Bulletin of the Iranian Mathematical Society Vol. 38 *No. 1* (*2012*)*, pp 101-112.*

ON A DECOMPOSITION OF HARDY–HILBERT'S TYPE INEQUALITY

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Communicated by Michel Waldschmidt

ABSTRACT. In this paper, two pairs of new inequalities are given, which decompose two Hilbert-type inequalities.

1. **Introduction**

In 1908, H. Weyl [3] published the following Hilbert inequality : If $\{a_n\}$, $\{b_n\}$ are real sequences, $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n b_m}{m+n} < \pi\Big(\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2\Big)^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor π is the best possible. In 1925, G. H. Hardy [1] extended (1.1) as : If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n \geq 0$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n b_m}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})}\Big(\sum_{n=1}^{\infty} a_n^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{\infty} b_n^q\Big)^{\frac{1}{q}}, \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible. We refer to (1.2) as the Hardy-Hilbert inequality. In 2005, Yang [5] gave an extension of(1.2)

MSC(2010): Primary: 26D15; Secondary: 26D10.
Keywords: Hilbert's inequality, Hilbert-type inequality, integral inequality.
Received: 23 May 2010, Accepted: 7 September 2010.
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with a pair of conjugate exponents $(p,q)(p>1)$ and a parameter $\lambda>0$ as: suppose that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\phi_r>0(r=p,q)$, $\phi_p+\phi_q=\lambda$ and u is a differentiable strictly increasing function on (a,b) $(-\infty\le a<b\le\infty)$, such that $u(a^+)=0$ and $u(b^-)=\infty$ and also $f,g\ge 0$ satisfy

$$0<\int_a^b (u(x))^{p(1-\phi_q)-1}(u'(x))^{1-p}f^p(x)\,dx<\infty,$$

$$0<\int_a^b (u(y))^{q(1-\phi_p)-1}(u'(y))^{1-q}g^q(y)\,dy<\infty.$$

Then

$$\int_a^b\int_a^b \frac{f(x)g(y)}{(u(x)+u(y))^\lambda}\,dxdy < K\Big(\int_a^b u(x)^{p(1-\phi_q)-1}(u'(x))^{1-p}f^p(x)\,dx\Big)^{\frac{1}{p}} \times\Big(\int_a^b (u(y))^{q(1-\phi_p)-1}(u'(y))^{1-q}g^q(y)\,dy\Big)^{\frac{1}{q}}, \tag{1.3}$$

where the constant factor $K=\beta(\phi_p,\phi_q)$, is the best possible. For $0<p<1$ with $\{\lambda:\phi_r>0\ (r=p,q),\phi_p+\phi_q=\lambda\}\ne\emptyset$, inequality (1.3) is reversed and the constant factor is still the best possible.

There are some kind of Hilbert-type inequalities. For instance, Dongmel Xin in [4] gave the following statement:

If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, $\lambda>0$ and $f,g\ge 0$ such that

$$0<\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)\,dx<\infty,$$

$$0<\int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)\,dy<\infty.$$

Then we have

$$\int_0^\infty\int_0^\infty \frac{\ln(\frac{x}{y})f(x)g(y)}{x^\lambda-y^\lambda}\,dxdy < \Big[\frac{\pi}{\lambda\sin(\frac{\pi}{r})}\Big]^2\Big(\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)\,dx\Big)^{\frac{1}{p}} \times\Big(\int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)\,dy\Big)^{\frac{1}{q}}, \tag{1.4}$$

where the constant factor is the best possible.

Recently, Yang [6] by the identity

$$\frac{1}{m+n} = \frac{\max\{m,n\}}{(m+n)^2} + \frac{\min\{m,n\}}{(m+n)^2} \qquad (m,n \in \mathbb{N})$$

gave a decomposition of Hilbert's inequality as follows

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\max\{m,n\}}{(m+n)^2} a_m b_n < (\frac{\pi}{2}+1)\Big(\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2\Big)^{\frac{1}{2}},$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\min\{m,n\}}{(m+n)^2} a_m b_n < (\frac{\pi}{2}-1)\Big(\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2\Big)^{\frac{1}{2}}.$$

The sum of two best constant factors is π (the constant factor of Hilbert's inequality).

