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ON SKEW ARMENDARIZ AND SKEW QUASI-ARMENDARIZ MODULES

A. ALHEVAZ AND A. MOUSSAVI*

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ABSTRACT. Let α be an endomorphism and δ an α-derivation of a ring R. In this paper we study the relationship between an R-module M_R and the general polynomial module $M[x]$ over the skew polynomial ring $R[x;\alpha,\delta]$. We introduce the notions of skew-Armendariz modules and skew quasi-Armendariz modules which are generalizations of α-Armendariz modules and extend the classes of non-reduced skew-Armendariz modules. An equivalent characterization of an α-skew Armendariz module is given. Some properties of this generalization are established, and connections of properties of a skew-Armendariz module M_R with those of $M[x]_{R[x;\alpha,\delta]}$ are investigated. As a consequence we extend and unify several known results related to Armendariz modules.

1. Introduction

Throughout this paper R denotes an associative ring with unity, α is a ring endomorphism and δ an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b+\alpha(a)\delta(b)$, for all $a, b \in R$. We denote $R[x;\alpha,\delta]$ the Ore extension (skew polynomial ring) whose elements are the polynomials over R, the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x + \delta(a)$ for any $a \in R$.

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∗Corresponding author

A ring R is called *Baer* (respectively, *quasi-Baer*) if the right annihilator of every nonempty subset (respectively, right ideal) of R is generated, as a right ideal, by an idempotent of R. Kaplansky [23], introduced the Baer rings to abstract various properties of rings of operators on a Hilbert space. Clark [13] introduced the quasi-Baer rings and used them to characterize a finite dimensional twisted matrix units semigroup algebra over an algebraically closed field. All modules are assumed to be unitary right modules. Let $ann_R(X) = \{r \in R \mid Xr = 0\}$, where X is a subset of a module M_R.

In [29], Lee and Zhou introduced Baer, quasi-Baer and p.p.-modules as follows:
(1) M_R is called *Baer* (respectively, *quasi-Baer*) if, for any subset (respectively, submodule) X of M, $ann_R(X) = eR$ where $e^2 = e \in R$.
(2) M_R is called *principally projective* (or simply *p.p.*) *module* (respectively, *principally quasi-Baer* (or simply *p.q.-Baer*) *module*) if, for any element $m \in M$, $ann_R(m) = eR$ (respectively, $ann_R(mR) = eR$) where $e^2 = e \in R$.
Clearly, a ring R is Baer (respectively, p.p. or quasi-Baer) if and only if R_R is Baer (respectively, p.p. or quasi-Baer) module. If R is a Baer (respectively, p.p. or quasi-Baer) ring, then for any right ideal I of R, I_R is Baer (respectively, p.p. or quasi- Baer) module. It is clear that R is a right p.q.-Baer ring if and only if R_R is a p.q.-Baer module. Every submodule of a p.q.-Baer module is p.q.-Baer and every Baer module is quasi-Baer.

A ring is called *reduced* if it has no nonzero nilpotent elements and M_R is called *reduced* by Lee and Zhou [29] if, for any $m \in M$ and $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. Lee and Zhou have extended various results of reduced rings to reduced modules and Agayev et al. [1] introduced and studied abelian modules as a generalization of abelian rings.

Zhang and Chen [43] introduced the notion of α-skew Armendariz modules. Namely, an R-module M_R is called *α-skew Armendariz*, if for polynomials $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]$ and $f(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x;\alpha]$, $m(x)f(x) = 0$ implies $m_i\alpha^i(b_j) = 0$ for each $0 \leq i \leq k$ and $0 \leq j \leq n$. According to Lee and Zhou [29], a module M_R is called *α-Armendariz* if M_R is α-compatible and α-skew-Armendariz. If α is equal to the identity, then the above definition boils down to the standard notion of Armendariz module. Moreover, they proved that R is an α-skew Armendariz ring if and only if every flat right R-module is α-skew Armendariz. By [29], a module M_R is α-*reduced* if M_R is α-compatible and reduced.

The polynomial extensions of Baer, quasi-Baer, right p.q.-Baer and p.p.-rings and modules have been investigated by many authors [5-10, 15-21, 34-43]. Most of these have worked either with the case $\delta = 0$ and α an automorphism or the case where α is the identity. With the impetus of quantized derivations, renewed interest in the general Ore extension $R[x;\alpha,\delta]$ has arisen during the last few years.

In this paper, we study the relationship between an R-module M_R and the general polynomial module $M[x]$ over the skew polynomial ring $R[x;\alpha,\delta]$. We introduce the notions of skew-Armendariz modules and skew quasi-Armendariz modules which are generalizations of α-skew Armendariz modules [43] and α-reduced modules [29]. An equivalent characterization of an α-skew-Armendariz module is given, which is useful to simplify the proofs. Also new families of non-reduced skew-Armendariz modules are presented. Among other results, we show that there is a strong connection of the Baer, quasi-Baer and the p.p.-property of the two modules, respectively.

Furthermore, we show that for an endomorphism α and an α-derivation δ of a ring R, (1) A right R-module M_R is α-*skew-Armendariz* if and only if for polynomials $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]$ and $f(x) = a_0 + a_1x + \cdots + a_nx^n$ in $R[x;\alpha]$, $m(x)f(x) = 0$ implies $m_0b_j = 0$ for each $0 \leq j \leq n$; (2) An α-compatible module M_R is reduced if and only if $M[x]/M[x](x^n)$ is an α-skew Armendariz module over $R[x]/(x^n)$ for any integer $n \geq 2$. This result shows that α-compatible reduced modules play so important roles in the study of skew-Armendariz modules (and hence skew-Armendariz rings) as that of reduced modules in the study of Armendariz modules. (3) An (α,δ)-compatible module M_R is quasi-Baer (respectively, p.q.-Baer) if and only if $M[x]$ is a quasi-Baer (respectively, p.q.-Baer) module over $R[x;\alpha,\delta]$; (4) If M_R is skew-Armendariz with $R \subseteq M$, then M_R is Baer (respectively, p.p) if and only if $M[x]$ is a Baer (respectively, p.p.-) module over $R[x;\alpha,\delta]$; (5) A necessary and sufficient condition for the trivial extension $T(R,R)$ to be skew quasi-Armendariz is obtained. Examples to illustrate the concepts and results are included.

We also study the relations between the set of annihilators in M and the set of annihilators in $M[x]_{R[x;\alpha,\delta]}$. We give a sufficient condition for a module to be skew quasi-Armendariz and study the structure of the skew quasi-Armendariz modules. This work extends and unifies several known results related to Armendariz rings and modules, in particular the landmark results of Hong et al. [20, 21], parallels results of the second author and A.R. Nasr-Isfahani [35] on Ore extensions, and complements later results of E. Hashemi [16] and Zhang and Chen [43] to general polynomial modules over Ore polynomial extension $R[x;\alpha,\delta]$.

2. **Skew-Armendariz Modules**

In this section the notion of an skew-Armendariz module is introduced as a generalization of skew-Armendariz rings to modules and its properties are studied. We prove that many results of skew-Armendariz rings can be extended to modules with this general settings. We show that the notion of skew-Armendariz module generalizes that of α-skew Armendariz modules of Zhang and Chen [43] as well as α-Armendariz modules and α-reduced modules of Lee and Zhou [29]. Moreover we extend the classes of skew-Armendariz modules.

We will be working here with general right modules M_R rather than just R_R, and the restrictions on α and δ we require are best phrased as conditions on the module M_R that arise from the use of general α and δ. Let us formally define these conditions here:

From the Ore commutation law, an inductive argument can be made to calculate an expression for $x^j a$, for all $j \in \mathbb{N}$ and $a \in R$. To record this result, we shall use some convenient notation introduced in [3, 27]:
Notation. Given α and δ as above and integers $j \geq i \geq 0$, let us write f_i^j for the sum of all "*words*" in α and δ in which there are i factors of α and $j-i$ factors of δ. For instance, $f_j^j = \alpha^j$, $f_0^j = \delta^j$, and $f_{j-1}^j = \alpha^{j-1}\delta + \alpha^{j-2}\delta\alpha + \cdots + \delta\alpha^{j-1}$.

Using recursive formulas for the f_i^j's and induction, as done in [27], one can show with a routine computation that

$$x^j a = \sum_{i=0}^{j} f_i^j(a) x^i, \tag{2.1}$$

for all $a \in R$, where $j \geq i \geq 0$. This formula uniquely determines a general product of (left) polynomials in $S = R[x;\alpha,\delta]$ and will be used freely in what follows. More generally, given a right R-module M_R, we can form the polynomial module $M[x]_S$ over S as follows. Elements of $M[x]$ have the form $\sum m_i x^i$ ($m_i \in M$), and the action of S on such elements is basically dictated by (2.1), since it suffices to define the action of monomials of S on monomials in $M[x]_S$ via

$$(mx^j)(ax^l) = m\sum_{i=0}^{j} f_i^j(a)x^{i+l}$$

for all $a \in R$ and $j, l \in \mathbb{N}$. It is readily verified that this makes $M[x]$ into an S-module.

A ring R is called *Armendariz* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. Following Anderson and Camillo [2], a module M_R is called Armendariz if, whenever $m(x)f(x) = 0$, where $m(x) = \sum_{i=0}^{s} m_ix^i \in M[x]$ and $f(x) = \sum_{j=0}^{t} a_jx^j \in R[x]$, we have $m_ia_j = 0$ for all i, j.

The term Armendariz was introduced by Rege and Chhawchharia [41]. This nomenclature was used by them since it was Armendariz [5], who initially showed that a reduced ring always satisfies this condition.

The more comprehensive study of Armendariz rings was carried out recently (see, e.g., [1-2, 5-6, 11-12, 15-22, 28-29]. The interest of this notion lies in its natural and useful role in understanding the relation between the annihilators of the ring R and the annihilators of the polynomial ring $R[x]$. The reason behind these is the fact that there is a natural bijection between the set of annihilators of R and the set of annihilators of $R[x]$ (see Hirano, [19]).

In [21], C.Y. Hong, N.K. Kim and T.K. Kwak extended the Armendariz property of rings to skew polynomial rings $R[x;\alpha]$: For an endomorphism α of a ring R, R is called an *α-skew Armendariz ring* (or, a skew-Armendariz ring with the endomorphism α) if for polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ in $R[x;\alpha]$, $f(x)g(x) = 0$ implies $a_i\alpha^i(b_j) = 0$ for each $0 \le i \le n$ and $0 \le j \le m$.

