BIVARIATE MEAN VALUE INTERPOLATION ON CIRCLES OF THE SAME RADIUS

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ABSTRACT. We consider bivariate mean-value interpolation problem, where the integrals over circles are interpolation data. In this case the problem is described over circles of the same radius and with centers are on a straight line and it is shown that in this case the interpolation is not correct.

1. Introduction

Denote by $\Pi_n^2 = \Pi_n(\mathbb{R}^2)$ the space of interpolation polynomials in 2 variables of total degree not exceeding n:

$$\Pi_n^2 = \{p(x,y) = \sum_{i+j \le n} a_{ij} x^i y^j : i,j \in Z_+\}.$$

Set

$$N = \dim\Pi_n^2 = \binom{n+2}{2}.$$

Let us fix the set of distinct points

$$\mathcal{X}_s = \{(x_1, y_1), \ldots, (x_s, y_s)\} \subset \mathbb{R}^2$$

as the set of interpolation nodes.

The classic Lagrange interpolation problem $(\Pi_n^2, \mathcal{X}_s)$ is to find a unique polynomial $p \in \Pi_n^2$ such that

(1.1) $$p(x_k, y_k) = c_k, k = 1, \ldots, s,$$

where $c_k, k = 1, \ldots, s$ are real numbers.

Definition 1.1. *The Lagrange interpolation problem $(\Pi_n^2, \mathcal{X}_s)$ is called correct if, for any real values $c_k, k = 1, \ldots, s$, there exists a unique polynomial $p \in \Pi_n^2$ satisfying the conditions* (1.1).

In other words, the Lagrange interpolation problem is to find a unique polynomial $p(x, y) = \sum_{i+j \leq n} a_{ij} x^i y^j \in \Pi_n^2$ which reduces the conditions (1.1) to the following linear system

(1.2) $$p(x_k, y_k) = \sum_{i+j \leq n} a_{ij} x_k^i y_k^j = c_k, k = 1, ..., s.$$

The correctness of interpolation means that the linear system (1.2) has a unique solution for arbitrary right hand side values. A necessary condition for this is that the number of unknowns is equal to the number of equations, i.e.,

$$s = N.$$

We know that in this case the linear system (1.2) has a unique solution for arbitrary values $\{c_1, \ldots, c_s\}$ if and only if the corresponding homogeneous system has only trivial solution. In other words we have:

Proposition 1.2. *The interpolation problem $(\Pi_n^2, \mathcal{X}_N)$ is correct if and only if*

$$p \in \Pi_n^2, p \mid_{\mathcal{X}_N} = 0 \Rightarrow p = 0.$$

Equivalently: The interpolation problem $(\Pi_n^2, \mathcal{X}_N)$ is not correct if and only if

(1.3) $$\exists p \in \Pi_n^2 \backslash \{0\} \text{ such that } p \mid_{\mathcal{X}_N} = 0.$$

In this paper a mean-value interpolation problem is considered where interpolation parameters are integrals over circles. Here we are going to find a unique polynomial $p \in \Pi_n^2$ such thatt

(1.4) $$\frac{1}{\mu_2(D_k)} \int \int_{D_k} p(x, y) dx dy = c_k, \ k = 1, \ldots, N,$$

where c_k's are arbitrary given numbers, D_k's are circles and $\mu_2(D_k)$ is the area of D_k. We denote this mean-value interpolation problem by

$(\Pi_n^2, \mathbb{D})^{m.v.}$, where $\mathbb{D}$ is the set of above circles:

$$\mathbb{D} = \{D_k : k = 1, \ldots, N\}.$$

Similar to Definition 1.1 we call the problem $(\Pi_n^2, \mathbb{D})^{m.v.}$ correct if for any number $c_k, k = 1, \ldots, N$ there exists a unique polynomial $p \in \Pi_n^2$ satisfying (1.4).

Note that for a Lebesgue integrable function f it is convenient to use this interpolation, while the Lagrange interpolation in this case is not suitable since values of f may not be determined. It is worth mentioning that even the mean-value interpolations where the interpolation parameters are obtained through integration over sets of $n-$dimensional Lebesgue measure zero, are not appropriate for integrable functions. (see, [1, p. 204], [2])

An example of correct interpolation problem is presented in [3]. To introduce it we need some preliminaries.

