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EXTENSIONS OF STRONGLY α-REVERSIBLE RINGS

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ABSTRACT. We introduce the notion of strongly α-reversible rings which is a strong version of α-reversible rings, and investigate its properties. We first give an example to show that strongly reversible rings need not be strongly α-reversible. We next argue about the strong α-reversibility of some kinds of extensions. A number of properties of this version are established. It is shown that a ring R is strongly right α-reversible if and only if its polynomial ring $R[x]$ is strongly right α-reversible if and only if its Laurent polynomial ring $R[x, x^{-1}]$ is strongly right α-reversible. Moreover, we introduce the concept of Nil-α-reversible rings to investigate the nilpotent elements in α-reversible rings. Examples are given to show that right Nil-α-reversible rings need not be right α-reversible.

1. Introduction

Throughout this paper, R denotes an associative ring with identity and α denotes a nonzero and non-identity endomorphism, unless specified otherwise. In [3], Cohn introduced the notion of a reversible ring, a ring R is said to be reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Anderson-Camillo [1], observing the rings whose zero products commute, used the term ZC_2 for what is called reversible; while Krempa-Niewieczerzal [10] took the term C_0 for it. In [13], the reversible property

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of a ring is extended to polynomial rings as follows: a ring R is called strongly reversible, if whenever polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $g(x)f(x) = 0$. A ring R is called reduced if it has nonzero nilpotent elements. Every reduced ring is strongly reversible and every strongly reversible ring is reversible by [13]. According to [2], an endomorphism α of a ring R is called right (respectively, left) reversible if whenever $ab = 0$ for $a, b \in R$, $b\alpha(a) = 0$ (respectively, $\alpha(b)a = 0$). A ring R is called right (respectively, left) α-reversible if there exists a right (respectively, left) reversible endomorphism α of R. A ring is α-reversible if it is both left and right α-reversible.

Recall that if α is an endomorphism of a ring R, then the map $R[x] \to R[x]$ defined by $\sum_{i=0}^{m} a_i x^i \to \sum_{i=0}^{m} \alpha(a_i)x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends α. We shall also denote the extended map $R[x] \to R[x]$ by α and the image of $f(x) \in R[x]$ by $\alpha(f(x))$. We consider the α-reversibility over which polynomial rings are α-reversible and call them strongly α-reversible rings, i.e., if α is an endomorphism of R, then α is called strongly right (respectively, left) reversible if whenever $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $g(x)\alpha(f(x)) = 0$ $(respectively, \alpha(g(x))f(x) = 0)$. A ring R is called strongly right (respectively, left) α-reversible if there exists a strongly right (respectively, left) reversible endomorphism α of R. A ring is strongly α-reversible if it is both strongly right and left α-reversible.

It is clear that if R is an Armendariz right α-reversible ring, then R is strongly right α-reversible. It is shown in [13] that every reduced ring is strongly reversible. We shall give an example to show that there exists a reduced ring which is not strongly right α-reversible. Moreover, we shall show that strongly reversible rings need not be strongly α-reversible in general.

In [13, Corollary 3.6], it is claimed that if R is a reduced ring, then the trivial extension $T(R, R)$ of R by R is strongly reversible. We show that if R is a reduced right α-reversible, then $T(R, R)$ is strongly right α-reversible. For an endomorphism α of a ring R, we prove that R is strongly right α-reversible if and only if $R[x]$ is strongly right α-reversible. Moreover, we introduce the concept of Nil-α-reversible rings to investigate the nilpotent elements in α-reversible rings. We do this by considering the nilpotent elements instead of the zero element in

α-reversible rings. This provides us with an opportunity to study α-reversible rings in a general setting. We also investigate connections to other related conditions. Examples to illustrate the concepts and results are included.

2. **Extensions of strongly α-reversible rings**

Our focus in this section is to introduce the concept of a strongly α-reversible ring and investigate its properties. Some examples needed in the process are also given. We start with the following definition.

Definition 2.1. *An endomorphism α of a ring R is called strongly right (respectively, left) reversible if whenever $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$, then $g(x)\alpha(f(x)) = 0$ $(respectively, \alpha(g(x))f(x) = 0)$. A ring R is called strongly right (respectively, left) α-reversible if there exists a strongly right (respectively, left) reversible endomorphism α of R. A ring is strongly α-reversible if it is both strongly right and left α-reversible.*

Clearly, every strongly α-reversible ring is α-reversible. It can be easily checked that if R is a strongly reversible ring then it is a one-sided strongly I_R-reversible ring for identity endomorphism I_R of R. It is easy to see that every subring S with $\alpha(S) \subseteq S$ of a strongly α-reversible ring is also strongly α-reversible.

The following example shows that there exists an endomorphism α of a strongly reversible ring R such that R is not strongly right α-reversible.