M. Başer in [6] studied relations between the set of annihilators in M_R and the set of annihilators in $M[x]$. In [43], Zhang and Chen extended a result of [42] and they showed that, a ring R is α-skew Armendariz if and only if every flat right R-module is α-skew Armendariz. Some other properties of Armendariz rings and modules have been studied in Armendariz [5], Rege and Chhawchharia [41], Rege and Buhphang [42], Anderson and Camillo [2], Hong et al. [20, 21], Kim and Lee

[25], Chen and Tong [12], Hashemi and Moussavi [17, 18], Huh, Lee and Smoktunowicz [22], Lee and Zhou [29], Nasr-Isfahani and Moussavi [35-39] and some other authors.

According to Krempa [26], an endomorphism α of a ring R is called to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is said to be α-*rigid* if there exists a rigid endomorphism α of R. Hong et al. [20], studied Ore extensions of Baer rings over α-rigid rings, and show that a ring R is α-rigid if and only if $R[x; \alpha, \delta]$ is reduced. Clearly a reduced ring is Baer if and only if it is quasi-Baer.

In [35], the second author and A.R. Nasr-Isfahani, introduced the concept of a skew-Armendariz ring and studied its properties. Our focus in this section is to introduce the concept of a skew-Armendariz module and study its properties. We prove that the notion of skew-Armendariz module generalizes that of α-skew Armendariz rings of Hong et al. [21] and Krempa's α-rigid rings [26] as well as that of the second author and A.R. Nasr-Isfahani's skew-Armendariz rings [35] to general polynomial modules over Ore polynomial extension $R[x; \alpha, \delta]$.

Definition 2.1. (*Zhang and Chen* [43]) *Let R be a ring with an endomorphism α and M_R an R-module. A module M_R is called an α-skew Armendariz module, if for polynomials $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]$ and $f(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x; \alpha]$, $m(x)f(x) = 0$ implies $m_i\alpha^i(b_j) = 0$ for each $0 \leq i \leq k$ and $0 \leq j \leq n$.*

Definition 2.2. Let R be a ring with an endomorphism α and α-derivation δ. Let M_R be an R-module. We say that M_R is an (α, δ)-*skew Armendariz module* if, for polynomials $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]$ and $f(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x; \alpha, \delta]$, $m(x)f(x) = 0$ implies $m_ix^ib_jx^j = 0$ for each $0 \leq i \leq k$ and $0 \leq j \leq n$.

Notice that in the case when $\delta = 0$, the above definition boils down to the notion of α-skew Armendariz of Zhang and Chen [43].

Definition 2.3. *Let R be a ring with an endomorphism α and α-derivation δ. Let M_R be an R-module. We say that M_R is a skew-Armendariz module, if for polynomials $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]$ and $f(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x; \alpha, \delta]$, $m(x)f(x) = 0$ implies $m_0b_j = 0$ for each $0 \leq j \leq n$.*

It is clear that (α, δ)-skew Armendariz modules are skew-Armendariz, and each Armendariz module is α-skew Armendariz, where $\alpha = id_R$, and every submodule of a skew-Armendariz module is skew-Armendariz. It is also clear that R is a skew-Armendariz ring if R_R is an skew-Armendariz module. In [35], the second author and A.R. Nasr-Isfahani provided numerous examples of non-semiprime (and hence non-reduced) skew-Armendariz rings.

The following equivalent characterization of an α-skew-Armendariz module is useful to simplify the proofs of results in the context of Armendariz rings and modules. It is shown that our definition of a skew-Armendariz module is a generalization of Hong et al.'s α-skew Armendariz ring [21] and Zhang and Chen's α-skew Armendariz module [43], for the more general setting.

The following result shows that our definition of a skew-Armendariz module is a generalization of the notion of an α-skew-Armendariz module for the more general setting:

Theorem 2.4. *Let M_R be a module and α an endomorphism of R. Then M_R is α-skew Armendariz if and only if for every polynomials $m(x) = m_0+m_1x+\cdots+m_kx^k \in M[x]$ and $f(x) = b_0+b_1x+\cdots+b_nx^n \in R[x;\alpha]$, $m(x)f(x) = 0$ implies $m_0b_j = 0$ for each $0 \leq j \leq n$.*

Proof. The forward direction is clear that if M_R is an α-skew Armendariz, then for every polynomials $m(x) = m_0+m_1x+\cdots+m_kx^k \in M[x]$ and $f(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x;\alpha]$, $m(x)f(x) = 0$ implies $m_0b_j = 0$ for each $0 \leq j \leq n$. For the backward direction, suppose that for every polynomials $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]$ and $f(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x;\alpha]$, $m(x)f(x) = 0$ implies $m_0b_j = 0$ for each $0 \leq j \leq n$. We show that M_R is α-skew Armendariz. We have, $0 = (m_0 + m_1x + \cdots + m_kx^k)(b_0 + b_1x + \cdots + b_nx^n) =$ $m_0(b_0+b_1x+\cdots+b_nx^n)+(m_1+m_2x+\cdots+m_kx^{k-1})x(b_0+b_1x+\cdots+b_nx^n)$. So $(m_1 + m_2x + \cdots + m_kx^{k-1})(\alpha(b_0)x + \alpha(b_1)x^2 + \cdots + \alpha(b_n)x^{n+1}) = 0$. Hence $m_1\alpha(b_j) = 0$ for each $0 \leq j \leq n$. Inductively, we can see that $m_i\alpha^i(b_j) = 0$ for each $0 \leq i \leq k$ and $0 \leq j \leq n$ and the result follows. $\square$

Corollary 2.5. *A ring R with an endomorphism α is α-skew Armendariz if and only if for every polynomials $f(x) = a_0 + a_1x + \cdots +$*

a_kx^k , $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x;\alpha]$, $f(x)g(x) = 0$ *implies* $a_0b_j = 0$ *for each* $0 \leq j \leq n$.

If we take $\alpha = id_R$, we deduce the following equivalent condition for a module to be Armendariz.

Corollary 2.6. *A module* M_R *is Armendariz if and only if for every polynomials* $m(x) = m_0+m_1x+\cdots+m_kx^k \in M[x]$ *and* $f(x) = b_0+b_1x+\cdots+b_nx^n \in R[x]$, $m(x)f(x) = 0$ *implies* $m_0b_j = 0$ *for each* $0 \leq j \leq n$.

Corollary 2.7. *A ring* R *is Armendariz if and only if for every polynomials* $f(x) = a_0+a_1x+\cdots+a_nx^n$, $g(x) = b_0+b_1x+\cdots+b_mx^m \in R[x]$, $f(x)g(x) = 0$ *implies* $a_0b_j = 0$ *for each* $0 \leq j \leq m$.

Definition 2.8. *Let* R *be a ring with an endomorphism* α *and an* α-*derivation* δ. *We say that* M_R *is a linearly skew-Armendariz module, if for linear polynomials* $m(x) = m_0 + m_1x \in M[x]$ *and* $g(x) = b_0 + b_1x \in R[x;\alpha,\delta]$, $m(x)g(x) = 0$ *implies* $m_0b_0 = m_0b_1 = 0$.

It is clear that each skew-Armendariz module is linearly skew-Armendariz and that every submodule of a linearly skew-Armendariz module is also linearly skew-Armendariz.

By [12, Example 2.2], there exists an α-skew Armendariz ring R such that α is not a monomorphism and R is not a reduced ring:

Example 2.9. *Let* D *be a domain and* $R_n(D)$ *a subring of* $M_n(D)$, *where* $n \geq 2$ *and*

$$R_n(D) := \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in D \right\}.$$

Let α *be an endomorphism of* $R_n(D)$ *such that*

$$\alpha\left(\begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix}\right) = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix}.$$

Clearly, α is not a monomorphism and $R_n(D)$ is not a reduced ring. In [12, Example 2.2] it is proved that $R_n(D)$ is an α-skew Armendariz ring.

Let R be a subring of a ring S with $1_S \in R$ and $M_R \subseteq L_S$. Let α be an endomorphism and δ an α-derivation of S such that $\alpha(R) \subseteq R$ and $\delta(R) \subseteq R$. If L_S is (α, δ)-skew Armendariz, then M_R is also (α, δ)-skew Armendariz.

We can deduce the following result, using the definition of skew-Armendariz modules.

Proposition 2.10. *Let α be an endomorphism and δ an α-derivation of a ring R. The class of skew-Armendariz modules is closed under submodules, direct products and direct sums.*

Definition 2.11. (*Annin,* [3]) *Given a module M_R, an endomorphism $\alpha : R \to R$ and an α-derivation $\delta : R \to R$, we say that M_R is α-compatible if for each $m \in M$ and $r \in R$, we have $mr = 0 \Leftrightarrow m\alpha(r) = 0$. Moreover, we say M_R is δ-compatible if for each $m \in M$ and $r \in R$, we have $mr = 0 \Rightarrow m\delta(r) = 0$. If M_R is both α-compatible and δ-compatible, we say that M_R is (α, δ)-compatible.*

The (α, δ)-compatibility condition on M_R is a natural, independently interesting condition from which we can derive a number of interesting properties, and it will be of invaluable service in the proof of our main results. After a few quick remarks about Definition 2.11, we will present some results on modules and annihilators in Ore extension rings that can be deduced for these (α, δ)-compatible modules. These fundamental properties of (α, δ)-compatible modules will lay the groundwork for our main results.

Remark 2.12. (a) *It is important to note that the α-compatibility assumption requires an "if and only if" while the δ-compatibility assumption is only a one-sided implication. The reason for the stronger assumption on α is that we will often need to consider the leading coefficient of an expression $m(x)r$, where $m(x) \in M[x]$ and $r \in R$, where by (2.1) will involve powers of α but will be free of δ. Finally, observe that in the classical case where $\delta = 0$, one never has the reverse implication to the δ-compatibility condition for a nonzero module M_R, so we certainly do not expect a two-sided implication for the condition on δ.*

(b) If M_R is α-compatible (respectively, δ-compatible), then so is any submodule of M_R.

(c) If M_R is α-compatible (respectively, δ-compatible), then for all $i \geq 1$, M_R is α^i-compatible (respectively, δ^i-compatible).