Definition 1.3. *We call a set of lines in the plane to be in general position if any two lines intersect at a point and no three lines are coincident.*

For a set of lines in general position we call cut-regions the bounded regions cut by the given set of lines.

As it is shown in [3] there are exactly N cut-regions if the lines $L_1, \ldots, L_{n+3}$ are in general position, where $n \geq 0$.

In [3] we consider the following conjecture:

Conjecture 1.1. *Suppose that the lines $L_1, ..., L_{n+3}$ are in general position. Then, the mean-value interpolation with Π_n^2 and N cut-regions is correct, i.e., for any $c_1, \ldots, c_N$ there exists a unique polynomial $p \in \Pi_n^2$ such that*

$$\frac{1}{\mu_2(G_k)} \int \int_{G_k} p \, dx dy = c_k, k = 1, \ldots, N,$$

where G_k's are the cut- regions.

The Conjecture 1.1 was established in [3] in the case of $n = 1$. Another special case is considered in [4]. To introduce it let us take a set $\triangle$ to be a measurable set in $\mathbb{R}^2$ with finite non-zero measure.

The following set is called λ-shift of $\triangle$

$$\triangle^\lambda := \{\mathbf{y} + \lambda : \mathbf{y} \in \triangle, \lambda \in \mathbb{R}^n\}.$$

Let us fix a set of N distinct nodes

$$\Lambda := \{\lambda_i : i = 1, \ldots, N\} \subset \mathbb{R}^2.$$

The following set is called the set of Λ-shifts of $\triangle$

$$\Lambda(\triangle) := \{\triangle^\lambda : \lambda \in \Lambda\}.$$

Theorem 1.4. *Suppose that $\mu_2(\triangle) \neq 0$. Then, the mean-value interpolation $(\Pi_n^2, \Lambda(\Delta))^{m.v.}$ is correct if and only if the Lagrange pointwise interpolation problem (Π_n^2, Λ) is correct.*

Consider an arbitrary set of N distinct balls of the same radius r : $\mathbb{B} := \{B_{\mathbf{a}_i, r} : i = 1, \ldots, N\}$. Let $A = \{\mathbf{a}_i : i = 1, \ldots, N\}$ be the set of centers of the balls.

Corollary 1.5. *The mean-value interpolation $(\Pi_n^2, \mathbb{B})$ is correct if and only if the Lagrange pointwise interpolation (Π_n^2, A) is correct.*

Note that Theorem 1.4 and Corollary 1.5 are both proved in arbitrary dimension in [4].

In the next section we consider the bivariate mean-value interpolation for polynomials of arbitrary degree with regions obtained by circles of the same radius and with centers on a straight line. We conclude that in this case the problem is not correct. For other version of this interpolation see [1, 5].

2. **Mean-value interpolation with circles**

Let us consider mean-value interpolation with polynomials of arbitrary degree and the regions, obtained the above circles, i.e., the problem $(\Pi_n^2, \mathbb{D})^{m.v.}$.

Theorem 2.1. *Suppose that among regions of the interpolation problem $(\Pi_n^2, \mathbb{D})^{m.v.}$ there are $n + 2$ circles with the same radius whose centers lie on a straight line, where $n \geq 1$. Then the mean-value interpolation problem $(\Pi_n^2, \mathbb{D})^{m.v.}$ is not correct.*

Proof. Let $L : y = \alpha x + \beta$ be the straight line of the centers of circles. Suppose that the $N - (n + 2)$ other regions are arbitrary. Let also that the centers of $n+2$ circles D_l on the line L be the points $(\gamma_l, \gamma_l \alpha + \beta), l = 1, ..., n + 2$. Let us verify separately that theorem holds for $n = 1$. We have

$$\int\int_{D_l} p(x, y)dxdy = \int\int_{D_l} [a_{0,0} + a_{1,0}x + a_{0,1}y]dxdy =$$

(2.1) $= \pi r^2[a_{1,0}\gamma_l + a_{0,1}\gamma_l^0(\gamma_l\alpha + \beta) + a_{0,0}], \; l = 1, 2, 3,$

then the corresponding coefficients matrix for the respective system (2.1) is

$$A = \begin{pmatrix} 1 & \gamma_1 & \gamma_1\alpha + \beta \\ 1 & \gamma_2 & \gamma_2\alpha + \beta \\ 1 & \gamma_3 & \gamma_3\alpha + \beta \end{pmatrix}.$$

Therefore $\det A = 0$ and the interpolation problem is not correct.