Example 2.2. *Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the ring of integers modulo 2. Since R is a commutative reduced ring, it is strongly reversible. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha((a, b)) = (b, a)$. For $f(x) = (1, 0) + (1, 0)x$ and $g(x) = (0, 1) + (0, 1)x$, it is clear that $f(x)g(x) = 0$. But $g(x)\alpha(f(x))=((0, 1) + (0, 1)x)((0, 1) + (0, 1)x) \neq 0$, thus R is not strongly right α-reversible.*

Note. It is proved in [2] that the concepts of reversible rings and right α-reversible rings are independent of each other. Since every strongly reversible ring is reversible, one may suspect that every strongly reversible ring is right α-reversible. But this is not true by [2, *Example* 2.3]. We notice that Example 2.2 also shows that there exists a commutative reduced ring which is not strongly right α-reversible. But it is shown in [13] that every reduced ring is strongly reversible.

Proposition 2.3. *Let R be an Armendariz ring. Then R is right α-reversible if and only if R is strongly right α-reversible.*

Proof. It is straightforward. □

Let R be a ring and $\triangle$ a multiplicative monoid in R consisting of central regular elements, and let $\triangle^{-1}R = \{u^{-1}a | u \in \triangle, a \in R\}$. Then $\triangle^{-1}R$ is a ring. For it, we have the following result for the strongly right α-reversible property, relating to idempotents.

Proposition 2.4. *Let R be a ring, e a central idempotent of R. Then the following statements are equivalent:*

(1) *R is strongly right α-reversible.*
(2) *eR and $(1\text{-}e)R$ are strongly right α-reversible.*
(3) *$\triangle^{-1}R$ is strongly right α-reversible.*

Proof. (1)⇔(2) This is straightforward since subrings and finite direct products of strongly right α-reversible rings are strongly right α-reversible.

(3)⇒(1) This is obvious since R is a subring of $\triangle^{-1}R$.

(1)⇒(3) Let $f(x) = \sum_{i=0}^{m} u_i^{-1} a_i x^i$, $g(x) = \sum_{j=0}^{n} v_j^{-1} b_j x^j \in \triangle^{-1}R[x]$ with $f(x)g(x) = 0$. Then

$$F(x) = (u_m u_{m-1} \cdots u_0) f(x),\ G(x) = (v_n v_{n-1} \cdots v_0) g(x) \in R[x].$$

Since R is strongly right α-reversible and $F(x)G(x) = 0$, this implies that $G(x)\alpha(F(x)) = 0$, and so $g(x)\alpha(f(x)) = 0$. This is because $\triangle$ is a multiplicative monoid in R consisting of central regular elements and $u_i, v_j \in \triangle$ for all i, j. This shows that $\triangle^{-1}R$ is strongly right α-reversible. □

The ring of Laurent polynomials in x, with coefficients in a ring R, consists of all formal sum $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers. Denote it by $R[x; x^{-1}]$.

Corollary 2.5. *For a ring R, $R[x]$ is strongly right α-reversible if and only if $R[x;x^{-1}]$ is strongly right α-reversible.*

Proof. It suffices to establish necessity since $R[x]$ is a subring of $R[x;x^{-1}]$. Suppose that $R[x]$ is strongly right α-reversible. Let $\triangle = \{1, x, x^2, \cdots\}$, then clearly $\triangle$ is a multiplicative closed subset of $R[x]$. Since $R[x;x^{-1}] = \triangle^{-1}R[x]$, it follows that $R[x;x^{-1}]$ is strongly right α-reversible by Proposition 2.4.

$\square$

Corollary 2.6. *A commutative ring R is strongly right α-reversible if and only if so is the total quotient ring of R.*

It is proved in [9, Proposition 2.4] that if R is an Armendariz ring, then R is reversible if and only if $R[x]$ is reversible if and only if $R[x;x^{-1}]$ is reversible. Accordingly, we have the following immediate corollary.

Corollary 2.7. *Let R be an Armendariz ring, then the following are equivalent:*

(1) *R is right α-reversible.*
(2) *R is strongly right α-reversible.*
(3) *$R[x;x^{-1}]$ is strongly right α-reversible.*

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R,M) = R\bigoplus M$ with the usual addition and the following multiplication $(r_1,m_1)(r_2,m_2) = (r_1r_2, r_1m_2 + m_1r_2)$.

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$, $m \in M$ and the usual matrix operations are used.

For an endomorphism α of a ring R and the trivial extension $T(R,R)$ of R, α can be extended to the endomorphism $\bar{\alpha} : T(R,R) \to T(R,R)$ defined by $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$.

The next example shows that even for a right α-reversible ring R, $T(R,R)$ need not be strongly right α-reversible.

Example 2.8. *Let $\mathbb{Z}_4$ be the ring of integers modulo 4. Consider the ring*

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in \mathbb{Z}_4 \right\}.$$

Let $\alpha : R \to R$ defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

Then R is right α-reversible by [2, *Example* 2.7]*. But $T(R, R)$ is not strongly right α-reversible since $T(R, R)$ is not right $\bar{\alpha}$-reversible by* [2, *Example* 2.17]*.*

However, we have the following.