2. **Main results**

In this section, by the following identities we give two pairs of new Hilbert-type inequalities which decompose inequalities (1.3) and (1.4), respectively:

$$\frac{1}{(x+y)^\lambda} = \frac{x}{(x+y)^{\lambda+1}} + \frac{y}{(x+y)^{\lambda+1}},$$

$$\frac{\ln(\frac{x}{y})}{x^\lambda - y^\lambda} = \frac{x^\lambda \ln(\frac{x}{y})}{x^{2\lambda} - y^{2\lambda}} + \frac{y^\lambda \ln(\frac{x}{y})}{x^{2\lambda} - y^{2\lambda}}.$$

At first, by using the idea of Lemma 2.3 at [2] one can easily prove the following Lemma.

Lemma 2.1. *Let $0 \le ps < 1$ and $0 \le sq < 2$ and $\lambda > 2 - \min\{p,q\}$. Define a function Φ by*

$$\Phi(s) = \Big(\beta(\lambda+ps, 1-ps)\Big)^{\frac{1}{p}} \Big(\beta(\lambda-(1-qs), 2-qs)\Big)^{\frac{1}{q}},$$

where $\beta(m,n)$ is beta function. Then $\Phi(s)$ attains its minimum at $s = \frac{2-\lambda}{pq}$.

Theorem 2.2. *Assume that* $p>1$, $\lambda>2-\min\{p,q\}$, $\frac{1}{p}+\frac{1}{q}=1$ *and* u,v *are two strict increasing differentiable functions such that* $u(0)=v(0)=0$, $u(\infty)=v(\infty)=\infty$,

$$0<\int_0^\infty (u(x))^{1-\lambda}(u'(x))^{1-p}f^p(x)\,dx<\infty,$$

$$0<\int_0^\infty (v(y))^{1-\lambda}(v'(y))^{1-q}g^q(y)\,dy<\infty.$$

Then

$$\int_0^\infty\int_0^\infty \frac{u(x)}{(u(x)+v(y))^{1+\lambda}}f(x)g(y)\,dxdy < K_1(\lambda)\Big(\int_0^\infty (u(x))^{1-\lambda}(u'(x))^{1-p}f^p(x)\,dx\Big)^{\frac{1}{p}} \times\Big(\int_0^\infty (v(y))^{1-\lambda}(v'(y))^{1-q}g^q(y)\,dy\Big)^{\frac{1}{q}},$$

where $\phi_r(\lambda)=1-\frac{2-\lambda}{r}$ *and* $K_1(\lambda)=\frac{\phi_p(\lambda)}{\lambda}\beta(\phi_p(\lambda),\phi_q(\lambda))$. *The constant factor is the best possible.*

Proof. Put $f(x)=F(x)(u'(x))^{\frac{1}{q}}$ and $g(y)=G(y)(v'(y))^{\frac{1}{p}}$. Then