The following lemma shows that the (α, δ)-compatibility property on a module M_R is inherited by the polynomial module $M[x]$.

Lemma 2.13. [3, Lemma 2.16] *A module M_R is (α, δ)-compatible if and only if the polynomial extension $M[x]_R$ is (α, δ)-compatible.*

Lemma 2.14. *The following are equivalent for a module M_R.*

(i) *M_R is reduced and (α, δ)-compatible;*

(ii) *The following conditions hold. For any $m \in M$ and $a \in R$,*

(a) *$ma = 0$ implies $mRa = 0$,*

(b) *$ma = 0$ implies $m\delta(a) = 0$,*

(c) *$ma = 0$ if and only if $m\alpha(a) = 0$,*

(d) *$ma^2 = 0$ implies $ma = 0$.*

Proof. The proof is straightforward.

□

Lemma 2.15. *Let M_R be an (α, δ)-compatible module. Let $m \in M$ and $a, b \in R$. Then we have the following:*

(i) *If $ma = 0$, then $m\alpha^j(a) = 0 = m\delta^j(a)$ for any positive integer j;*

(ii) *If $mab = 0$, then $m\alpha(\delta^j(a))\delta(b) = 0 = m\alpha^i(\delta(a))\delta^j(b)$, and hence $ma\delta^j(b) = 0 = m\delta^j(a)b$ for any positive integer i, j;*

(iii) *$ann_R(ma) = ann_R(m\alpha(a)) \subseteq ann_R(m\delta(a))$.*

Proof. (i) This follows from section (c) of Remark 2.12.
(ii) Suppose that $mab = 0$. Since M_R is δ-compatible, $ma\delta^j(b) = 0$ for each j.

Using α-compatibility of M_R, $m\alpha(ab) = 0$, so $m\alpha(a)b = 0$. Since M_R is δ-compatible, $m\alpha(a)\delta(b) = 0$.
Since M_R is δ-compatible, $mab = 0$ implies $0 = m\delta(a)b + m\alpha(a)\delta(b)$. By above, we deduce $m\delta(a)b = 0$.
Using α-compatibility of M_R, $m\alpha(\delta(a)b) = 0$ if and only if $m\alpha(\delta(a))\alpha(b) = 0$ if and only if $m\alpha(\delta(a))b = 0$. By δ-compatibility of M_R, we have $m\alpha(\delta(a))\delta(b) = 0$.
By above calculations, $m\delta(a)b = 0$ and by δ-compatibility of M_R, $0 = m\delta(\delta(a)b) = m\delta^2(a)b + m\alpha(\delta(a))\delta(b)$. So, $m\delta^2(a)b = 0$.
Therefore, inductively we get $m\delta^j(a)b = 0$ for each j. So, $ma\delta^j(b) = 0 = m\delta^j(a)b$. Also, we can similarly deduce that $m\alpha(\delta^j(a))\delta(b) = 0$. Now we show that $mab = 0$ implies that $m\alpha^i(\delta(a))\delta^j(b) = 0$. By above, $m\delta(a)b = 0$, and then α^i-compatibility of M_R implies $m\alpha^i(\delta(a)b) = 0$ and hence $m\alpha^i(\delta(a))\alpha^i(b) = 0$. Also using α^i-compatibility of M_R, it implies $m\alpha^i(\delta(a))b = 0$. Since M_R is δ^j-compatible, $m\alpha^i(\delta(a))\delta^j(b) = 0$. These computations impliy the result.
(iii) Note that α-compatibility of M_R yields $m\alpha(a)b = 0 \Leftrightarrow m\alpha(a)\alpha(b) = 0 \Leftrightarrow m\alpha(ab) = 0 \Leftrightarrow mab = 0$ for all $a, b \in R$. It remains only to show that $ann_R(ma) \subseteq ann_R(m\delta(a))$. To see this, let $mab = 0$ for some $b \in R$. Using δ-compatibility, we get $0 = m\delta(ab) = m\left(\delta(a)b + \alpha(a)\delta(b)\right) = 0$. Since we have already concluded that $m\alpha(a)b = 0$, δ-compatibility implies that $m\alpha(a)\delta(b) = 0$, and hence $m\delta(a)b = 0$, as desired.

$\square$

Lemma 2.16. *Let M_R be an (α,δ)-compatible module and $m(x) = m_0 + \cdots + m_k x^k \in M[x]$ and $r \in R$. Then $m(x)r = 0$ if and only if $m_i r = 0$ for all $0 \le i \le k$.*

Proof. Assume $m_i r = 0$ for all $0 \le i \le k$. An easy calculation using (2.1) shows that

$$m(x)r = \sum_{i=0}^{k}\left(\sum_{j=i}^{k} m_j f_i^j(r)\right)x^i. \tag{2.2}$$

By (α,δ)-compatibility of M_R, we have $m_j f_i^j(r) = 0$, for all i, j. Thus (2.2) yields $m(x)r = 0$. Conversely, assume that $m(x)r = 0$. We deduce from (2.2) that,

$$\sum_{j=i}^{k} m_j f_i^j(r) = 0, \tag{2.3}$$

for each $i \leq k$.
Starting with $i = k$, Eq. (2.3) yields $m_k\alpha^k(r) = 0$ and hence $m_j f_i^j(r) = 0$, for each $j > i$, by (α, δ)-compatibility of M_R. Using (2.3) again, we deduce that $m_i\alpha^i(r) = 0$, and that $m_i r = 0$ as desired. □

Proposition 2.17. *A module M_R is α-reduced if and only if the polynomial extension $M[x]_R$ is an α-reduced module.*

Proof. It is enough to prove the forward direction. By Lemma 2.13, M_R is α-compatible if and only if $M[x]_R$ is α-compatible. Now assume that, M_R is reduced, to show that $M[x]_R$ is reduced, using Lemma 2.14, we only need to show that $m(x)a = 0$ implies $m(x)Ra = 0$ and $m(x)a^2 = 0$ implies $m(x)a = 0$, where $m(x) = \sum_{i=0}^{k} m_i x^i \in M[x]$ and $a \in R$. First let $m(x)a = 0$. Since M_R is reduced and $m_i a = 0$ for each i, $m_i Ra = 0$ for each i and hence $m(x)Ra = 0$. Now suppose $m(x)a^2 = 0$. Since M_R is reduced and $m_i a^2 = 0$ for each i, $m_i a = 0$ for each i and hence $m(x)a = 0$. Thus $M[x]_R$ is reduced and the result follows by Lemma 2.14. □

Notice that, the concept of α-reduced for the regular module R_R coincides with that of reduced and α-compatible ring R, which in this case R is indeed an α-rigid ring; and note also that, a ring R is α-rigid if and only if R is reduced and (α, δ)-compatible. So we deduce the following:

Corollary 2.18. *A ring R is α-rigid if and only if $R[x]_R$ ($R[x;\alpha]$ or $R[x;\alpha,\delta]$) is an α-reduced R-module.*

Theorem 2.19. *Every (α, δ)-compatible and reduced module is skew-Armendariz.*

Proof. Let $m(x) = m_0 + \cdots + m_k x^k \in M[x]$, $f(x) = a_0 + \cdots + a_n x^n \in R[x;\alpha,\delta]$ and $m(x)f(x) = 0$. So $m_k\alpha^k(a_n) = 0$, because it is the leading coefficient of $m(x)f(x)$. By α-compatibility of M_R, we have $m_k a_n = 0$. By Lemma 2.14, $m_k R a_n = 0$, and by (α, δ)-compatibility of M_R, $m_k f_i^j(a_n) = 0$. Thus the coefficient of x^{k+n-1} in the equation $m(x)f(x) = 0$ is $m_k\alpha^k(a_{n-1}) + m_{k-1}\alpha^{k-1}(a_n) = 0$. Multiplying by a_n from right we get $m_{k-1}\alpha^{k-1}(a_n)a_n = 0$. Using α-compatibility repeatedly we obtain $m_{k-1}a_n^2 = 0$. Hence $m_{k-1}a_n = 0$, by Lemma 2.14. So $m_{k-1}Ra_n = 0$, by Lemma 2.14 and by (α, δ)-compatibility of M_R, $m_{k-1}f_i^j(a_n) = 0$. Therefore $m_k a_{n-1} = 0$. Continuing this process and using (α, δ)-compatibility of M_R, we obtain $m_i x^i a_j x^j = 0$ for each $0 \leq i \leq k$ and $0 \leq j \leq n$. Since (α, δ)-skew Armendariz modules are skew Armendariz, the result follows. $\square$

Zhang and Chen [43] proved that, for an endomorphism α of a ring R and $\alpha^\ell = id_R$ for some positive integer ℓ, M_R is α-reduced if and only if $M[x]/M[x](x^n)$ is an α-skew Armendariz module over $R[x]/(x^n)$ for integer $n \geq 2$. They also asked if the condition $\alpha^\ell = id_R$ superfluous.

For a right R-module M_R and $A = (a_{ij}) \in M_n(R)$, let $MA = \{(ma_{ij}) \mid m \in M\}$. For $n \geq 2$, let $V = \sum_{i=1}^{n-1} E_{i,i+1}$ where $\{E_{i,j} \mid 1 \leq i, j \leq n\}$ are the matrix units, and set $T(R, n) = RI_n + RV + \cdots + RV^{n-1}$, $T(M, n) = MI_n + MV + \cdots + MV^{n-1}$. Then $T(R, n)$ is a ring and $T(M, n)$ becomes a right module over $T(R, n)$ under usual addition and multiplication of matrices. There is a ring isomorphism $\psi : T(R, n) \to R[x]/(x^n)$ given by $\psi(r_0 I_n + r_1 V + \cdots + r_{n-1}V^{n-1}) = r_0 + r_1 x + \cdots + r_{n-1}x^{n-1} + (x^n)$ and an Abelian group isomorphism $\phi : T(M, n) \to M[x]/M[x](x^n)$ given by $\phi(m_0 I_n + m_1 V + \cdots + m_{n-1}V^{n-1}) = m_0 + m_1 x + \cdots + m_{n-1}x^{n-1} + M[x](x^n)$ such that $\phi(WA) = \phi(W)\psi(A)$ for all $W \in T(M, n)$ and $A \in T(R, n)$.