Proposition 2.9. *Let R be a reduced ring. If R is right α-reversible, then $T(R, R)$ is strongly right α-reversible.*

Proof. Let $p(x) = (p_0(x), p_1(x))$, $q(x) = (q_0(x), q_1(x)) \in T(R, R)[x]$ with $p(x)q(x) = 0$, we shall prove $q(x)\alpha(p(x)) = 0$. So we have

$$p_0(x)q_0(x) = 0, \tag{1}$$
$$p_0(x)q_1(x) + p_1(x)q_0(x) = 0. \tag{2}$$

Since R is reduced, $R[x]$ is reduced. Therefore, (1) implies $q_0(x)p_0(x) = 0$. Multiplying (2) by $q_0(x)$ on the left we get $p_1(x)q_0(x) = 0$, and so $p_0(x)q_1(x) = 0$. Let $p(x) = \Sigma_{i=0}^{n}(a_i, b_i)x^i, q(x) = \Sigma_{j=0}^{m}(a_j, b_j)x^j$, where $p_0(x) = \sum_{i=0}^{n} a_i x^i, p_1(x) = \sum_{i=0}^{n} b_i x^i, q_0(x) = \sum_{j=0}^{m} a_j x^j$ and $q_1(x) = \sum_{j=0}^{m} b_j x^j$. Since every reduced ring is an Armendariz ring, we obtain that $a_i a_j = 0$, $a_i b_j = 0$, $b_i a_j = 0$ for all i, j by the preceding results. With these facts and the fact that R is right α-reversible, we have $a_j\alpha(a_i) = 0, a_j\alpha(b_i) = 0, b_j\alpha(a_i) = 0$. Then $q(x)\alpha(p(x)) = 0$, implies that $T(R, R)$ is strongly right α-reversible. □

Corollary 2.10. [**13, *Corollary* 3.6**] *If R is a reduced ring, then $T(R, R)$ is a strongly reversible ring .*

Recall that an endomorphism α of a ring R is said to be rigid if $a\alpha(a) = 0$ implies $a = 0$ for all $a \in R$. A ring R is said to be α-rigid if there exists a rigid endomorphism α of R. We note that any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced. According to [4], a ring R is said to be α-compatible if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$, where α is an endomorphism of R. The notion of the α-compatible ring is a generalization of α-rigid rings to the more general case where R is not assumed to be reduced.

Note that all reversible rings are McCoy rings by [12, Theorem 2]. Unlike the reversibility of a ring R, the next example shows that there exists a right McCoy ring which is not right α-reversible.

Example 2.11. *Let α be an endomorphism of a ring R. If R is an α-rigid ring, then*

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} |a, b, c, d \in R \right\}$$

is a right McCoy ring by [14*, Example* 2.2]*, but S is not right α-reversible by* [2*, Example* 2.20]*.*

The trivial extension $T(R, R)$ of a ring R can be extended to a ring

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} |a, b, c, d \in R \right\}$$

and an endomorphism α of a ring R is also extended to the endomorphism $\bar{\alpha} : S \to S$ defined by $\bar{\alpha}(a_{ij}) = (\alpha(a_{ij}))$.

Note that Example 2.11 also shows that the ring S constructed above is not strongly right α-reversible, even if R is an α-rigid ring. However, we have the following proposition.

Proposition 2.12. *Let R be a reduced ring. If R is right α-reversible, then*

$$M = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} |a, b, c, d \in R \right\}$$

is strongly right α-reversible.

Proof. For $\begin{pmatrix} a_1 & 0 & b_1 \\ 0 & a_1 & c_1 \\ 0 & 0 & a_1 \end{pmatrix}, \begin{pmatrix} a_2 & 0 & b_2 \\ 0 & a_2 & c_2 \\ 0 & 0 & a_2 \end{pmatrix} \in M$. We can denote their addition and multiplication by

$$(a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2),$$
$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1a_2, a_1b_2 + b_1a_2, a_1c_2 + c_1a_2),$$

respectively. So every polynomial in $M[x]$ can be expressed in the form of (f_0, f_1, f_2) for some f_i's in $R[x]$.