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{u(x)}{(u(x)+v(y))^{\lambda+1}}f(x)g(y)\,dxdy\\
&=\int_0^\infty\int_0^\infty \frac{u(x)F(x)(u'(x))^{\frac{1}{q}}G(y)(v'(y))^{\frac{1}{p}}}{(u(x)+v(y))^{\lambda+1}}\,dxdy\\
&=\int_0^\infty\int_0^\infty \frac{F(x)(u(x)v'(y))^{\frac{1}{p}}}{(u(x)+v(y))^{\frac{\lambda+1}{p}}}\Big(\frac{u(x)}{v(y)}\Big)^s\\
&\quad\times\frac{G(y)(u(x)u'(x))^{\frac{1}{q}}}{(u(x)+v(y))^{\frac{\lambda+1}{q}}}\Big(\frac{v(y)}{u(x)}\Big)^s\,dxdy\\
&\le\Big(\int_0^\infty\int_0^\infty \frac{F^p(x)u(x)v'(y)}{(u(x)+v(y))^{\lambda+1}}\Big(\frac{u(x)}{v(y)}\Big)^{sp}\,dxdy\Big)^{\frac{1}{p}}\\
&\quad\times\Big(\int_0^\infty\int_0^\infty \frac{G^q(y)u(x)u'(x)}{(u(x)+v(y))^{\lambda+1}}\Big(\frac{v(y)}{u(x)}\Big)^{sq}\,dxdy\Big)^{\frac{1}{q}}\\
&=M^{\frac{1}{p}}N^{\frac{1}{q}},
\end{aligned}$$

where

$$M = \int_0^\infty \int_0^\infty \frac{F^p(x)u(x)v'(y)}{u^{\lambda+1}(x)(1+\frac{v(y)}{u(x)})^{\lambda+1}} \Big(\frac{u(x)}{v(y)}\Big)^{sp} \, dxdy.$$

By substituting $t = \frac{v(y)}{u(x)}$, one obtains

$$\begin{aligned} M &= \Big(\int_0^\infty F^p(x)u^{1-\lambda}(x)\,dx\Big)\int_0^\infty \frac{t^{-ps}}{(1+t)^{\lambda+1}}\,dt \\ &= \beta(1-ps,\lambda+ps)\int_0^\infty (u(x))^{1-\lambda}(u'(x))^{1-p}f^p(x)\,dx, \end{aligned}$$

providing $1-ps>0$ and $\lambda+ps>0$. Similarly,

$$N = \beta(2-sq,\lambda-1+sq)\int_0^\infty (v(y))^{1-\lambda}(v'(y))^{1-q}g^q(y)\,dy,$$

providing $2-qs>0$ and $\lambda-1+qs>0$. So

$$\begin{aligned} &\int_0^\infty\int_0^\infty \frac{u(x)}{(u(x)+v(y))^{\lambda+1}} f(x)g(y)\,dxdy \\ &\qquad \le K\Big(\int_0^\infty (u(x))^{1-\lambda}(u'(x))^{1-p}f^p(x)\,dx\Big)^{\frac{1}{p}} \\ &\qquad\quad \times\Big(\int_0^\infty (v(y))^{1-\lambda}(v'(y))^{1-q}f^p(y)\,dy\Big)^{\frac{1}{q}}, \end{aligned}$$

where

$$K = \beta^{\frac{1}{p}}(1-ps\ ,\lambda+ps)\beta^{\frac{1}{q}}(2-sq\ ,\lambda-1+sq).$$

We should choose the parameter s such that

$$\begin{cases} 1-ps>0 \\ \lambda+ps>0 \end{cases} \qquad \text{and} \qquad \begin{cases} 2-qs>0 \\ \lambda-1+qs>0; \end{cases}$$

By Lemma 2.1 K attains its minimum at $s = \frac{2-\lambda}{pq}$. In this case,

$$K = \Big(\beta(1-\frac{2-\lambda}{q}\ ,\ \lambda+\frac{2-\lambda}{q})\Big)^{\frac{1}{p}}\Big(\beta(2-\frac{2-\lambda}{p}\ ,\ \lambda-(1-\frac{2-\lambda}{p.})\Big)^{\frac{1}{q}}.$$

So by the identity $1-\frac{2-\lambda}{q} = \lambda-(1-\frac{2-\lambda}{p})$, we have

$$K(\lambda) = \frac{\phi_p(\lambda)}{\lambda}\beta(\phi_p(\lambda),\phi_q(\lambda)).$$

If the inequality mentioned in Theorem 2.2 takes the form of equality, then there exist constants c_1, c_2 such that ${c_1}^2+{c_2}^2\neq 0$ and

$$c_1 f^p(x)(u'(x))^{-p}(u(x))^{2-\lambda}=c_2 g^q(y)(v'(y))^{-q}(v(y))^{2-\lambda}=c$$

almost everywhere on $(0,\infty)\times(0,\infty)$, where c is constant. Without loss of generality, suppose that $c_1\neq 0$, then one has $f^p(x)=\frac{c}{c_1}(u'(x))^p(u(x))^{\lambda-2}$, almost everywhere on $(0,\infty)$.