Notice that

$$T(R,n) := \left\{ \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_1 \\ 0 & 0 & \cdots & 0 & a_0 \end{pmatrix} \mid a_i \in R \right\},$$

with $n \geq 2$, is a ring with point-wise addition and usual matrix multiplication. We can denote elements of $T(R, n)$ by $(a_0, a_1, \ldots, a_{n-1})$.

Lee and Zhou [29] proved that for each integer $n \geq 2$, $M[x]/M[x](x^n)$ is an Armendariz right module over $R[x]/(x^n)$ if and only if M_R is reduced. In the following we generalize this to α-reduced modules.

Let α be an endomorphism of a ring R. Then the map $T(R, n) \to T(R, n)$ defined by $a_0 I_n + a_1 V + \cdots + a_{n-1}V^{n-1} \to \alpha(a_0)I_n + \alpha(a_1)V + \cdots + \alpha(a_{n-1})V^{n-1}$ is an endomorphism of $T(R, n)$. Similarly it is easy to see that the map $R[x]/(x^n) \to R[x]/(x^n)$ defined by $a_0 + a_1 x + \cdots +$

$a_{n-1}x^{n-1} + (x^n) \to \alpha(a_0) + \alpha(a_1)x + \cdots + \alpha(a_{n-1})x^{n-1} + (x^n)$ is an endomorphism of $R[x]/(x^n)$. We will also denote the two maps above by α.

The following result shows that α-compatible reduced modules play so important roles in the study of skew-Armendariz modules (and hence skew-Armendariz rings) as that of reduced rings in the study of Armendariz rings.

Theorem 2.20. *An α-compatible module M_R is reduced if and only if $M[x]/M[x](x^n)$ is an α-skew Armendariz module over $R[x]/(x^n)$ for integer $n \geq 2$.*

Proof. First assume that $T(M,n)$ is an α-skew Armendariz module over $T(R,n)$ and let $ma = 0$ for $a \in R$ and $m \in M$. Let $p(x) = (m, 0, \ldots, 0) + (0, 0, \ldots, mr)x \in T(M,n)[x;\alpha]$, $q(x) = (a, 0, \ldots, 0) - (0, 0, \ldots, r\alpha(a))x \in T(R,n)[x;\alpha]$ with $p(x)q(x) = 0$. Since $T(M,n)$ is α-skew Armendariz, $(m, 0, \ldots, 0)(0, 0, \ldots, r\alpha(a)) = 0$ implies $mr\alpha(a) = 0$ for each $r \in R$. Hence $mR\alpha(a) = 0$ yields $mRa = 0$, because M_R is α-compatible. Thus M_R is reduced. Conversely, assume that M_R is reduced. Consider the following mapping
$\varphi_1 : T(M,n)[x;\alpha] \to T(M[x;\alpha], n)$, be given by $\varphi_1(A_0 + A_1x + \cdots + A_kx^k) = (f_1, f_2, \ldots, f_n)$, where $A_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \in T(M,n)$, $f'_i = a_{0i'} + a_{1i'}x + \cdots + a_{ki'}x^k \in M[x]$, $0 \leq i \leq k$ and $1 \leq i' \leq n$. Let $\varphi_2 : T(R,n)[x;\alpha] \to T(R[x;\alpha], n)$, given by $\varphi_2(B_0 + B_1x + \cdots + B_lx^l) = (g_1, g_2, \ldots, g_n)$, where $B_j = (b_{j1}, b_{j2}, \ldots, b_{jn}) \in T(R,n)$, $g_{j'} = b_{0j'} + b_{1j'}x + \cdots + b_{lj'}x^l \in R[x;\alpha]$, $0 \leq j \leq l$ and $1 \leq j' \leq n$. It is easy to see that φ_1, φ_2 are isomorphisms. Suppose that $p = A_0 + A_1x + \cdots + A_tx^t \in T(M,n)[x;\alpha]$ and $q = B_0 + B_1x + \cdots + B_mx^m \in T(R,n)[x;\alpha]$, where $A_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \in T(M,n)$, for each $0 \leq i \leq t$ and $B_j = (b_{j1}, b_{j2}, \ldots, b_{jn}) \in T(R,n)$ for each $0 \leq j \leq m$ and let $p(x)q(x) = 0$. Suppose that $p_i = a_{0i} + a_{1i}x + \cdots + a_{ti}x^t \in M[x;\alpha]$ and $q_j = b_{0j} + b_{1j}x + \cdots + b_{mj}x^m \in R[x;\alpha]$, then $p_iq_j = 0$ for $1 \leq i \leq n$ and $1 \leq j \leq n - i + 1$. We then have the system of equations

(A_0) $\quad a_{0i}b_{0j} = 0,$

(A_1) $\quad a_{0i}b_{1j} + a_{1i}\alpha(b_{0j}) = 0,$

(A_2) $\quad a_{0i}b_{2j} + a_{1i}\alpha(b_{1j}) + a_{2i}\alpha^2(b_{2j}) = 0,$

$\vdots$

(A_{t+m-1}) $\quad a_{(t-1)i}b_{mj} + a_{ti}\alpha^t(b_{(m-1)j}) = 0,$

(A_{t+m}) $\quad a_{ti}\alpha^t(b_{mj}) = 0.$

By (A_{t+m}), we have $a_{ti}\alpha^t(b_{mj}) = 0$, which implies $a_{ti}b_{mj} = 0$, by α-compatibility of M_R. Hence $a_{ti}Rb_{mj} = 0$. Multiplying (A_{t+m-1}) by b_{mj} from the right, (A_{t+m-1}) becomes $a_{(t-1)i}b_{mj}^2 + a_{ti}\alpha^t(b_{(m-1)j})b_{mj} = 0$. Since $a_{ti}Rb_{mj} = 0$, we get $a_{(t-1)i}b_{mj}^2 = 0$. But M_R is reduced, so $a_{(t-1)i}b_{mj} = 0$. Continuing this process, we have $a_{0i}b_{lj} = 0$, where $0 \le l \le m$, $1 \le i \le n$ and $1 \le j \le n - i + 1$. This shows that $A_0 B_s = 0$ for $0 \le s \le m$, proving that $T(M, n)$ is α-skew Armendariz module over $T(R, n)$. □

Corollary 2.21. [29, Theorem 1.9] *A module M_R is reduced if and only if $M[x]/M[x](x^n)$ is an Armendariz module over $R[x]/(x^n)$ for an integer $n \ge 2$.*

Next we recall a well-known result.

Proposition 2.22. *Suppose that M is a flat right R-module. Then for every exact sequence $0 \to K \to F \to M \to 0$, where F is R-free, we have $(FI) \cap K = KI$ for each left ideal I of R; in particular, we have $Fa \cap K = Ka$ for each element a of R.*

Proposition 2.23. *Let α be an endomorphism of a ring R and δ an α-derivation. Then R is a skew-Armendariz ring if and only if every flat R module M is skew-Armendariz.*

Proof. Let M be a flat R-module. Suppose $0 \to K \to F \to M \to 0$ is an exact sequence with F free over R. For an element $y \in F$, we denote $\bar{y} = y + K$ in M. Suppose that $f(x) = \sum_{i=0}^{t} \bar{y}_i x^i \in M[x]$ and $g(x) = \sum_{j=0}^{n} a_j x^j \in R[x; \alpha, \delta]$ with $f(x)g(x) = 0$. We show that $\bar{y_0}a_j = 0$ for $0 \le j \le n$. We have $f(x)g(x) = 0$, so we get,
The constant term: $\bar{y_0}a_0 + \bar{y_1}\delta(a_0) + \bar{y_2}\delta^2(a_0) + \cdots = 0$;
The coefficient of x: $\bar{y_0}a_1 + \bar{y_1}\alpha(a_0) + \bar{y_1}\delta(a_1) + \cdots = 0$;
$\vdots$
The coefficient of x^{t+n}, $\bar{y_t}\alpha^t(a_n) = 0$.

Since M is a flat R-module, there exists an R-module homomorphism $\beta : F \to K$ such that β fixes these coefficients. Write $w_i := \beta(y_i) - y_i$ for $i = 0, 1, \ldots, t$. Each w_i is an element of F, therefore the polynomial $h(x) = \sum_{j=0}^{t} w_i x^i \in F[x]$ and $h(x)g(x) = 0$. Since R is skew-Armendariz and F is a free R-module, F is skew-Armendariz by Proposition 2.10. Thus, we have $w_0 a_j = 0$ for all j. It follows that $y_0 a_j \in K$ for all j, so $\bar{y_0}a_j = 0$

in M, proving that M is skew-Armendariz.

$\square$

Put $Ann_R(2^{M_R}) = \{ann_R(U) \mid U \subseteq M_R\}$, where M_R is an R-module.

Theorem 2.24. *Let M_R be an (α, δ)-compatible module and $S = R[x; \alpha, \delta]$. Then the following statements are equivalent:*

(1) M_R is a skew-Armendariz module;

(2) The map $\psi : Ann_R(2^{M_R}) \to Ann_S(2^{M[x]_S})$, defined by $A \to AS$ for all $A \in Ann_R(2^{M_R})$, is bijective.

Proof. (1) $\Rightarrow$ (2). Consider the maps $\psi : \{ann_R(U) \mid U \subseteq M_R\} \to \{ann_S(U) \mid U \subseteq M[x]_S\}$ defined by $A \mapsto AS$ for every $A \in \{ann_R(U) \mid U \subseteq M_R\}$, and $\psi' : \{ann_S(U) \mid U \subseteq M[x]_S\} \to \{ann_R(U) \mid U \subseteq M_R\}$ defined by $B \mapsto B \cap R$. It is clear that ψ is well defined, because $ann_R(U)S = ann_S(U)$ for each $U \subseteq M_R$. Since M_R is (α, δ)-compatible, we see that $ann_S(V) \cap R = ann_R(V_0)$ for each $V \subseteq M[x]_S$, where V_0 is the set of coefficients of all elements of V. Hence ψ' is also well defined. Since $\psi'\psi = id$, ψ is injective. Assume that $B \in \{ann_S(U) \mid U \subseteq M[x]_S\}$, then $B = ann_S(J)$ for some $J \subseteq M[x]_S$. Let B_1 and J_1 denote the set of coefficients of elements of B and J, respectively. We claim that $ann_R(J_1) = B_1R$. Let $m(x) = m_0 + m_1x + \cdots + m_kx^k \in J$ and $f(x) = b_0 + b_1x + \cdots + b_nx^n \in B$. Then $m(x)f(x) = 0$. Since M_R is skew-Armendariz and (α, δ)-compatible, $m_ib_j = 0$ for all m_i and b_j. Thus $J_1B_1 = 0$, hence $B_1R \subseteq ann_R(J_1)$. Since M_R is (α, δ)-compatible, $ann_R(J_1) \subseteq B_1R$. Thus $ann_R(J_1) = B_1R$, and hence $ann_S(J) = B_1RS$. Therefore ψ is surjective.