Let $p(x) = (p_0(x), p_1(x), p_2(x)), q(x) = (q_0(x), q_1(x), q_2(x)) \in M[x]$ with $p(x)q(x) = 0$. Then

$p(x)q(x) = (p_0(x)q_0(x), p_0(x)q_1(x)+p_1(x)q_0(x), p_0(x)q_2(x)+p_2(x)q_0(x)).$

So we have the following system of equations:

$$p_0(x)q_0(x) = 0, \tag{1}$$
$$p_0(x)q_1(x) + p_1(x)q_0(x) = 0, \tag{2}$$
$$p_0(x)q_2(x) + p_2(x)q_0(x) = 0. \tag{3}$$

Use the fact that $R[x]$ is reduced. From Eq. (1), we get $q_0(x)p_0(x) = 0$. If we multiply Eq. (2) on the right side by $p_0(x)$, then $p_0(x)q_1(x)p_0(x)+p_1(x)q_0(x)p_0(x) = 0$. Hence $p_0(x)q_1(x) = 0$ and so $p_1(x)q_0(x) = 0$. Also if we multiply Eq. (3) on the right side by $p_0(x)$, then $p_0(x)q_2(x)p_0(x)+p_2(x)q_0(x)p_0(x) = 0$. So $p_0(x)q_2(x) = 0$ and hence $p_2(x)q_0(x) = 0$. Let

$$p(x) = \sum_{i=0}^{n} \begin{pmatrix} a_i & 0 & b_i \\ 0 & a_i & c_i \\ 0 & 0 & a_i \end{pmatrix} x^i, q(x) = \sum_{j=0}^{m} \begin{pmatrix} a'_j & 0 & b'_j \\ 0 & a'_j & c'_j \\ 0 & 0 & a'_j \end{pmatrix} x^j \in M[x],$$

where $p_0(x) = \sum_{i=0}^{n} a_ix^i, p_1(x) = \sum_{i=0}^{n} b_ix^i, p_2(x) = \sum_{i=0}^{n} c_ix^i$, $q_0(x) = \sum_{j=0}^{m} a'_jx^j, q_1(x) = \sum_{j=0}^{m} b'_jx^j$ and $q_2(x) = \sum_{j=0}^{m} c'_jx^j$. Since every reduced ring is an Armendariz ring, we obtain that $a_ia'_j = 0, a_ib'_j = 0, b_ia'_j = 0, a_ic'_j = 0, c_ia'_j = 0$ for all i, j by the preceding results. With these facts and the fact that R is right α-reversible, we have $a'_j\alpha(a_i) = 0, b'_j\alpha(a_i) = 0, a'_j\alpha(b_i) = 0, c'_j\alpha(a_i) = 0, a'_j\alpha(c_i) = 0$. Consequently, we get the equation:

$$\begin{aligned} &q(x)\alpha(p(x)) = (q_0(x), q_1(x), q_2(x))\alpha(p_0(x), p_1(x), p_2(x)) \\ &= (q_0(x)\alpha(p_0(x)), q_1(x)\alpha(p_0(x)) + q_0(x)\alpha(p_1(x)), q_2(x)\alpha(p_0(x))+ \\ &q_0(x)\alpha(p_2(x))) = 0. \end{aligned}$$

Therefore, M is strongly right α-reversible. $\square$

Corollary 2.13. *If R is a reduced ring, then*

$$M = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} | a, b, c, d \in R \right\}$$

is a strongly reversible ring.

We next show that $R[x]$ is strongly right α-reversible if and only if R is.

Proposition 2.14. *Let α be an endomorphism of a ring R. Then R is strongly right α-reversible if and only if $R[x]$ is strongly right α-reversible.*

Proof. Assume that R is strongly right α-reversible. Let $p(y) = f_0 + f_1 y + \cdots + f_m y^m$, $q(y) = g_0 + g_1 y + \cdots + g_n y^n$ be in $R[x][y]$ with $p(y)q(y) = 0$. We also let $f_i = a_{i_0} + a_{i_1} x + \cdots + a_{w_i} x^{w_i}$, $g_j = b_{j_0} + b_{j_1} x + \cdots + b_{v_i} x^{v_j}$ for each $0 \leq i \leq m$ and $0 \leq j \leq n$, where $a_{i_0}, a_{i_1}, \cdots, a_{w_i}, b_{j_0}, b_{j_1}, b_{v_j} \in R$. We claim that $R[x]$ is strongly right α-reversible. Take a positive integer k such that $k > max\{deg(f_i), deg(g_j)\}$ for any $0 \leq i \leq m$ and $0 \leq j \leq n$, where the degree is as polynomials in $R[x]$ and the degree of zero polynomial is taken to be zero. Let $p(x^k) = f_0 + f_1 x^k + \cdots + f_m x^{mk}$, $q(x^k) = g_0 + g_1 x^k + \cdots + g_n x^{nk}$ in $R[x]$. Then the set of coefficients of the $f_i^{'}$s (respectively, $g_j^{'}$s) is equal the set of coefficients of $p(x^k)$ (respectively, $q(x^k)$). Since $p(y)q(y) = 0$, x commutes with elements of R in the polynomial ring $R[x]$, $p(x^k)q(x^k) = 0$. Since R is strongly right α-reversible, $q(x^k)\alpha(p(x^k)) = 0$, and hence $R[x]$ is strongly right α-reversible. The converse is obvious since R is a subring of $R[x]$. □

Corollary 2.15. *Let R be a ring. Then R is strongly reversible if and only if $R[x]$ is strongly reversible.*

The next corollary gives more examples of strongly right α-reversible rings.

