and the deletion and contraction method of the Tutte polynomial. Let $m = 2$. Then, we have a flower graph with center C_n and the petals are 2-cycles. Thus, $F_{n_2}^1$ is just C_n with one pair of parallel edges. Let e be one of the parallel edges. If we delete e, then we get C_n and if we contract e, then we get C_{n-1} with a loop. Hence,

$$
\begin{aligned}
T(F_{n_m}^1; x, y) &= T(C_n; x, y) + yT(C_{n-1}; x, y) \\
&= \left(\sum_{i=0}^{m-2} x^i \right) T(C_n; x, y) + yT(C_{n-1}; x, y).
\end{aligned}
$$

This establishes the basis. Assume, then, that the statement is true for $m = k$ and $k > 2$. Let $m = k + 1$ and let e be the edge e_2 of the petal. By the deletion and contraction method, we have that

$$(4.1) \qquad T(F_{n_{k+1}}^1; x, y) = T(F_{n_{k+1}}^1 \backslash e; x, y) + T(F_{n_{k+1}}^1 / e; x, y).$$

But $F_{n_{k+1}}^1 \backslash e$ is isomorphic to $C_n \cdot P_{k-1}$. Thus, the $k - 1$ edges of P_{k-1} are isthmuses in $F_{n_{k+1}}^1 \backslash e$. Hence,

$$(4.2) \qquad T(F_{n_{k+1}}^1 \backslash e; x, y) = x^{k-1} T(C_n; x, y).$$

On the other hand, $F^1_{n_{k+1}}/e$ is isomorphic to $F^1_{n_k}$. Thus,

$$T(F^1_{n_{k+1}}/e; x, y) = T(F^1_{n_k}; x, y).$$

By the induction hypothesis,

$$(4.3)\ T(F^1_{n_{k+1}}/e; x, y) = \left(\sum_{i=0}^{k-2} x^i\right) T(C_n; x, y) + yT(C_{n-1}; x, y).$$

Substituting (4.2) and (4.3) into equation (4.1), yields the required result.

Recall that $F^i_{n_{m_l,k_j}}$ denotes a deformed flower graph with center C_n, l petals of size m and j petals of size k and $i = l+j$. The following lemma gives the recursive method for the Tutte polynomial of $F^i_{n_{k+1}}$.

Lemma 4.2. *Let $F^i_{n_m}$ be a flower graph with i petals, where $i \in \{1, 2, \cdots, n-1\}$. Then,*

$$T(F^i_{n_m}; x, y) = \left[\sum_{j=0}^{m-2} x^j\right] T(F^{i-1}_{n_m}; x, y) + yT(F^{i-1}_{(n-1)_m}; x, y).$$

Proof. We fix and consider one petal of $F^i_{n_m}$, say P. Let e be the edge e_2 of petal P. By the deletion and contraction formula of the Tutte polynomial, we have that

$$T(F^i_{n_m}; x, y) = T(F^i_{n_m} \backslash e; x, y) + T(F^i_{n_m}/e; x, y).$$

Now, $F^i_{n_m} \backslash e$ is isomorphic to $F^{i-1}_{n_m} \cdot P_{m-2}$ and the $m-2$ edges of P_{m-2} are isthmuses in $F^i_{n_m} \backslash e$. Hence,

$$T(F^i_{n_m} \backslash e; x, y) = x^{m-2}T(F^{i-1}_{n_m}; x, y).$$

On the other hand, $F^i_{n_m}/e$ is isomorphic to $F^i_{n_{m_{i-1},(m-1)_1}}$. We now repeat this process, using the deletion and contraction method on the petal P and the edge e, until this petal is removed to get the required result.

The following theorem is well known in the literature and is used in proving our result.

Theorem 4.3. *Let n and k be positive integers with $k \leq n - 1$. Then,*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

We are now in a position to present an explicit expression of the Tutte polynomial of an incomplete flower graph.

Theorem 4.4. *Let $F_{n_m}^i$ be a flower graph with i petals, where $i \in \{1, 2, \cdots, n-1\}$. Then,*

$$T(F_{n_m}^i; x, y) = \sum_{k=0}^{i} \binom{i}{k} \left(\sum_{j=0}^{m-2} x^j \right)^{i-k} y^k \left(y + \sum_{q=1}^{n-k-1} x^q \right).$$