(2) $\Rightarrow$ (1). Let $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]_S$ and $f(x) = b_0 + b_1x + \cdots + b_nx^n \in S = R[x; \alpha, \delta]$ satisfy $m(x)f(x) = 0$. Then $f(x) \in ann_S(m(x)) = AS$, where $A = ann_R(U)$ and $U \subseteq M_R$. Hence $b_0, \ldots, b_n \in A$ and so $m(x)b_j = 0$ for $0 \leq j \leq n$. Hence $m_0b_j = 0$ for each $0 \leq j \leq n$, and the result follows.

$\square$

Theorem 2.25. *If M_R is a linearly skew-Armendariz module with $R \subseteq M$, then for each idempotent $e \in R$, $\alpha(e) = e$ and $\delta(e) = 0$.*

Proof. Since M_R is a linearly skew-Armendariz module with $R \subseteq M_R$, then R_R is also linearly skew-Armendariz. Hence by [35, Theorem 3.1], the result follows. $\square$

N. Agayev et al. [1] introduced and studied the notion of abelian modules:
A module M_R is called *abelian* if, for any $m \in M$ and any $a \in R$, any idempotent $e \in R$, $mae = mea$. It is proved in [1] that every Armendariz module and hence every reduced module is abelian. The class of abelian modules is closed under direct sums, and a ring R is abelian if and only if every flat R-module M_R is abelian.

Theorem 2.26. *If M_R is a linearly skew-Armendariz module with $R \subseteq M$, then M_R is an abelian module.*

Proof. Let M_R be a linearly skew-Armendariz module. Consider the polynomials $m_1(x) = me - mer(1-e)x$ and $m_2(x) = m(1-e) - m(1-e)rex \in M[x]_{R[x;\alpha,\delta]}$ and $f_1(x) = (1-e) + er(1-e)x$ and $f_2(x) = e + (1-e)rex \in R[x;\alpha,\delta]$, where e is an idempotent in R, $r \in R$ and $m \in M$. Since $\alpha(e) = e$ and $\delta(e) = 0$, we have $m_1(x)f_1(x) = 0$ and $m_2(x)f_2(x) = 0$. Since M_R is linearly skew-Armendariz, we get $mere = mer$ and $mere = mre$. Thus $mer = mre$ for each $r \in R$, and hence M_R is an abelian module.

□

Corollary 2.27. *If M_R is a skew-Armendariz module with $R \subseteq M$, then M_R is an abelian module.*

Theorem 2.28. *Let M_R be a reduced module. Then M_R is a p.p.-module if and only if M_R is a p.q.-Baer module.*

Proof. Since M_R is reduced, by Lemma 2.14, for each $m \in M$ and $a \in R$, $ma = 0$ implies $mRa = 0$. So $ann_R(m) \subseteq ann_R(mR)$ and hence $ann_R(m) = ann_R(mR)$.

□

Theorem 2.29. *Let M_R be an (α,δ)-compatible and skew-Armendariz module with $R \subseteq M$. Then M_R is p.p. if and only if $M[x]_{R[x;\alpha,\delta]}$ is p.p.*

Proof. Suppose that M_R is a p.p.-module and $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]$. So $ann_R(m_i) = e_iR$ for idempotents $e_i \in R$ with $0 \leq i \leq k$. Set $e = e_0e_1 \cdots e_k$, then e is an idempotent, this is because M_R is abelian by Corollary 2.27. Hence $eR = \cap_{i=0}^{k} ann_R(m_i)$. By Theorem 2.25, $\alpha(e) = e$ and $\delta(e) = 0$. Thus $m(x)e = 0$ and hence $eS \subseteq ann_S(m(x))$, where $S = R[x;\alpha,\delta]$. Next, assume that $q(x) =$

$\sum_{j=0}^{n} b_j x^j \in ann_S(m(x))$. Since M_R is skew-Armendariz, $m_0 b_j = 0$ for $0 \leq j \leq n$. So $b_j \in eR$ and hence $q(x) \in eS$, so $ann_S(m(x)) = eS$. This shows that $M[x]$ is a p.p.-module over $R[x;\alpha,\delta]$.

Conversely, suppose that $M[x]$ is a p.p.-module over $R[x;\alpha,\delta]$ and $m \in M$. Let $e(x) = e_0 + e_1 x + \cdots + e_n x^n$ be an idempotent in $R[x;\alpha,\delta]$. Then from $e(1-e) = 0 = (1-e)e$, we get $(e_0 + e_1 x + \cdots + e_n x^n)(1 - e_0 - e_1 x - \cdots - e_n x^n) = 0$ and $(1 - e_0 - e_1 x - \cdots - e_n x^n)(e_0 + e_1 x + \cdots + e_n x^n) = 0$. Since M_R is skew-Armendariz, $e_0(1-e_0) = 0$, $(1-e_0)e_i = 0$. So $e_0 e_i = 0$, $e_i = e_0 e_i$, and hence $e_i = 0$. Thus $e(x) = e_0^2 = e_0 \in R$, and $ann_S(m) = eS$, which yields $ann_R(m) = eR$ and the result follows. □

Theorem 2.30. *Let M_R be an (α,δ)-compatible skew-Armendariz module with $R \subseteq M$. Then M_R is Baer if and only if $M[x]_{R[x;\alpha,\delta]}$ is Baer.*

Proof. Assume that M_R is a Baer module and $J \subseteq M[x]$. First suppose $J_0 = \{m \in M | m$ is a leading coefficient of some non-zero element of J$\}$. Clearly, J_0 is a subset of M. Since M_R is Baer, there exists $e^2 = e \in R$ such that $ann_R(J_0) = eR$. Hence $eS \subseteq ann_S(J)$ by Lemma 2.15. Let $f(x) = b_0 + b_1 x + \cdots + b_n x^n \in ann_S(J)$. Then $J_0 b_j = 0$ for each $j = 0, \ldots, n$, because M_R is skew-Armendariz. Hence $b_j = eb_j$ for each $j = 0, \ldots, n$ and $f(x) = ef(x) \in eS$. Thus $ann_S(J) = eS$ and $M[x]_S$ is a Baer module. Conversely, assume that $M[x]_S$ is a Baer module and $A \subseteq M$. Then $A[x] \subseteq M[x]$. Since $M[x]$ is Baer, there exists an idempotent $e(x) = e_0 + \cdots + e_n x^n \in S$ such that $ann_S(A[x]) = e(x)S$. Hence $Ae_0 = 0$ and $e_0 R \subseteq ann_R(A)$. Next, let $t \in ann_R(A)$. Then $A[x]t = 0$ by Lemma 2.16. Hence $t = e(x)t$ and so $t = e_0 t \in e_0 R$. Thus $ann_R(A) = e_0 R$ and M_R is a Baer module. □

Example 2.31. *Let F be a filed and $R = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$ and let $M_R = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$ be a right R-module. Let $\alpha : R \to R$ be the automorphism given by $\alpha\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$, for each $a, b \in F$. Note that R is an abelian ring and M_R is an abelian module. But we see that M_R is not α-skew Armendariz. For this let $m(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} x \in$*

$M[x]$ and $f(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} x \in R[x;\alpha]$. *Then, we can easily see that* $m(x)f(x) = 0$. *But we have,* $m_0 a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \neq 0$.

McCoy [31, Theorem 2] proved that if R is a commutative ring, then whenever $g(x)$ is a zero-divisor in $R[x]$ there exists a nonzero $c \in R$ such that $cg(x) = 0$. We shall extend this result as follows.

Proposition 2.32. *Let M_R be an (α, δ)-compatible and reduced module. If $m(x)$ is a torsion element in $M[x]$ (i.e., $m(x)h(x) = 0$ for some $0 \neq h(x) \in R[x;\alpha,\delta]$), then there exists a non-zero element c of R such that $m(x)c = 0$.*

Proof. Let $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x]$ and $h(x) = \sum_{j=0}^{s} h_j x^j \in R[x;\alpha,\delta]$ and $m(x)h(x) = 0$. Then $m_n \alpha^n(h_s) = 0$, and since M is α-compatible, we have $m_n h_s = 0$. By Lemma 2.14, we get $m_n R h_s = 0$. Since M_R is (α, δ)-compatible, it is (α^i, δ^j)-compatible for each i, j and hence $m_n f_i^j(h_s) = 0$ for each $j \geq i \geq 0$. Hence the coefficient of x^{n+s-1} in $m(x)h(x) = 0$ is $m_n \alpha^n(h_{s-1}) + m_{n-1}\alpha^{n-1}(h_s) = 0$.
Multiply the above equation from right by h_s, we get $m_{n-1}\alpha^{n-1}(h_s)h_s = 0$. Using α-compatibility repeatedly, we obtain $m_{n-1}h_s^2 = 0$, and then by Lemma 2.14, we have $m_{n-1}h_s = 0$. Using Lemma 2.14 again, we have $m_{n-1}Rh_s = 0$, and by (α, δ)-compatibility of M_R, $m_{n-1} f_i^j(h_s) = 0$ for each $j \geq i \geq 0$. Hence the coefficient of x^{n+s-2} in $m(x)h(x) = 0$ is $m_n\alpha^n(h_{s-2}) + m_{n-1}\alpha^{n-1}(h_{s-1}) + m_n f_{n-1}^n(h_{s-1}) + m_{n-2}\alpha^{n-2}(h_s) = 0$.
Multiplying the above equation from right by h_s, we get $m_{n-2}\alpha^{n-2}(h_s)h_s = 0$. Using α-compatibility repeatedly we obtain $m_{n-2}h_s^2 = 0$, and then by Lemma 2.14, we have $m_{n-2}h_s = 0$. Continuing this process we deduce that $m_j h_s = 0$ for each j. Since $h(x) \neq 0$ we may assume that $c = h_s \neq 0$. Then by Lemma 2.16, we get $m(x)c = 0$.