Now, we have

$$\int_0^\infty (u(x))^{1-\lambda}(u'(x))^{1-p}f^p(x)\,dx=\frac{c}{c_1}\int_0^\infty \frac{du}{u},$$

which contradicts the fact that

$$0<\int_0^\infty (u(x))^{1-\lambda}(u'(x))^{1-p}f^p(x)\,dx<\infty.$$

If the constant factor is not the best possible, then there is a positive number K with $K<K(\lambda)$ such that

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(u(x)+v(y))^\lambda}\,dxdy<K\Big(\int_0^\infty (u(x))^{1-\lambda}(u'(x))^{1-p}f^p(x)\,dx\Big)^{\frac{1}{p}}$$
$$\times\Big(\int_0^\infty (v(y))^{1-\lambda}(v'(y))^{1-q}g^q(y)\,dy\Big)^{\frac{1}{q}}.$$

Assume that $0<\epsilon<q+\lambda-2$ and

$$f_\epsilon(x)=\begin{cases}0 & 0\le x<u^{-1}(1)\\ (u(x))^{-\frac{2+\epsilon-\lambda}{p}}u'(x) & x\ge u^{-1}(1)\end{cases}$$

and

$$g_\epsilon(y)=\begin{cases}0 & 0\le y<v^{-1}(1)\\ (v(y))^{-\frac{2+\epsilon-\lambda}{q}}v'(y) & y\ge v^{-1}(1)\end{cases}.$$

One can show that

$$\int_0^\infty (u(x))^{1-\lambda}(u'(x))^{1-p}f_\epsilon^p(x)\,dx=\int_0^\infty (v(y))^{1-\lambda}(v'(y))^{1-q}g_\epsilon^q(y)\,dy$$
$$=\frac{1}{\epsilon}.$$

On the other hand, we have

$$\int_0^\infty\int_0^\infty \frac{u(x)f_\epsilon(x)g_\epsilon(y)}{(u(x)+v(y))^{\lambda+1}}\,dxdy$$

$$= \int_{u^{-1}(1)}^{\infty} (u(x))^{-1-\epsilon} u'(x) \Big(\int_{\frac{1}{u(x)}}^{\infty} \frac{t^{-\frac{2+\epsilon-\lambda}{q}}}{(1+t)^{\lambda+1}} \, dt \Big) \, dx$$

$$> \frac{1}{\epsilon}(k(\lambda) + o(1)) - \int_{u^{-1}(1)}^{\infty} (u(x))^{-1-\epsilon} u'(x) \Big(\int_{0}^{\frac{1}{u(x)}} \frac{t^{-\frac{2+\epsilon-\lambda}{q}}}{t^{\lambda+1}} \, dt \Big) \, dx$$

$$= \frac{1}{\epsilon}(k(\lambda) + o(1)) - \frac{1}{(1-\lambda-\frac{2+\epsilon-\lambda}{q})(\epsilon-\lambda-\frac{2+\epsilon-\lambda}{q})}.$$

Hence, we deduce that $K > K(\lambda)$ as ϵ tends to zero. □

Similarly, one may prove the following theorem:

Theorem 2.3. *Assume that* $p > 1$, $\lambda > 2 - \min\{p, q\}$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* u, v *are two strict increasing differentiable functions such that* $u(0) = v(0) = 0$, $u(\infty) = v(\infty) = \infty$,

$$0 < \int_0^{\infty} (u(x))^{1-\lambda} (u'(x))^{1-p} f^p(x) \, dx < \infty$$

and

$$0 < \int_0^{\infty} (v(y))^{1-\lambda} (v'(y))^{1-q} g^q(y) \, dy < \infty.$$

Then

$$\int_0^{\infty} \int_0^{\infty} \frac{v(y)}{(u(x)+v(y))^{1+\lambda}} f(x) g(y) \, dx dy$$

$$< K_2(\lambda) \Big(\int_0^{\infty} (u(x))^{1-\lambda} (u'(x))^{1-p} f^p(x) \, dx \Big)^{\frac{1}{p}}$$

$$\times \Big(\int_0^{\infty} (v(y))^{1-\lambda} (v'(y))^{1-q} g^q(y) \, dy \Big)^{\frac{1}{q}},$$

where $\phi_r(\lambda) = 1 - \frac{2-\lambda}{r}$ *and* $K_2(\lambda) = \frac{\phi_q(\lambda)}{\lambda} \beta(\phi_p(\lambda), \phi_q(\lambda))$*. The constant factor is the best possible.*

Remark 2.4. *By* $\lambda = \phi_p(\lambda) + \phi_q(\lambda)$*, the sum of two best constant factors in Theorems 2.2 and 2.3 is* $\beta(\phi_p, \phi_q)$*, the best constant factor in inequality (1.3). So the above mentioned two theorems are decompositions of inequality (1.3) due to Yang.*

Theorem 2.5. *If* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>1$, $\frac{1}{r}+\frac{1}{s}=1$ *and* u,v *are two strictly increasing differentiable functions,* $u(0)=v(0)=0$, $u(\infty)=v(\infty)=\infty$, $\lambda>0$ *and* $f,g\geq 0$ *and*

$$\int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p}f^p(x)\,dx<\infty,$$

$$\int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q}g^q(y)\,dy<\infty.$$

Then

$$\int_0^\infty\int_0^\infty \frac{u^\lambda(x)\ln\left(\frac{u(x)}{v(y)}\right)}{u^{2\lambda}(x)-v^{2\lambda}(y)}f(x)g(y)\,dxdy < \Big[\frac{\pi}{2\lambda\sin(\pi(1-\frac{1}{2s}))}\Big]^2\Big(\int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p}f^p(x)\,dx\Big)^{\frac{1}{p}} \times\Big(\int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q}g^q(y)\,dy\Big)^{\frac{1}{q}}.$$

The constant factor is the best possible.

Proof. Let $f(x)=F(x)(u'(x))^{\frac{1}{q}}$ and $g(y)=G(y)(v'(y))^{\frac{1}{p}}$, then

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{u^\lambda(x)\ln\left(\frac{u(x)}{v(y)}\right)f(x)g(y)}{u^{2\lambda}(x)-v^{2\lambda}(y)}\,dxdy\\
&=\int_0^\infty\int_0^\infty \frac{u^\lambda(x)\ln\left(\frac{u(x)}{v(y)}\right)F(x)G(y)(u'(x))^{\frac{1}{q}}(v'(y))^{\frac{1}{p}}}{u^{2\lambda}(x)-v^{2\lambda}(y)}\,dxdy\\
&=\int_0^\infty\int_0^\infty\Big(\frac{u^\lambda\ln(\frac{u}{v})}{u^{2\lambda}-v^{2\lambda}}\Big)^{\frac{1}{p}}\times\frac{u^{\frac{(1-\frac{\lambda}{r})}{q}}}{v^{\frac{(1-\frac{\lambda}{s})}{p}}}F(x)(v')^{\frac{1}{p}}\\
&\quad\times\Big(\frac{u^\lambda\ln(\frac{u}{v})}{u^{2\lambda}-v^{2\lambda}}\Big)^{\frac{1}{q}}\times\frac{v^{\frac{(1-\frac{\lambda}{s})}{p}}}{u^{\frac{(1-\frac{\lambda}{r})}{q}}}G(y)(u')^{\frac{1}{q}}\,dxdy\\
&\leq\Big(\int_0^\infty\int_0^\infty\frac{u^\lambda\ln(\frac{u}{v})}{u^{2\lambda}-v^{2\lambda}}\frac{u^{(p-1)(1-\frac{\lambda}{r})}}{v^{(1-\frac{\lambda}{s})}}F^p(x)v'(y)\,dxdy\Big)^{\frac{1}{p}}\\
&\quad\times\Big(\int_0^\infty\int_0^\infty\frac{u^\lambda\ln(\frac{u}{v})}{u^{2\lambda}-v^{2\lambda}}\frac{v^{(q-1)(1-\frac{\lambda}{s})}}{u^{(1-\frac{\lambda}{r})}}G^q(y)u'(x)\,dxdy\Big)^{\frac{1}{q}}
\end{aligned}$$