Now we turn to the case of general $n(n \geq 2)$. Consider an arbitrary polynomial $p \in \Pi_n^2$. We have

$$\int\int_{D_l} p(x,y)dxdy = r^2 \int\int_{D:x^2+y^2\leq 1} p(rx + \gamma_l, ry + (\alpha\gamma_l + \beta))dxdy$$

$$\int\int_{D_l} p(x,y)dxdy = r^2 \int\int_{D:x^2+y^2\leq 1} p(rx + \gamma_l, ry + (\alpha\gamma_l + \beta))dxdy$$

$$= r^2 \int\int_D \sum_{i+j\leq n} a_{ij}(rx + \gamma_l)^i(ry + (\alpha\gamma_l + \beta))^j dxdy = r^2$$

$$\{\int\int_D [a_{n,0}\gamma_l^n + a_{n-1,1}\gamma_l^{n-1} \cdot (\gamma_l\alpha + \beta)(\gamma_l)^i(rx)^{n-i}dxdy + a_{n-1,1}$$

$$\int\int_D (ry + (\alpha\gamma_l + \beta)).\sum_{i=0}^{n-2} \binom{n-1}{i}(\gamma_l)^i(rx)^{n-1-i}dxdy + \cdots + a_{0,n}$$

$$\int\int_D \sum_{i=0}^{n-1} \binom{n}{i}(\alpha\gamma_l + \beta)^i(ry)^{n-i}dxdy] + \int\int_D \sum_{i+j<n} a_{ij}$$

$$\cdot(rx + \gamma_l)^i(ry + (\gamma_l\alpha + \beta))^j dxdy\}, l = 1, \ldots, n + 2.$$

Therefore, by separating the highest degree in γ_l and expanding the above sums, we have

$$\int\int_{D_l} p(x,y)dxdy = \pi r^2[a_{n,0}\gamma_l^n + \ldots + a_{0,n}\gamma_l^0(\gamma_l\alpha + \beta)^n] + r^2\{[a_{0,0}$$

$$\int\int_D dxdy + a_{2,0}r^2 \int\int_D x^2 dxdy + \cdots + a_{0,n}r^n \int\int_D y^n dxdy] \times 1+$$

$$[a_{1,0} \int\int_D dxdy + 2a_{2,0}r \int\int_D x dxdy + \cdots + na_{n,0}r^{n-1} \int\int_D x^{n-1}$$

$$dxdy] \times \gamma_l + [a_{0,1} \int\int_D dxdy + a_{1,1}r \int\int_D x dxdy + 2a_{2,0}r \int\int_D y dxdy$$

$$+ \cdots + r^{n-1}a_{0,n} \int\int_D y^{n-1}dxdy] \times (\gamma_l\alpha + \beta) + \cdots + [a_{0,n-1}\cdot$$

$$\int\int_D dxdy + nra_{0,n} \int\int_D ydxdy] \times (\gamma_l\alpha + \beta)^{n-1}\}, \ l = 1, 2, ..., n+2.$$

Let us put

$$A_{n,0} = a_{n,0}, \cdots, A_{0,n} = a_{0,n},$$

also denote the above brackets by

$$A_{i,j}, i, j = 0, \ldots, n-1$$

respectively, i.e.,

$$A_{0,0} = a_{0,0} \int\int_D dxdy + a_{2,0}r^2 \int\int_D x^2 dxdy +$$

$$\cdots + a_{0,n}r^n \int\int_D y^n dxdy,$$

$$A_{1,0} = a_{1,0} \int\int_D dxdy + 2a_{2,0}r \int\int_D xdxdy +$$

$$\cdots + na_{n,0}r^{n-1} \int\int_D x^{n-1}dxdy$$

$$A_{0,1} = a_{0,1} \int\int_D dxdy + a_{1,1}r \int\int_D xdxdy + 2a_{2,0}r \int\int_D ydxdy +$$