Corollary 2.16. *Let α be an endomorphism of a ring R, then the following are equivalent:*

(1) *R is strongly right α-reversible.*
(2) *$R[x]$ is strongly right α-reversible.*

(3) $R[x;x^{-1}]$ *is strongly right* α*-reversible.*

Let $A(R,\alpha)$ or A be the subset $\{x^{-i}rx^i | r \in R, i \geq 0\}$ of the skew Laurent polynomial ring $R[x,x^{-1};\alpha]$, where $\alpha : R \to R$ is an injective ring endomorphism of a ring R (see [8] for more details). Elements of $R[x,x^{-1};\alpha]$ are finite sums of elements of the form $x^{-i}rx^j$, where $r \in R$ and i,j are non-negative integers. Multiplication is subject to $xr = \alpha(r)x$ and $rx^{-1} = x^{-1}\alpha(r)$ for all $r \in R$.

Note that for each $j \geq 0$, $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{(i+j)}$. It follows that the set $A(R,\alpha)$ of all such elements forms a subring of $R[x,x^{-1};\alpha]$ with

$$\begin{aligned} &x^{-i}rx^i + x^{-j}sx^j = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{(i+j)} \\ &(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}(\alpha^j(r)\alpha^i(s))x^{(i+j)} \end{aligned}$$

for $r,s \in R$ and $i,j \geq 0$. Note that α is actually an automorphism of $A(R,\alpha)$.

Let $A(R,\alpha)$ be the ring defined above, then for the endomorphism α in $A(R,\alpha)$, the map $A(R,\alpha)[t] \to A(R,\alpha)[t]$ defined by $\sum_{i=0}^m (x^{-i}rx^i)t^i \to \sum_{i=0}^m (x^{-i}\alpha(r)x^i)t^i$ is an endomorphism of the polynomial ring $A(R,\alpha)[t]$.

Proposition 2.17. *Let* $A(R,\alpha)$ *be an Armendariz ring. If* R *is right* α*-reversible, then* $A(R,\alpha)$ *is strongly right* α*-reversible.*

Proof. Let $f(t) = \sum_{i=0}^m (x^{-i}rx^i)t^i$, $g(t) = \sum_{j=0}^n (x^{-j}sx^j)t^j \in A(R,\alpha)[t]$ with $f(t)g(t) = 0$. Since $A(R,\alpha)$ is Armendariz, $(x^{-i}rx^i)(x^{-j}sx^j) = 0$, and so $x^{-(i+j)}(\alpha^j(r)\alpha^i(s))x^{(i+j)} = 0$. This implies that $\alpha^j(r)\alpha^i(s) = 0$. Then we obtain $\alpha^i(s)\alpha^{j+1}(r) = 0$ since R is right α-reversible. Then

$$\begin{aligned} &g(t)\alpha(f(t)) \\ &= (\textstyle\sum_{j=0}^n (x^{-j}sx^j)t^j)(\sum_{i=0}^m (x^{-i}\alpha(r)x^i)t^i) \\ &= \textstyle\sum_{k=0}^{m+n} (x^{-j}sx^j)(x^{-i}\alpha(r)x^i)t^k \\ &= \textstyle\sum_{k=0}^{m+n} (x^{-(i+j)}(\alpha^i(s)\alpha^j(\alpha(r))x^{(i+j)})t^k \\ &= \textstyle\sum_{k=0}^{m+n} (x^{-(i+j)}(\alpha^i(s)\alpha^{j+1}(r))x^{(i+j)})t^k. \end{aligned}$$

Since $\alpha^i(s)\alpha^{j+1}(r) = 0$, this yields $g(t)\alpha(f(t)) = 0$, and thus $A(R,\alpha)$ is strongly right α-reversible. $\square$

Corollary 2.18. *Let $A(R,\alpha)$ be an Armendariz ring. If R is reversible, then $A(R,\alpha)$ is strongly reversible.*

Recall that a ring R is called right Ore if given $a, b \in R$ with b regular, there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is well-known that R is a right Ore ring if and only if the classical right quotient ring $Q(R)$ of R exists. Let F be a field and $R = F\{x, y\}$ the free algebra in two indeterminates over F. For x and y, there is no $a, b \in R$ such that $y^{-1}x = ab^{-1}(xy^{-1} = b^{-1}a)$. So the domain R cannot have its classical right (left) quotient ring.

If we suppose that the classical right quotient ring $Q(R)$ of R exists. Then for an automorphism α of R and any $ab^{-1} \in Q(R)$ where $a, b \in R$ with b regular, the induced map $\bar{\alpha} : Q(R) \to Q(R)$ defined by $\bar{\alpha}(ab^{-1}) = \alpha(a)\alpha(b)^{-1}$ is also an endomorphism.