□

Corollary 2.33. *Let M_R be an (α, δ)-compatible and reduced module. Then M_R is Baer (respectively, p.p.) if and only if so is $M[x]_{R[x;\alpha,\delta]}$.*

Proof. This follows from Theorems 2.19, 2.29 and 2.30.

□

Corollary 2.34. *Let R be an α-compatible and reduced ring. Then R is Baer (respectively, p.p.) if and only if $R[x;\alpha,\delta]$ is Baer (respectively, p.p.).*

Proof. Since R_R is α-compatible and reduced, by definition, R is an α-rigid ring. Hence the result follows by Theorems 11 and 14 of [20].

$\square$

Example 2.35. *Let R_0 be a domain with characteristic 0 and let R be the polynomial ring $R_0[t]$. Let α be the automorphism of R which is invariant on R_0 and $\alpha(t)=-t$. For each fixed element $a\in R_0$, let δ be the derivation on R given by $\delta(at^n)=\begin{cases} at^{n-1} & \text{if } n \text{ is odd,}\\ 0 & \text{if } n \text{ is even.}\end{cases}$ Assume that $M:=R_0\oplus R_0\oplus\cdots$. Then M is a right R module given by $(m_0,m_1,\cdots)r=(0,m_0k_0,m_1k_1,\cdots)$ for each $(m_0,m_1,\cdots)\in M$ and $r\in R$ and fixed non-zero integers $k_0,k_1,k_2,\cdots$. First we show that M_R is (α,δ)-compatible. It is enough to show that for each $0\neq m\in M$, $ann(m)=0$. Suppose that $(a_0,a_1,a_2,\cdots)(b_rt^r+b_{r+1}t^{r+1}+\cdots)=0$, where $a_i,b_i\in R_0$ for each $i\in\mathbb{N}_0$ and $b_r\neq 0$. So we have $(0,0,\cdots,0,a_0k_0k_1\cdots k_{r-1},a_1k_1k_2\cdots k_r,\cdots)(b_r+b_{r+1}t+\cdots)=0$. This implies that $a_0k_0k_1\cdots k_{r-1}b_r=0$. Since R_0 is of characteristic 0, R is a domain. Since $b_r\neq 0$ and hence $k_0k_1\cdots k_{r-1}b_r\neq 0$, we get $a_0=0$. By induction we can see that $a_i=0$ for each i. Now we show that M_R is (α,δ)-skew Armendariz. To see this let $m(x)=m_0+m_1x+\cdots+m_kx^k\in M[x]$ and $f(x)=b_0+b_1x+\cdots+b_nx^n\in R[x;\alpha,\delta]$ with $0=m(x)f(x)=\sum_{p=0}^{k+n}\left(\sum_{i+l=p}\sum_{j=i}^{k} m_jf_i^j(b_l)\right)x^p$. So $m_k\alpha^k(a_n)=0$. By α-compatibility of M_R, we have $m_ka_n=0$. Since M_R is reduced module, $m_kRa_n=0$. On the other hand, by (α,δ)-compatibility of M_R, $m_kf_i^j(a_n)=0$. Thus the coefficient of x^{k+n-1} in equation $m(x)f(x)=0$ is $m_k\alpha^k(a_{n-1})+m_{k-1}\alpha^{k-1}(a_n)=0$. Multiplying by a_n from right we get $m_{k-1}\alpha^{k-1}(a_n)a_n=0$. Using α-compatibility repeatedly we obtain $m_{k-1}a_n^2=0$. Hence $m_{k-1}a_n=0$. Since M_R is reduced, $m_{k-1}Ra_n=0$, and by (α,δ)-compatibility of M_R, $m_{k-1}f_i^j(a_n)=0$. Therefore $m_ka_{n-1}=0$. Continuing this process and using (α,δ)-compatibility of M_R, we obtain $m_ix^ia_jx^j=0$ for each $0\leq i\leq k$ and $0\leq j\leq n$, as desired.*

In the following, we show by an example that the "(α,δ)-compatibility condition" in Lemma 2.16, is not superfluous.

Example 2.36. *Let R_0 be a domain and $R = R_0[t_1, t_2]$, where t_1, t_2 are commuting indeterminates. Let α be the R_0-automorphism defined by $\alpha(t_1) = t_2$ and $\alpha(t_2) = t_1$. Let M be the polynomial ring $R_0[t_1]$. Consider M to be a right R-module given by ordinary polynomial multiplication subject to the condition $Mt_2 = 0$. Then it is easy to see that M_R is not α-compatible. Now take $0 \neq m(x) = g_0(t_1)+g_1(t_1)x+\cdots+g_r(t_1)x^r \in M[x]$ and $t_2 \in R$. Then $0 = m(x)t_2 = g_0(t_1)t_2 + g_1(t_1)xt_2 + \cdots + g_r(t_1)x^r t_2 = g_1(t_1)t_1x + g_3(t_1)t_1x^3 + \cdots$. Thus for odd integers i, $g_i(t_1)t_1 = 0$ which implies that $g_i(t_1) = 0$, as R_0 is a domain. But $0 \neq m(x)$, so for some even number j, $0 \neq g_j(t_1)$ and hence $g_j(t_1)t_2 \neq 0$ for some j.*

3. Skew Quasi-Armendariz Modules

Following Hirano [19], a module M_R is called quasi-Armendariz if, whenever $m(x)R[x]f(x) = 0$, where $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{t} a_j x^j \in R[x]$, we have $m_iRa_j = 0$ for all i, j.

In this section, we generalize the notions of quasi-Armendariz rings and quasi-Armendariz modules and consider the relations between the set of annihilators in M_R and the set of annihilators in $M[x]_{R[x;\alpha,\delta]}$.

We give a sufficient condition for a module to be skew quasi-Armendariz and study the structure of the skew quasi-Armendariz modules.

By Hirano in [19], a ring R is called a quasi-Armendariz ring if, whenever $f(x)R[x]g(x) = 0$ where $f(x) = a_0 + a_1x + \cdots + a_mx^m \in R[x]$ and $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$, it implies that $a_iRb_j = 0$ for all i and j. Every semiprime ring is a quasi-Armendariz ring, by [19].

In [19], a module M_R is called a quasi-Armendariz module if whenever $m(x)R[x]f(x) = 0$, where $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]$ and $f(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$, it implies that $m_iRb_j = 0$ for all i and j.

Definition 3.1. Let M_R be a module, α an endomorphism of R and δ an α-derivation. We say M_R is *skew quasi-Armendariz*, if whenever $m(x) = \sum_{i=0}^{k} m_i x^i \in M[x]$, $f(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\alpha,\delta]$ satisfy $m(x)R[x;\alpha,\delta]f(x) = 0$, we have $m_ix^iRx^tb_jx^j = 0$ for $t \geq 0$, $i = 0, 1, \ldots, k$ and $j = 0, 1, \ldots, n$.

Theorem 3.2. *Let M_R be an α-compatible module and $S = R[x;\alpha]$. Then,*
(1) *The following statements are equivalent:*
(*a*) *for any $m(x) \in M[x]_S$, $(ann_S(m(x)S) \cap R)[x;\alpha] = ann_S(m(x)S)$.*
(*b*) *for any $m(x) = \sum_{i=0}^{k} m_i x^i \in M[x]_S$ and $f(x) = \sum_{j=0}^{t} a_j x^j \in S$, $m(x)Sf(x) = 0$ implies $m_i R a_j = 0$, for each i, j.*
(2) *Let M_R be an skew quasi-Armendariz module and $m(x) \in M[x]_S$. If $ann_S(m(x)S) \neq 0$, then $ann_S(m(x)S) \cap R \neq 0$.*

Proof. (1). $(a) \Rightarrow (b)$ Let $m(x) = \sum_{i=0}^{k} m_i x^i \in M[x]_S$, $f(x) = \sum_{j=0}^{t} a_j x^j$ $\in S$ and assume that $m(x)Sf(x) = 0$. By (a), $f(x) \in (ann_S(m(x)S) \cap R)[x;\alpha]$, and we deduce that $a_j \in ann_S(m(x)S) \cap R$ for each $0 \leq j \leq t$. So $m(x)Sa_j = 0$ and then by α-compatibility of M_R, we obtain $m_i R a_j = 0$ for each i, j.

$(b) \Rightarrow (a)$ Let $g(x) = \sum_{j=0}^{s} b_j x^j \in (ann_S(m(x)S) \cap R)[x;\alpha]$, so $b_j \in ann_S(m(x)S) \cap R$. So $m(x)Sb_j = 0$ for each j and hence $m(x)Sg(x) = 0$. Thus $g(x) \in ann_S(m(x)S)$. Now assume that $h(x) = \sum_{j=0}^{k} c_j x^j \in ann_S(m(x)S)$. So $m(x)Sh(x) = 0$ and by (b) we get $m_i R c_j = 0$. By α-compatibility of M_R, $m(x)Rc_j = 0$. So $c_j \in ann_S(m(x)S) \cap R$ for each j and hence $h(x) \in (ann_S(m(x)S) \cap R)[x;\alpha]$. So $ann_S(m(x)S) = (ann_S(m(x)S \cap R))[x;\alpha]$.
(2). The proof follows by Lemma 2.15 and (1) $(b) \Rightarrow (a)$. □

In the following result, we give relations between the set of annihilators in M_R and the set of annihilators in $M[x]_{R[x;\alpha]}$.