Proof. We use induction on the number of petals i. Let $i = 1$. Then, by Lemma 4.1 and part (ii) of Theorem 3.1,

$$
\begin{aligned}
T(F_{n_m}^1; x, y) &= \left(\sum_{i=0}^{m-2} x^i \right) T(C_n; x, y) + yT(C_{n-1}; x, y) \\
&= \sum_{k=0}^{1} \binom{1}{k} \left(\sum_{i=0}^{m-2} x^i \right)^{1-k} y^k T(C_{n-k}; x, y) \\
&= \sum_{k=0}^{1} \binom{1}{k} \left(\sum_{i=0}^{m-2} x^i \right)^{1-k} y^k \left(y + \sum_{q=1}^{n-k-1} x^q \right).
\end{aligned}
$$

Hence, it is true for the basis. Assume, then, that the statement is true for $i = l$, where $1 < l < n - 1$. Let $i = l + 1 \leq n - 1$. Let $b = \sum_{j=0}^{m-2} x^j$. By Lemma 4.2,

$$T(F_{n_m}^{l+1}; x, y) = bT(F_{n_m}^l; x, y) + yT(F_{(n-1)_m}^l; x, y).$$

By the induction hypothesis,

$$T(F_{n_m}^{l+1}; x, y) = b \sum_{k=0}^{l} \binom{l}{k} b^{l-k} y^k T(C_{n-k}; x, y)$$

$$+ \quad y \sum_{k=0}^{l} \binom{l}{k} b^{l-k} y^k T(C_{n-1-k}; x, y)$$

$$= \quad \sum_{k=0}^{l} \binom{l}{k} b^{l+1-k} y^k T(C_{n-k}; x, y)$$

$$+ \quad \sum_{k=0}^{l} \binom{l}{k} b^{l-k} y^{k+1} T(C_{n-1-k}; x, y).$$

By expanding the summation and gathering like terms, we have that

$$T(F_{n_m}^{l+1}; x, y)$$

$$= \quad b^{l+1} y^0 T(C_n; x, y) + \binom{l}{1} b^l y^1 T(C_{n-1}; x, y)$$

$$+ \quad \binom{l}{2} b^{l-1} y^2 T(C_{n-2}; x, y) + \cdots$$

$$+ \quad \binom{l}{l-1} b^2 y^{l-1} T(C_{n-l+1}; x, y) + \binom{l}{l} b^1 y^l T(C_{n-l}; x, y)$$

$$+ \quad \binom{l}{0} b^l y^1 T(C_{n-1}; x, y) + \binom{l}{1} b^{l-1} y^2 T(C_{n-2}; x, y)$$

$$+ \quad \binom{l}{2} b^{l-2} y^3 T(C_{n-3}; x, y) + \cdots$$

$$+ \quad \binom{l}{l-1} b^1 y^l T(C_{n-l}; x, y) + b^0 y^{l+1} T(C_{n-(l+1)}; x, y)$$

$$= \quad b^{l+1} y^0 T(C_n; x, y) + \left[\binom{l}{1} + \binom{l}{0} \right] b^l y^1 T(C_{n-1}; x, y)$$

$$+ \quad \left[\binom{l}{2} + \binom{l}{1} \right] b^{l-1} y^2 T(C_{n-2}; x, y) + \cdots$$

$$+ \quad \left[\binom{l}{l} + \binom{l}{l-1} \right] b^1 y^l T(C_{n-l}; x, y) + b^0 y^{l+1} T(C_{n-l+1}; x, y).$$

Applying Theorem 4.3 and part (ii) of Theorem 3.1 part (ii), we have that

$$
\begin{aligned}
T(F_{n_m}^{l+1}; x, y) &= \binom{l+1}{0} b^{l+1} y^0 T(C_{n-0}; x, y) \\
&\quad + \binom{l+1}{1} b^{l+1-1} y^1 T(C_{n-1}; x, y) \\
&\quad + \binom{l+1}{2} b^{l+1-2} y^2 T(C_{n-2}; x, y) + \cdots \\
&\quad + \binom{l+1}{l} b^{l+1-l} y^l T(C_{n-l}; x, y) \\
&\quad + \binom{l+1}{l+1} b^{l+1-(l+1)} y^{l+1} T(C_{n-(l+1)}; x, y) \\
&= \sum_{k=0}^{l+1} \binom{l+1}{k} b^{l+1-k} y^k T(C_{n-k}; x, y) \\
&= \sum_{k=0}^{l+1} \binom{l+1}{k} \left(\sum_{j=0}^{m-2} x^j \right)^{l+1-k} y^k \left(y + \sum_{q=1}^{n-k-1} x^q \right).
\end{aligned}
$$

Hence, the statement is true for any $i \in \{1, 2, \cdots, n-1\}$.