Proposition 2.19. *Let R be a right Ore ring and Q the classical right quotient ring of R. If $R[x]$ is an α-compatible ring, then R is strongly right α-reversible if and only if Q is strongly right α-reversible.*

Proof. Assume that R is strongly right α-reversible. Let $f(x) = \Sigma_{i=0}^{m}\alpha_i x^i$, $g(x) = \Sigma_{j=0}^{n}\beta_j x^j \in Q[x]$ with $f(x)g(x) = 0$. Then by [11, Proposition 2.1.16], we may assume that $\alpha_i = a_i u^{-1}$ and $\beta_j = b_j v^{-1}$ with $a_i, b_j \in R$ for all i, j and some regular $u, v \in R$. Moreover, for each j, there exists $c_j \in R$ and regular $s \in R$ such that $u^{-1}b_j = c_j s^{-1}$ also by [11, Proposition 2.1.16]. Suppose $f_1(x) = \Sigma_{i=0}^{m}a_i x^i$, $g_1(x) = \Sigma_{j=0}^{n}b_j x^j$ and $g_2(x) = \Sigma_{j=0}^{n}c_j x^j$. From $f(x)g(x) = 0$, we have the following equation: $f(x)g(x) = \Sigma_{i=0}^{m}\Sigma_{j=0}^{n}\alpha_i\beta_j x^{i+j} = \Sigma_{i=0}^{m}\Sigma_{j=0}^{n}a_i(u^{-1}b_j)v^{-1}x^{i+j} = \Sigma_{i=0}^{m}\Sigma_{j=0}^{n}a_i c_j(vs)^{-1}x^{i+j} = f_1(x)g_2(x)(vs)^{-1} = 0$, which implies that $f_1(x)g_2(x) = 0$.

Since $f_1(x)g_2(x) = 0$, we get $\alpha(f_1(x)g_2(x)) = 0$, i.e., $\alpha(f_1(x))\alpha(g_2(x)) = 0$. This implies that $\alpha(f_1(x))g_2(x) = 0$ since $R[x]$ is an α-compatible ring. Since R is strongly right α-reversible and α is an automorphism of R, it follows that $R[x]$ is semicommutative by [2, Proposition 2.5]. Therefore, $\alpha(f_1(x))ug_2(x) = \alpha(f_1(x))g_1(x)s = 0$, and hence we have $\alpha(f_1(x))g_1(x) = 0$. Using [11, Proposition 2.1.16] again, for each i there exists $d_i \in R$ and regular element $t \in R$ such that $v^{-1}\alpha(a_i) = d_i t^{-1}$.

Since α is an automorphism of R, there exists $k_i \in R$ such that $d_i = \alpha(k_i)$ for each i. Let $f_2(x) = \Sigma_{i=0}^{m} k_i x^i$, then

$$\alpha(f_1(x))tg_1(x) = \Sigma_{i+j=k}\alpha(a_i)tb_jx^{i+j} = \Sigma_{i+j=k}vd_ib_jx^{i+j}$$
$$= v\alpha(f_2(x))g_1(x) = 0.$$

This shows that $\alpha(f_2(x))g_1(x) = 0$. Since R is strongly right α-reversible, we have $g_1(x)\alpha(\alpha(f_2(x))) = 0$. Therefore, $g_1(x)\alpha(f_2(x)) = 0$ since $R[x]$ is an α-compatible ring. Then

$$g(x)\alpha(f(x)) = (\Sigma_{j=0}^{n}\beta_jx^j)(\Sigma_{i=0}^{m}\alpha(\alpha_i)x^i)$$
$$= (\Sigma_{j=0}^{n}b_jv^{-1}x^j)(\Sigma_{i=0}^{m}\alpha(a_i)\alpha(u)^{-1}x^i)$$
$$= \Sigma_{i+j=k}b_j(v^{-1}\alpha(a_i))\alpha(u)^{-1}x^{i+j}$$
$$= \Sigma_{i+j=k}b_jd_i(\alpha(u)t)^{-1}x^{i+j}$$
$$= g_1(x)\alpha(f_2(x))(\alpha(u)t)^{-1} = 0.$$

Therefore, Q is strongly right α-reversible. □

In particular, we obtain the following.

Corollary 2.20. [**13,** ***Theorem*** **3.9**] *Let R be a right Ore ring and Q the classical right quotient ring of R. Then R is strongly reversible if and only if Q is strongly reversible.*

3. **Related topics**

Now we investigate a weak form of α-reversible rings in the sense of the following definition. We call them Nil-α-reversible rings. We do this by considering the nilpotent elements instead of the zero element in α-reversible rings. For a ring R, we denote by $nil(R)$ the set of all nilpotent elements of R.