$$\cdots + r^{n-1}a_{0,n} \int\int_D y^{n-1}dxdy, \ldots,$$

$$A_{0,n-1} = a_{0,n-1} \int\int_D dxdy + nra_{0,n} \int\int_D ydxdy.$$

Namely $A_{i,j}$ is a coefficient of γ_l^k or $(\gamma_l\alpha + \beta)^k$, $k = 0, \ldots, n-1$. Then $A_{i,j}$ is a linear combination of $a_{i,j}$ which contains the coefficients of γ_l^k or $(\gamma_l\alpha + \beta)^k$ in the above relations. It is important to know that all coefficients before γ_l^k or $(\gamma_l\alpha + \beta)^k$ are zero. We denote the coefficients of any $a_{i,j}, i, j = 0, \ldots, n-1$ by $\zeta_{i,j,k}, \ i,j, = 0, \ldots, n, \ k = 1, \ldots, n-1$. Then

$$A_{0,0} = \pi a_{0,0} + a_{2,0}\zeta_{2,0,1} + \ldots + a_{0,n}\zeta_{0,n,1},$$

$$A_{1,0} = \pi a_{1,0} + a_{2,0}\zeta_{2,0,2} + \ldots + a_{n,0}\zeta_{n,0,2}$$

$$, \ldots, A_{0,n-1} = \pi a_{0,n-1} + a_{0,n}\zeta_{0,n,n-1},$$

where $\zeta_{i,j,k}$'s are real numbers. Hence, new variables are written by $a_{i,j}, i, j = 0, \ldots, n$. Also, in any new variable we have exactly one $a_{i,j}, i, j = 0, \ldots, n-1$ that is not repeated in other sums and also its

coefficient is not zero. Thus by backward recursive relation one can find the following linear system

$$a_{0,n-1} = \frac{1}{\pi}(A_{0,n-1} - A_{0,n}\eta_{0,n,n-1}), \ldots,$$

$$a_{0,0} = \frac{1}{\pi}(A_{0,0} - A_{1,0}\eta_{1,0,n-1} - \cdots - A_{0,n}\eta_{0,n,n-1}),$$

where $\eta_{i,j,k}$, $i, j, = 0, \ldots, n$, $k = 1, \ldots, n-1$ are real numbers. Therefore the two corresponding linear system are equivalent.

Now if we consider the linear systems with the variables $A_{i,j}$ then the $(n+2) \times N$ corresponding coefficients matrix of this linear system is as follows:

$$T = \begin{pmatrix} c_{0,1} & c_{1,1}\gamma_1 & \cdots & (\gamma_1\alpha + \beta)^n \\ . & . & \cdots & . \\ . & . & \cdots & . \\ c_{0,n+2} & c_{1,n+2}\gamma_{n+2} & \cdots & (\gamma_{n+2}\alpha + \beta)^n \end{pmatrix},$$

where $c_{i,l}$ $i = 0, \ldots, N - n - 2$ are real numbers and $l = 1, \ldots, n + 2$. In view of the case $n = 1$ in the above matrix by using the elementary operations after expanding the expressions

$$(\gamma_l\alpha + \beta)^j, j = 1, \ldots, n \ and \ l = 1, \ldots, n + 2,$$

the coefficients matrix can be reduced to the following matrix

$$R = \begin{pmatrix} 1 & \gamma_1 & 0 & \cdots & \gamma_1^n & 0 & \cdots & 0 \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ 1 & \gamma_{n+2} & 0 & \cdots & \gamma_{n+2}^n & 0 & \cdots & 0 \end{pmatrix}.$$

We know that the elementary operations do not change the rank of matrix. Consequently, in degree n the rank of the corresponding matrix is equal to $n + 1$ whenever $l = 1, \ldots, n + 2$. Namely in this submatrix one row is a linear combination of other rows. Therefore in the square linear system the rank of coefficients matrix is less than or equal to $N - 1$ whenever the number of unknowns is N. □

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