Corollary 4.5. *Let $\mathcal{F}^i$ be a class of all non-isomorphic incomplete flower graphs with i petals. The Tutte polynomial of any graph G in $\mathcal{F}^i$ is:*

$$
T(G; x, y) = T(F_{n_m}^i; x, y).
$$

5. Tutte polynomial of complete flower graph

We remark that the Tutte polynomial of a complete flower graph can be calculated using the formula of the Tutte polynomial of the tensor product given in Section 3. Here, we use an alternative and efficient method to calculate the Tutte polynomial of a complete flower graph and give a simplified explicit expression of the Tutte polynomial of a complete flower graph. For this purpose, we shall need the following lemma.

Lemma 5.1. *Let F_{2_m} be a complete flower graph. Then,*

$$T(F_{2_m}; x, y) = \left(\sum_{i=0}^{m-2} x^i \right) T(F_{2_m}^1; x, y) + y^2 T(C_{m-1}; x, y).$$

Proof. We consider only one petal of F_{2_m}, say P, and repeatedly use the deletion and contraction formula of the Tutte polynomial on the edges of the petal P which are not in the center. We then remove isthmuses and loops to get the required result.

The following lemma gives the recursive method of the Tutte polynomial of a complete flower graph.

Lemma 5.2. *Let F_{n_m} be a complete flower graph. Then,*

$$T(F_{n_m}; x, y) = \left(\sum_{i=0}^{m-2} x^i \right) T(F_{n_m}^{n-1}; x, y) + y T(F_{(n-1)_m}; x, y).$$

Proof. Consider one petal of F_{n_m}, fix it and call it petal P. Then, use the deletion and contraction method repeatedly on the petal P until all the edges of petal P which are not in the center are removed.

The following theorem gives an explicit expression of the Tutte polynomial of a complete flower graph.

Theorem 5.3. *Let F_{n_m} be a complete flower graph. Then,*

$$T(F_{n_m}; x, y)$$
$$= \sum_{r=0}^{n-2} y^r \left[\sum_{k=0}^{n-1-r} \binom{n-1-r}{k} \left(\sum_{i=0}^{m-2} x^i \right)^{n-r-k} y^k \left(y + \left[\sum_{i=1}^{n-1-k-r} x^i \right] \right) \right]$$
$$+ \quad y^n \left(y + \sum_{i=1}^{m-2} x^i \right).$$

Proof. We proceed by induction on the number of vertices of the center n. The basis corresponds to $n = 2$. By Lemma 5.1,

$$T(F_{2_m}; x, y) = \left(\sum_{i=0}^{m-2} x^i \right) T(F_{2_m}^1; x, y) + y^2 T(C_{m-1}; x, y).$$

Applying Theorem 4.4 and part (ii) of Theorem 3.1, we have,

$$T(F_{2_m}; x, y)$$

$$= \left(\sum_{i=0}^{m-2} x^i\right) \sum_{k=0}^{1} \binom{1}{k} \left(\sum_{i=0}^{m-2} x^i\right)^{1-k} y^k \left(y + \sum_{q=1}^{n-k-1} x^q\right) + y^2 \left(\sum_{q=1}^{m-2} x^q\right)$$

$$= \sum_{k=0}^{1} \binom{1}{k} \left(\sum_{i=0}^{m-2} x^i\right)^{2-k} y^k \left(y + \sum_{q=1}^{n-k-1} x^q\right) + y^2 \left(\sum_{q=1}^{m-2} x^q\right)$$

$$= \sum_{r=0}^{n-2} y^r \left[\sum_{k=0}^{n-1-r} \binom{n-1-r}{k} \left(\sum_{j=0}^{m-2} x^j\right)^{n-r-k} y^k \left(y + \left[\sum_{q=1}^{n-1-k-r} x^q\right]\right)\right]$$

$$+ \quad y^n \left(y + \sum_{q=1}^{m-2} x^q\right).$$

Hence, the statement is true for a complete flower with 2-cycle center. This establishes the basis. Assume, then, that the statement is true for some flower graph with n-cycle center and let $n = j+1$. By Lemma 5.2,

$$(5.1) \quad T(F_{(j+1)_m}; x, y) = \left(\sum_{i=0}^{m-2} x^i\right) T(F_{(j+1)_m}^j; x, y) + y T(F_{j_m}; x, y).$$

By Theorem 4.4, we have,

$$(5.2) \quad T(F_{(j+1)_m}^j; x, y) = \sum_{k=0}^{j} \binom{j}{k} \left(\sum_{i=0}^{m-2} x^i\right)^{j-k} y^k \left(y + \sum_{q=1}^{j-k} x^q\right).$$