Definition 3.1. *Let R be a ring and α an endomorphism of R. Then R is said to be a right (respectively, left) Nil-α-reversible ring if $ab = 0$ implies that $r_1br_2\alpha(a)r_3 \in nil(R)$ (respectively, $r_1\alpha(b)r_2ar_3 \in nil(R)$) for all $a, b \in R$ and $r_1, r_2, r_3 \in R$. A ring is Nil-α-reversible if it is both right and left Nil-α-reversible.*

In particular, if $\alpha \equiv I_R$, then we call the ring weakly reversible since reversible rings are clearly weakly reversible. We shall observe briefly, see Example 3.7, that weakly reversible rings are not necessarily reversible.

According to [9], a ring R is called semicommutative if $ab = 0$ implies $aRb = 0$ for all $a, b \in R$. It is well-known that reversible rings are semicommutative. Unlike the situation of reversible rings, the next example shows that there exists a semicommutative ring R such that R is not right α-reversible, and hence R is not strongly right α-reversible.

Example 3.2. *Let α be a an endomorphism of an α-rigid ring R and let*

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} | a, b, c, d \in R \right\}.$$

Since every α-rigid ring is reduced, it follows that S is a semicommutative ring by [9, *Proposition* 1.2]. *But S is not right α-reversible by* [2, *Example* 2.20].

Clearly, every semicommutative right α-reversible ring is right Nil-α-reversible. It is proved in [2, Proposition 2.5] that if α is a monomorphism of a ring R, then every right α-reversible ring is semicommutative. In this case, we know that if α is a monomorphism of R, then every right α-reversible ring is right Nil-α-reversible. We shall give example to show that there exists a right Nil-α-reversible ring which is not right α-reversible. It is easy to see that every subring S with $\alpha(S) \subseteq S$ of an Nil-α-reversible ring is also Nil-α-reversible.

The following example shows that there exists an endomorphism α of a ring R such that R is a right Nil-α-reversible ring which is not reversible.

Example 3.3. *Let $\mathbb{Z}$ be the ring of integers. Consider the ring*

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{Z} \right\}.$$

Let $\alpha : R \to R$ be an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that R is not reversible (see, e.g., [2, *Example* 2.2]*). If $AB = 0$ for $A = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$ and $B = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$ in R, then we get $a_1a_2 = 0$ and $c_1c_2 = 0$. Let $C_i = \begin{pmatrix} c_i & d_i \\ 0 & e_i \end{pmatrix}$ be some elements in R, where $c_i, d_i, e_i \in R$ and $i = 1, 2, 3$. It can be easily checked that*

$$C_1BC_2\alpha(A)C_3 = 0,$$

which implies that R is right Nil-α-reversible.

Let $T_n(R)$ be the ring of n-by-n upper triangular matrices over R. Note that an endomorphism α of a ring R can also be extended to the endomorphism $\bar{\alpha} : T_n(R) \to T_n(R)$ defined by $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$. The following proposition gives more examples of right Nil-α-reversible rings by matrix extensions.

Proposition 3.4. *If R is a right Nil-α-reversible ring, then for any n, $T_n(R)$ is right Nil-α-reversible.*

Proof. Let $A_{i_1} = (a_{ij}^{i_1}), A_{i_2} = (a_{ij}^{i_2}) \in T_n(R)$ with $A_{i_1}A_{i_2}$=0, and let $B_{j_k} = (b_{ij}^{j_k}) \in T_n(R), k = 1, 2, 3$. Then we have $a_{ii}^{i_1}a_{ii}^{i_2} = 0$ for any $1 \leq i \leq n$. Since R is right Nil-α-reversible, there exists $m_i \in N$ such that $(b_{ii}^{j_1}a_{ii}^{i_2}b_{ii}^{j_2}\alpha(a_{ii}^{i_1})b_{ii}^{j_3})^{m_i} = 0$ for any $i, i = 1, 2, \cdots, n$. Let $m = max\{m_1, m_2, \cdots, m_n\}$, then $((B_{j_1}A_{i_2}B_{j_2}\alpha(A_{i_1})B_{j_3})^m)^n = 0$, this implies that $T_n(R)$ is right Nil-α-reversible.

The next example shows that there exists a right Nil-α-reversible ring which is not right α-reversible. $\square$

Example 3.5. *Let R be the ring in Example* 2.8. *Then R is right α-reversible by* [2, *Example* 2.7]. *We claim that $T(R, R)$ is right Nil-α-reversible. In fact, since α is a monomorphism, this implies that R is a semicommutative ring by* [2, *Proposition* 2.5]*, and hence R is a right Nil-α-reversible ring. Therefore, $T(R, R)$ is right Nil-α-reversible by Proposition* 3.4. *But $T(R, R)$ is not right α-reversible by* [2, *Example* 2.17]*, and so $T(R, R)$ is not strongly right α-reversible.*

For an endomorphism α of R, we define the map $R[x;x^{-1}] \to R[x;x^{-1}]$ by the same endomorphism as in the polynomial ring $R[x]$ above. The next two results extend the class of right Nil-α-reversible rings.