Theorem 3.3. *Let M_R be an α-compatible module and $S = R[x;\alpha]$. Then the following statements are equivalent:*
(1) *M_R is a skew quasi-Armendariz module;*
(2) *The map $\psi : Ann_R(sub(M_R)) \to Ann_S(sub(M[x]_S))$, defined by $\psi(ann_R(N)) = ann_S(N) = ann_S(N[x])$ for all $N \in sub(M_R)$, is bijective, where $sub(M_R)$ and $sub(M[x]_S)$ denote the sets of submodules.*

Proof. (1) $\Rightarrow$ (2) Assume that M_R is skew quasi-Armendariz. Obviously ψ is injective. Therefore, it is enough to show ψ is surjective. Let $V \in sub(M[x]_S)$ and C_V denotes the set of all coefficients of elements of V. Then for $ann_R(C_V R) \in Ann_R(sub(M))$, we have $\psi(ann_R(C_V R)) = ann_S(C_V R) = ann_S(V)$. In fact, let $f(x) \in ann_S(C_V R)$. Then $C_V R f(x) = 0$ and hence $Vf(x) = 0$. So $f(x) \in ann_S(V)$. Conversely, let $g(x) = b_0 + \cdots + b_k x^k \in ann_S(V)$. Then $Vg(x) = 0$. Since V is a submodule of $M[x]_S$, $VSg(x) = 0$. So $v(x)Sg(x) = 0$ for all $v(x) =$

$v_0 + v_1x + \cdots + v_lx^l \in V$. Since M_R is α-compatible and skew quasi-Armendariz, $v_iRb_j = 0$ for all i, j. Hence $C_VRg(x) = 0$ and therefore $g(x) \in ann_S(C_VR)$. Consequently ψ is surjective.

(2) $\Rightarrow$ (1) Assume $m(x)Sf(x) = 0$, where $m(x) = m_0 + m_1x + \cdots + m_tx^t \in M[x]$ and $f(x) = a_0 + a_1x + \cdots + a_kx^k \in S$. By hypothesis, $ann_S(m(x)S) = ann_R(N)[x;\alpha]$ for some submodule N of M. Then $f(x) \in ann_R(N)[x;\alpha]$ and hence $a_j \in ann_R(N)$ for all j. So $a_j \in ann_R(N) \subseteq ann_R(N)[x;\alpha] = ann_S(m(x)S)$ and then $m(x)Sa_j = 0$. In particular $m(x)Ra_j = 0$ and hence $m_iRa_j = 0$ for all i, j. Since M_R is α-compatible, $m_ix^iRx^ta_jx^j = 0$, for $t \geq 0$, $i = 0, 1, \ldots, t$ and $j = 0, 1, \ldots, k$. Therefore M_R is skew quasi-Armendariz . □

Let R be a ring. The trivial extension of R is given by:
$T(R,R) = \left\{ \begin{pmatrix} a & r \\ 0 & a \end{pmatrix} \mid a, r \in R \right\}$. Clearly, $T(R,R)$ is a subring of the ring of 2×2 matrices over R. The endomorphism α of R and the α-derivation δ on R are extended to $\bar{\alpha} : T(R,R) \to T(R,R)$ by $\bar{\alpha}\left(\begin{pmatrix} a & r \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} \alpha(a) & \alpha(r) \\ 0 & \alpha(a) \end{pmatrix}$, $\bar{\delta}\left(\begin{pmatrix} a & r \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} \delta(a) & \delta(r) \\ 0 & \delta(a) \end{pmatrix}$. One can show that $\bar{\delta}$ is an $\bar{\alpha}$-derivation on $T(R,R)$ and also we can see $T(R,R)[x;\bar{\alpha},\bar{\delta}] \cong T(R[x;\alpha,\delta], R[x;\alpha,\delta])$.

Proposition 3.4. *If the trivial extension of R, $T(R,R)$, is skew-quasi Armendariz, then so is R.*

Proof. Let $f(x) = a_0 + \cdots + a_nx^n, g(x) = b_0 + \cdots + b_mx^m \in R[x;\alpha,\delta]$ and $f(x)R[x;\alpha,\delta]g(x) = 0$. For each $a, r \in R$ and $t \geq 0$, we have the following equation:
$$0 = \begin{pmatrix} f(x) & 0 \\ 0 & f(x) \end{pmatrix} \begin{pmatrix} ax^t & rx^t \\ 0 & ax^t \end{pmatrix} \begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & f(x)ax^tg(x) \\ 0 & 0 \end{pmatrix}.$$
Since $T(R,R)$ is skew quasi-Armendariz, it implies that $a_ix^iax^tb_jx^j = 0$, for each i, j, t. Therefore R is skew quasi-Armendariz. □

When the trivial extension $T(R,R)$ is skew quasi-Armendariz?

Theorem 3.5. *Let R be a ring such that*

(i) R is skew quasi-Armendariz;

(ii) If $f(x)R[x;\alpha,\delta]g(x) = 0$, then $f(x)R[x;\alpha,\delta] \cap R[x;\alpha,\delta]g(x) = 0$.

Then the trivial extension $T = T(R,R)$ is skew quasi-Armendariz.

Proof. Suppose that $\alpha(x)T[x;\bar{\alpha},\bar{\delta}]\beta(x) = 0$, where

$\alpha(x) = \begin{pmatrix} a_0 & r_0 \\ 0 & a_0 \end{pmatrix} + \begin{pmatrix} a_1 & r_1 \\ 0 & a_1 \end{pmatrix} x + \cdots + \begin{pmatrix} a_n & r_n \\ 0 & a_n \end{pmatrix} x^n$ and
$\beta(x) = \begin{pmatrix} b_0 & s_0 \\ 0 & b_0 \end{pmatrix} + \begin{pmatrix} b_1 & s_1 \\ 0 & b_1 \end{pmatrix} x + \cdots + \begin{pmatrix} b_m & s_m \\ 0 & b_m \end{pmatrix} x^m \in T[x;\bar{\alpha},\bar{\delta}]$.
Let $f(x) = a_0 + a_1x + \cdots + a_nx^n, r(x) = r_0 + r_1x + \cdots + r_nx^n$,
$g(x) = b_0+b_1x+\cdots+b_mx^m$ and $s(x) = s_0+s_1x+\cdots+s_mx^m \in R[x;\alpha,\delta]$.
For each $\begin{pmatrix} a & r \\ 0 & a \end{pmatrix} x^t \in T[x;\bar{\alpha},\bar{\delta}]$, it follows that
$0 = \begin{pmatrix} f(x) & r(x) \\ 0 & f(x) \end{pmatrix} \begin{pmatrix} ax^t & rx^t \\ 0 & ax^t \end{pmatrix} \begin{pmatrix} g(x) & s(x) \\ 0 & g(x) \end{pmatrix} =$
$\begin{pmatrix} f(x)ax^tg(x) & f(x)ax^ts(x) + f(x)rx^tg(x) + r(x)ax^tg(x) \\ 0 & f(x)ax^tg(x) \end{pmatrix}$. Hence

$$f(x)ax^tg(x) = 0, \tag{3.1}$$

and

$$f(x)ax^ts(x) + f(x)rx^tg(x) + r(x)ax^tg(x) = 0. \tag{3.2}$$

Since $\begin{pmatrix} a & r \\ 0 & a \end{pmatrix} x^t$ is an arbitrary element of $T(R,R)[x;\bar{\alpha},\bar{\delta}]$ and $T(R,R)[x;\bar{\alpha},\bar{\delta}] \cong T(R[x;\alpha,\delta], R[x;\alpha,\delta])$, by (3.1) we get

$$f(x)R[x;\alpha,\delta]g(x) = 0. \tag{3.3}$$

Since R is skew quasi-Armendariz, $a_ix^iRx^tb_jx^j = 0$, for all i,j,t. Thus by (3.2), $f(x)[ax^ts(x) + rx^tg(x)] + [r(x)ax^t]g(x) = 0$. Hence by (3.2) and (3.3), we have
$f(x)[ax^ts(x)+rx^tg(x)] = -[r(x)ax^t]g(x) \in f(x)R[x;\alpha,\delta] \cap R[x;\alpha,\delta]g(x) = 0$. So $f(x)[ax^ts(x) + rx^tg(x)] = 0 = r(x)ax^tg(x)$, and hence we have $r(x)R[x;\alpha,\delta]g(x) = 0$, since ax^t is an arbitrary element. Thus $r_ix^iRx^tb_jx^j = 0$ for all i,j,t, since R is skew quasi-Armendariz. Also we have $f(x)[ax^ts(x)] = -[f(x)rx^t]g(x) \in f(x)R[x;\alpha,\delta] \cap R[x;\alpha,\delta]g(x) = 0$. Thus $f(x)ax^ts(x) = 0$. So we have $f(x)R[x;\alpha,\delta]s(x) = 0$. Since R is skew quasi-Armendariz, we deduce $a_ix^iRx^ts_jx^j = 0$ for all i,j,t. Hence
$\begin{pmatrix} a_i & r_i \\ 0 & a_i \end{pmatrix} x^i \begin{pmatrix} a & r \\ 0 & a \end{pmatrix} x^t \begin{pmatrix} b_j & s_j \\ 0 & b_j \end{pmatrix} x^j =$

$\begin{pmatrix} a_ix^iax^tb_jx^j & a_ix^irx^tb_jx^j + a_ix^irx^tb_jx^j + r_ix^iax^tb_jx^j \\ 0 & a_ix^iax^tb_jx^j \end{pmatrix} = 0$ for all i, j and each $\begin{pmatrix} a & r \\ 0 & a \end{pmatrix} x^t \in T(R,R)$. Therefore the trivial extension $T(R,R)$ is skew quasi-Armendariz. □

Kerr [24] constructed an example of a commutative Goldie ring R whose polynomial ring $R[x]$ has an infinite ascending chain of annihilator ideals.