By the induction hypothesis,

$$T(F_{j_m}; x, y)$$

$$= \sum_{k=0}^{j-1} \binom{j-1}{k} \left(\sum_{i=0}^{m-2} x^i\right)^{j-k} y^k \left(y + \sum_{q=1}^{j-1-k} x^q\right)$$

$$+ \quad y \sum_{k=0}^{j-2} \binom{j-2}{k} \left(\sum_{i=0}^{m-2} x^i\right)^{j-1-k} y^k \left(y + \sum_{q=1}^{j-2-k} x^q\right)$$

$$+ \quad \cdots\cdots\cdots$$

$$+ \quad y^{j-2} \sum_{k=0}^{2} \binom{2}{k} \left(\sum_{i=0}^{m-2} x^i \right)^{3-k} y^k \left(y + \sum_{q=1}^{2-k} x^q \right)$$

$$+ \quad y^{j-1} \sum_{k=0}^{1} \binom{1}{k} \left(\sum_{i=0}^{m-2} x^i \right)^{2-k} y^k \left(y + \sum_{q=1}^{1-k} x^q \right)$$

(5.3)
$$+ \quad y^{j} \left(y + \sum_{q=1}^{m-2} x^q \right).$$

Hence, if we substitute Equation (5.2) and Equation (5.3) into Equation (5.1), we get,

$$T(F_{(j+1)_m}$$

$$= \quad \sum_{k=0}^{j} \binom{j}{k} \left(\sum_{i=0}^{m-2} x^i \right)^{j+1-k} y^k \left(y + \sum_{q=1}^{j-k} x^q \right)$$

$$+ \quad y \sum_{k=0}^{j-1} \binom{j-1}{k} \left(\sum_{i=0}^{m-2} x^i \right)^{j-k} y^k \left(y + \sum_{q=1}^{j-1-k} x^q \right)$$

$$+ \quad y^2 \sum_{k=0}^{j-2} \binom{j-2}{k} \left(\sum_{i=0}^{m-2} x^i \right)^{j-1-k} y^k \left(y + \sum_{q=1}^{j-2-k} x^q \right)$$

$$+ \quad \cdots\cdots\cdots$$

$$+ \quad y^{j-1} \sum_{k=0}^{2} \binom{2}{k} \left(\sum_{i=0}^{m-2} x^i \right)^{3-k} y^k \left(y + \sum_{q=1}^{2-k} x^q \right)$$

$$+ \quad y^{j} \sum_{k=0}^{1} \binom{1}{k} \left(\sum_{i=0}^{m-2} x^i \right)^{2-k} y^k \left(y + \sum_{q=1}^{1-k} x^q \right)$$

$$+ \quad y^{j+1} \left(y + \sum_{q=1}^{m-2} x^q \right).$$

Hence, this yields:

$$T(F_{(j+1)_m}$$

$$= \quad \sum_{r=0}^{j-1} y^r \left[\sum_{k=0}^{j-r} \binom{j-r}{k} \left(\sum_{i=0}^{m-2} x^i \right)^{j+1-r-k} y^k \left(y + \left[\sum_{i=1}^{j-k-r} x^i \right] \right) \right]$$

$$+ \quad y^{j+1} \left(y + \sum_{i=1}^{m-2} x^i \right).$$

Therefore, the statement is true for any flower graph with center c_n, where $n > 1$.

Acknowledgments

References

[1] J.E. Bonin and H. Qin, Tutte polynomials of q-cones, *Discrete Math.* **232** (2001) 95-103.

[2] T.H. Brylawski and J.G. Oxley, *Encyclopedia of Mathematics and its Applications*, Matroid Applications: The Tutte polynomial and its applications, Cambridge University Press,Cambridge, 1992.

[3] F. Jaeger, D. Vertigan and D.J.A. Welsh, On the computational complexity of the Jones and Tutte polynomials, *Mathematical Proceedings of the Cambridge Philosophical society* **108** (1990) 35-53.

[4] E.G. Mphako, Tutte polynomials of perfect matroid designs, *Combinatorics, Probability and computing* **9** (2000) 363- 367.

[5] E.G. Mphako-Banda, Some polynomials of flower graphs, *International Mathematical Forum* **2** (2007) 2511-2518.

[6] T.W. Tutte, A contribution to the theory of polynomials, *Can. J. Math.* **6** (1954) 80-91.

[7] T.W.Tutte, On dichromatic polynomials, *J. Combin. Theory* **2** (1967) 301-320.

[8] T.W. Tutte, *Encyclopedia of Mathematics and its Applications*, Graph Theory, Vol. 21, Addison-Wesley Publishing Company, Menlo Park California, 1984.

[9] D.J.A. Welsh, Counting, colourings and flows in random graphs, *in Combinatorics, Paul Erdös is Eighty* **2** (1993) 1-15.

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