$$= M^{\frac{1}{p}} N^{\frac{1}{q}}.$$

Note that

$$M = \int_0^\infty \Big(\int_0^\infty \frac{\ln(\frac{u}{v})}{u^\lambda\Big(1-\Big(\frac{v}{u}\Big)^{2\lambda}\Big)} \frac{u^{(p-1)(1-\frac{\lambda}{r})}}{v^{(1-\frac{\lambda}{s})}} v'(y)\,dy \Big) F^p(x)\,dx.$$

By substitution $t = \Big(\frac{v(y)}{u(x)}\Big)^{2\lambda}$, one obtains

$$M = \frac{1}{4\lambda^2} \int_0^\infty \Big(\int_0^\infty \frac{\ln(t)}{t^{1-\frac{1}{2s}}(t-1)}\,dt \Big) u^{p(1-\frac{\lambda}{r})-1} F^p(x)\,dx$$

$$= \Big[\frac{\pi}{2\lambda \sin(\pi(1-\frac{1}{2s}))}\Big]^2 \int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1} (u'(x))^{1-p} f^p(x)\,dx.$$

By the same way one obtains

$$N = \Big[\frac{\pi}{2\lambda \sin(\pi(1-\frac{1}{2s}))}\Big]^2 \int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1} (v'(y))^{1-q} g^q(y)\,dy.$$

If inequality mentioned in Theorem 2.5 takes the form of equality, then there exist constants c_1, c_2 such that ${c_1}^2 + {c_2}^2 \neq 0$ and

$$c_1 f^p(x)(u'(x))^{-p}(u(x))^{2-\lambda} = c_2 g^q(y)(v'(y))^{-q}(v(y))^{2-\lambda} = c$$

almost everywhere on $(0,\infty)\times(0,\infty)$, where c is constant. Without loss of generality, suppose that $c_1 \neq 0$, then one has $f^p(x) = \frac{c}{c_1}(u'(x))^p(u(x))^{\lambda-2}$, almost everywhere on $(0,\infty)$.

Now, we have

$$\int_0^\infty (u(x))^{1-\lambda}(u'(x))^{1-p} f^p(x)\,dx = \frac{c}{c_1}\int_0^\infty \frac{du}{u},$$

which contradicts

$$0 < \int_0^\infty (u(x))^{1-\lambda}(u'(x))^{1-p} f^p(x)\,dx < \infty.$$

If the constant factor is not the best possible, then there is a positive number K with $K < \Big[\frac{\pi}{2\lambda\sin(\pi(1-\frac{1}{2s}))}\Big]^2$ such that

$$\int_0^\infty \int_0^\infty \frac{u^\lambda \ln\Big(\frac{u(x)}{v(y)}\Big) f(x) g(y)}{u^{2\lambda}(x) - v^{2\lambda}(y)}\,dxdy$$

$$< K\Big(\int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p} f^p(x)\,dx\Big)^{\frac{1}{p}}$$

$$\times\Big(\int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q}g^q(y)\,dy\Big)^{\frac{1}{q}}.$$