Theorem 3.6. *Let M_R be an skew quasi-Armendariz module. If M_R is (α, δ)-compatible, then M_R satisfies the ascending chain condition on annihilator of submodules if and only if so does $M[x]_S$, where $S = R[x; \alpha, \delta]$.*

Proof. Assume that M_R satisfies the ascending chain condition on annihilator of submodules. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ be a chain of annihilator of submodules of $M[x]_S$. Then there exist submodules K_i of $M[x]_S$ such that $ann_S(K_i) = I_i$, for all $i \geq 1$ and $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$. Let $M_i = \{\text{all coefficients of elements of} \, K_i\}$. Since M is skew quasi-Armendariz, M_i is submodule of M for all $i \geq 1$. Clearly $M_i \supseteq M_{i+1}$ for all $i \geq 1$. Thus $ann_R(M_1) \subseteq ann_R(M_2) \subseteq ann_R(M_3) \subseteq \cdots$. Since M_R satisfies the ascending chain condition on annihilator of submodules, there exists $n \geq 1$ such that $ann_R(M_i) = ann_R(M_n)$ for all $i \geq n$. We show that $ann_S(K_i) = ann_S(K_n)$ for all $i \geq n$. Let $f(x) = a_0 + a_1x + \cdots + a_mx^m \in ann_S(K_i)$. Then $M_ia_j = 0$ for $j = 0, \dots, m$, because M is skew quasi-Armendariz. Thus $M_na_j = 0$ for $j = 0, \dots, m$ and so $K_nf(x) = 0$ by Lemma 2.16. Therefore $ann_S(K_i) = ann_S(K_n)$ for all $i \geq n$ and $M[x]_S$ satisfies the ascending chain condition on annihilator of submodules. Now assume $M[x]_S$ satisfies the ascending chain condition on annihilator of submodules. Let $J_1 \subseteq J_2 \subseteq J_3 \subseteq \dots$ be a chain of annihilator of submodules of M_R. Then there exist submodules M_i of M such that $ann_R(M_i) = J_i$ and $M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$ for all $i \geq 1$. Hence $M_i[x]$ is a submodule of $M[x]$ and $M_i[x] \supseteq M_{i+1}[x]$ and $ann_S(M_i[x]) \subseteq ann_S(M_{i+1}[x])$ for all $i \geq 1$. Since $M[x]_S$ satisfies the ascending chain condition on annihilator of submodules, there exists $n \geq 1$ such that $ann_S(M_i[x]) = ann_S(M_n[x])$ for all $i \geq n$. Since M is skew quasi-Armendariz, by a similar argument as used in the previous paragraph, one can show that $ann_R(M_i) = ann_R(M_n)$ for all $i \geq n$.

□

Following [3], the second author and E. Hashemi [17] introduced (α, δ)-compatible rings and studied its properties. A ring R is α-compatible if for each $a, b \in R$, $ab = 0$ if and only if $a\alpha(b) = 0$. Moreover, R is said to be δ-compatible if for each $a, b \in R$, $ab = 0$ implies $a\delta(b) = 0$. A ring R is (α, δ)-compatible if it is both α-compatible and δ-compatible. In this case, clearly the endomorphism α is injective. Also by [17, Lemma 2.2], a ring R is (α, δ)-compatible and reduced if and only if R is α-rigid in the sense of Krempa [26]. Thus the α-compatible ring is a generalization of α-rigid ring to the more general case where R is not assumed to be reduced.

Corollary 3.7. *Let R be an (α, δ)-compatible and skew quasi-Armendariz ring. Then R satisfies the ascending chain condition on right annihilators if and only if so does $R[x; \alpha, \delta]$.*

Corollary 3.8. [19, Corollary 3.3] *Let R be an Armendariz ring. Then R satisfies the ascending chain condition on right annihilators if and only if so does $R[x]$.*

Theorem 3.9. *Let M_R be an (α, δ)-compatible module. Then M_R is quasi-Baer (respectively, p.q.-Baer) if and only if $M[x]_{R[x;\alpha,\delta]}$ is quasi-Baer (respectively, p.q.-Baer). In this case M_R is skew quasi-Armendariz.*

Proof. Assume M_R is quasi-Baer. First we shall prove that M_R is skew quasi-Armendariz. Suppose that $(m_0 + m_1x + \cdots + m_kx^k)R[x; \alpha, \delta](b_0 + b_1x + \cdots + b_nx^n) = 0$, with $m_i \in M, b_j \in R$. In particular case we have

$$(3.4) \quad (m_0 + m_1x + \cdots + m_kx^k)R(b_0 + b_1x + \cdots + b_nx^n) = 0.$$

Thus $m_kRb_n = 0$ and $b_n \in ann_R(m_kR)$. Then $m_kx^kRx^tb_nx^n = 0$, by Lemma 2.15. Since M_R is quasi-Baer, there exists $e_k^2 = e_k \in R$ such that $ann_R(m_kR) = e_kR$ and so $b_n = e_kb_n$. Replacing R by Re_k in (3.4) and using Lemma 2.15, we obtain $(m_0 + m_1x + \cdots + m_{k-1}x^{k-1})Re_k(b_0 + b_1x + \cdots + b_nx^n) = 0$. Hence $m_{k-1}Re_kb_n = m_{k-1}Rb_n = 0$ and $b_n \in ann_R(m_{k-1}R)$. Then $m_{k-1}x^{k-1}Rx^tb_nx^n = 0$, by Lemma 2.15. Hence $b_n \in ann_R(m_kR) \cap ann_R(m_{k-1}R)$. Since M_R is quasi-Baer, there exists $f^2 = f \in R$ such that $ann_R(m_kR) = fR$ and so $b_n = fb_n$. If we put $e_{k-1} = e_kf$, then $e_{k-1}b_n = e_kfb_n = e_kb_n = b_n$ and $e_{k-1} \in ann_R(m_kR) \cap ann_R(m_{k-1}R)$. Next, replacing R by Re_{k-1} in (3.4), and using Lemma 2.15, we obtain $(m_0 + m_1x + \cdots + m_{k-2}x^{k-2})Re_{k-1}(b_0 +$

$b_1x + \cdots + b_nx^n) = 0$. Hence we have $m_{k-2}Re_{k-1}b_n = m_{k-2}Rb_n = 0$ and that $b_n \in ann_R(m_{k-2}R)$ and so $m_{k-2}x^{k-2}Rx^tb_nx^n = 0$, by Lemma 2.15. Continuing this process, we get $m_ix^iRx^tb_nx^n = 0$ for $i = 0, \ldots, k$. Using induction on $k+n$, we obtain $m_ix^iRx^tb_jx^j = 0$ for all i, j, t. Therefore M_R is skew quasi-Armendariz. Let J be a S-submodule of $M[x]$. Let $N = \{m \in M \mid m$ is a leading coefficient of some non-zero element of J$\}$ $\cup \{0\}$. Clearly, N is a submodule of M. Since M_R is quasi-Baer, there exists $e^2 = e \in R$ such that $ann_R(N) = eR$. Hence $eS \subseteq ann_S(J)$ by Lemma 2.15. Let $f(x) = b_0 + b_1x + \cdots + b_nx^n \in ann_S(J)$. Then $Nb_j = 0$ for each $j = 0, \ldots, n$, because M_R is skew quasi-Armendariz. Hence $b_j = eb_j$ for each $j = 0, \ldots, n$ and $f(x) = ef(x) \in eS$. Thus $ann_S(J) = eS$ and $M[x]_S$ is quasi-Baer. Now assume that $M[x]_S$ is quasi-Baer and I is a submodule of M. Then $I[x]$ is a submodule of $M[x]$. Since $M[x]$ is quasi-Baer, there exists an idempotent $e(x) = e_0 + \cdots + e_nx^n \in S$ such that $ann_S(I[x]) = e(x)S$. Hence $Ie_0 = 0$ and $e_0R \subseteq ann_R(I)$. Let $t \in ann_R(I)$. Then $I[x]t = 0$, by Lemma 2.16. Hence $t = e(x)t$ and so $t = e_0t \in e_0R$. Thus $ann_R(I) = e_0R$ and M_R is quasi-Baer.

□

It is clear that R is a right p.q.-Baer ring if and only if R_R is a p.q.-Baer module. But, there exists a p.q.-Baer right R-module such that R is not right p.q.-Baer.

Example 3.10. *Let $R = Z_2[x]/(x^2)$, where $Z_2[x]$ is the polynomial ring over the field Z_2 of two elements and (x^2) is the ideal of $Z_2[x]$ generated by x^2. It is easy to see that R is a quasi-Armendariz ring. Since right annihilator of $x + (x^2)$ is not generated by any idempotent, R is not a right p.q.-Baer ring. Now let $e = 1 + (x^2)$ and $I = ReR$. Then $e^2 = e$, and for each $a \in R$, $ann_R((a + I)R) = eR$. Therefore R/I is p.q.-Baer right R-module.*

Corollary 3.11. [17, Corollary 2.8] *Let R be an (α, δ)-compatible ring. Then R is quasi-Baer (respectively, right p.q.-Baer) if and only if $R[x; \alpha, \delta]$ is quasi-Baer (respectively, right p.q.-Baer). In this case R is a skew quasi-Armendariz ring.*

Corollary 3.12. [9, Corollary 2.8] *A ring R is quasi-Baer (respectively, right p.q.-Baer) if and only if $R[x]$ is quasi-Baer (respectively, right p.q.-Baer).*

Corollary 3.13. [20, Theorems 12, 15] *Let R be an α-rigid ring. Then R is quasi-Baer (respectively, right p.q.-Baer) if and only if $R[x;\alpha,\delta]$ is quasi-Baer (respectively, right p.q.-Baer).*

The following example shows that "(α,δ)-compatibility condition" on M_R in Theorem 3.9 is not superfluous.

Example 3.14. [5, Example 11] There is a ring R and a derivation δ of R such that $R[x;\delta]$ is a Baer (hence quasi-Baer) ring, but R is not quasi-Baer. In fact let $R=\mathbb{Z}_2[t]/(t^2)$ with the derivation δ such that $\delta(\bar{t})=1$ where $\bar{t}=t+(t^2)$ in R and $\mathbb{Z}_2[t]$ is the polynomial ring over the field $\mathbb{Z}_2$ of two elements. Consider the Ore extension $R[x;\delta]$. If we set $e_{11}=\bar{t}x, e_{12}=\bar{t}, e_{21}=\bar{t}x^2+x$, and $e_{22}=1+\bar{t}x$ in $R[x;\delta]$, then they form a system of matrix units in $R[x;\delta]$. Now the centralizer of these matrix units in $R[x;\delta]$ is $\mathbb{Z}_2[x^2]$. Therefore $R[x;\delta]\cong M_2(\mathbb{Z}_2[x^2])\cong M_2(\mathbb{Z}_2)[y]$, where $M_2(\mathbb{Z}_2)[y]$ is the polynomial ring over $M_2(\mathbb{Z}_2)$. So the ring $R[x;\delta]$ is a Baer ring, but R is not quasi-Baer.

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Abdollah Alhevaz
Department of Pure Mathematics, Faculty of Mathematical Sciences,
Tarbiat Modares University, P.O. Box 14115-134, Tehran, Iran
Email: a.alhevaz@yahoo.com and a.alhevaz@gmail.com.

Ahmad Moussavi
Department of Pure Mathematics, Faculty of Mathematical Sciences,
Tarbiat Modares University, P.O. Box 14115-134, Tehran, Iran
Email: moussavi.a@modares.ac.ir and moussavi.a@gmail.com.