Proposition 3.6. *Let R be a ring, then $R[x]$ is right Nil-α-reversible if and only if $R[x;x^{-1}]$ is right Nil-α-reversible.*

Proof. It is sufficient to show the necessity. Let $f(x), g(x) \in R[x;x^{-1}]$ with $f(x)g(x) = 0$, and let $h_i(x) \in R[x;x^{-1}]$ be some elements with $i = 1, 2, 3$. Then there exists $s \in N$ such that

$f_1(x) = f(x)x^s, g_1(x) = g(x)x^s$ and $h_i'(x) = h_i(x)x^s \in R[x], i = 1, 2, 3$.

Since $R[x]$ is right Nil-α-reversible and $f_1(x)g_1(x) = 0$ by the hypothesis, there exists $n \in N$ such that $(h_1'(x)g_1(x)h_2'(x)\alpha(f_1(x))h_3'(x))^n = 0$. Then we have

$$(h_1(x)g(x)h_2(x)\alpha(f(x))h_3(x))^n = (x^{-5s}(h_1'(x)g_1(x)h_2'(x)\alpha(f_1(x))h_3'(x)))^n$$

$= (x^{-5s})^n(h_1'(x)g_1(x)h_2'(x)\alpha(f_1(x))h_3'(x))^n = 0$. This shows that $R[x;x^{-1}]$ is right Nil-α-reversible. □

Since every reversible ring is semicommutative, it is clear that every reversible ring is weakly reversible. The next example shows that there exists a weakly reversible ring which is not reversible.

Example 3.7. *Let R be a reduced ring and let*

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} |a, b, c \in R \right\}.$$

Note that for $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in S, we have $AB = 0$ but
$BA \neq 0$, so S is not reversible.

On the other hand, let $A = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$ and $B = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$ in S with $AB = 0$, then $a_1a_2 = 0$ and $c_1c_2 = 0$. Let

$$C_i = \begin{pmatrix} d_i & e_i \\ 0 & f_i \end{pmatrix} \in S,$$

where $d_i, e_i, f_i \in R$, $i = 1, 2, 3$. Since every reduced ring is weakly reversible, there exist $n_1, n_2 \in \mathbb{N}$ such that

$$(d_1a_2d_2a_1d_3)^{n_1} = (f_1c_2f_2c_1f_3)^{n_2} = 0$$

for all $d_i, f_i \in R$, i=$1, 2, 3$. *It follows from the following equation that* S *is weakly reversible:*

$$(C_1BC_2AC_3)^{max\{n_1,n_2\}+2} = 0.$$

For an automorphism α of R and any $u^{-1}a \in \triangle^{-1}R$ where $u \in \Delta, a \in R$ with u central regular, the induced map $\bar{\alpha} : \triangle^{-1}R \to \triangle^{-1}R$ defined by $\bar{\alpha}(u^{-1}a) = \alpha(u)^{-1}\alpha(a)$ is also an endomorphism. In general, we can get the following result.

Proposition 3.8. *Let* δ *be an automorphism of a ring* R. *Then* R *is right Nil-*δ*-reversible if and only if* $\triangle^{-1}R$ *is right Nil-*δ*-reversible.*

Proof. It suffices to show the necessity. Let $\alpha\beta = 0$ with $\alpha = u^{-1}a, \beta = v^{-1}b, u, v \in \triangle$ and $a, b \in R$, and let $r_i = w_i^{-1}c_i$ be any element of $\triangle^{-1}R, i = 1, 2, 3, w_i \in \triangle, c_i \in R$. Since $\triangle$ is contained in the center of R, we have $0 = \alpha\beta = u^{-1}av^{-1}b = (u^{-1}v^{-1})ab = (vu)^{-1}ab$, and so $ab = 0$. But R is right Nil-δ-reversible, so there exists $n \in N$ such that $(c_1bc_2\delta(a)c_3)^n = 0$. Then we obtain the following:

$$\begin{aligned}
&(w_1^{-1}c_1v^{-1}bw_2^{-1}c_2\delta(u^{-1}a)w_3^{-1}c_3)^n \\
&= ((w_1^{-1}v^{-1}w_2^{-1}(\delta u)^{-1}w_3^{-1})(c_1bc_2\delta(a)c_3))^n \\
&= ((w_3\delta(u)w_2vw_1)^{-1})^n(c_1bc_2\delta(a)c_3)^n \\
&= (r_1\beta r_2\delta(\alpha)r_3)^n = 0
\end{aligned}$$

This implies that $\triangle^{-1}R$ is right Nil-δ-reversible. □

Corollary 3.9. *Let* δ *be an automorphism of a commutative ring* R. *Then* R *is right Nil-*δ*-reversible if and only if so is the total quotient ring of* R.

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