Assume that $0<\epsilon<(\frac{s+1}{s})\lambda q$ and

$$f_\epsilon(x)=\begin{cases}0 & 0\le x<u^{-1}(1)\\ (u(x))^{\frac{\lambda}{r}-1-\frac{\epsilon}{p}}u'(x) & x\ge u^{-1}(1)\end{cases}$$

and

$$g_\epsilon(y)=\begin{cases}0 & 0\le y<v^{-1}(1)\\ (v(y))^{\frac{\lambda}{s}-1-\frac{\epsilon}{q}}v'(y) & y\ge v^{-1}(1)\end{cases}.$$

One can show that

$$\begin{aligned}&\int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p}f_\epsilon^p(x)\,dx\\ &=\int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q}g_\epsilon^q(y)\,dy\\ &\qquad e=\frac{1}{\epsilon}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{u^\lambda \ln\left(\frac{u(x)}{v(y)}\right)f_\epsilon(x)g_\epsilon(y)}{u^{2\lambda}(x)-v^{2\lambda}(y)}\,dxdy\\ &=\int_{u^{-1}(1)}^\infty \frac{(u(x))^{-1-\epsilon}u'(x)}{4\lambda^2}\Big(\int_{\frac{1}{u^{2\lambda}(x)}}^\infty \frac{\ln(z)}{z^\alpha(z-1)}\,dz\Big)\,dx\\ &>\frac{1}{\epsilon}\Big(\Big[\frac{\pi}{2\lambda\sin(\pi(1-\frac{1}{2s}))}\Big]^2+o(1)\Big)-\int_{u^{-1}(1)}^\infty \frac{(u(x))^{-1-\epsilon}u'(x)}{4\lambda^2}\\ &\qquad\times\Big(\int_0^{\frac{1}{u^{2\lambda}(x)}} z^{-\alpha}dz\Big)\,dx\\ &=\frac{1}{\epsilon}\Big(\Big[\frac{\pi}{2\lambda\sin(\pi(1-\frac{1}{2s}))}\Big]^2+o(1)\Big)-\frac{1}{4\lambda^2(1-\alpha)(\epsilon+2\lambda(1-\alpha))},\end{aligned}$$

where $\alpha=\frac{\epsilon}{2\lambda q}+1-\frac{1}{2s}$. Hence, we deduces that $K>\left[\frac{\pi}{2\lambda\sin(\pi(1-\frac{1}{2s}))}\right]^2$ as ϵ tends to zero. □

By the same manner, one may prove the following theorem:

Theorem 2.6. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$ *and* u, v *are two strictly increasing differentiable functions,* $u(0) = v(0) = 0$, $u(\infty) = v(\infty) = \infty$, $\lambda > 0$ *and* $f, g \geq 0$ *and*

$$\int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p} f^p(x)\,dx < \infty,$$

$$\int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q} g^q(y)\,dy < \infty.$$

Then

$$\int_0^\infty \int_0^\infty \frac{v^\lambda(y)\ln\left(\frac{u(x)}{v(y)}\right)}{u^{2\lambda}(x) - v^{2\lambda}(y)} f(x)g(y)\,dxdy < \left[\frac{\pi}{2\lambda\sin(\pi(1-\frac{1}{2r}))}\right]^2 \left(\int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p} f^p(x)\,dx\right)^{\frac{1}{p}} \times \left(\int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q} g^q(y)\,dy\right)^{\frac{1}{q}}.$$

The constant factor is the best possible.

Remark 2.7. *One may easily verify that*

$$\left[\frac{\pi}{2\lambda\sin(\pi(1-\frac{1}{2s}))}\right]^2 + \left[\frac{\pi}{2\lambda\sin(\pi(1-\frac{1}{2r}))}\right]^2 = \left[\frac{\pi}{\lambda\sin(\frac{\pi}{r})}\right]^2.$$

So the above mentioned two theorems are decompositions of the inequality (1.4).

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