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THE TWO PARAMETER QUANTUM GROUPS $U_{r,s}(\mathfrak{g})$ ASSOCIATED TO GENERALIZED KAC-MOODY ALGEBRA AND THEIR EQUITABLE PRESENTATION

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Abstract. We construct a family of two parameter quantum groups $U_{r,s}(\mathfrak{g})$ associated with a generalized Kac-Moody algebra corresponding to symmetrizable admissible Borcherds Cartan matrix. We also construct the **A**-form $U_{\mathbf{A}}$ and the classical limit of $U_{r,s}(\mathfrak{g})$. Furthermore, we display the equitable presentation for a subalgebra $U^{b-}_{r,s}(\mathfrak{g})$ of $U_{r,s}(\mathfrak{g})$ and show that this presentation has the attractive feature that all of its generators act semisimply on finite dimensional irreducible $U_{r,s}(\mathfrak{g})$-modules associated with the Kac-Moody algebra.

1. Introduction

Since early 1990s, the two parameter quantum groups and multiparameter quantum groups have drawn much attention both in mathematics and mathematical physics. Since then, a rich mathematical theory was developed for these objects and their representations with connections to many areas of both mathematics and physics. Much work has been done in this field; for example, see [1, 6, 14, 17]. Recently, Hu and Pei [8] gave a simpler definition for a class of two parameter quantum

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groups $U_{r,s}(\mathfrak{g})$ associated with semisimple Lie algebras in terms of the Euler form (or Ringel form). As in [1] and [9], these quantum groups also possess Drinfel'd double structures and the triangular decompositions. We shall restrict our attention to this kind of two parameter quantum groups.

In [10], Ito et al. introduced the equitable presentation for the one parameter quantum group $U_q(\mathfrak{sl}_2)$. Terwilliger in [19] displayed an analogous equitable presentation for one parameter quantum group $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra. In the usual Chevellay presentation for $U_q(\mathfrak{g})$, the various generators play different roles, while in the equitable presentation, the generators are on a more equal footing. For $\mathfrak{g} = \mathfrak{sl}_2$, the equitable presentation has generators $X^{\pm 1}$, Y, Z with relations $XX^{-1} = X^{-1}X = 1$,

$$\frac{qXY - q^{-1}YX}{q - q^{-1}} = 1, \frac{qYZ - q^{-1}ZY}{q - q^{-1}} = 1, \frac{qZX - q^{-1}XZ}{q - q^{-1}} = 1.$$

More importantly, they are related to Koornwinder's twisted primitive elements [16, 15]. And this presentation has the attractive feature that all of its generators act semisimply on finite dimensional irreducible $U_q(\mathfrak{g})$-modules associated with an affine Kac-Moody algebra $\mathfrak{g}$, as proved in [2]. In 1988, Borcherds gave the concept of generalized Kac-Moody algebra [4]. For such an algebra $\mathfrak{g}$, one parameter quantum deformation $U_q(\mathfrak{g})$ was constructed in [13]. Here, we give the definition of two parameter quantum groups $U_{r,s}(\mathfrak{g})$ associated with a generalized Kac-Moody algebra and prove that $U_{r,s}(\mathfrak{g})$ also has a triangular decomposition. We also present the $\mathbf{A}$-form $U_{\mathbf{A}}$ and the classical limit of $U_{r,s}(\mathfrak{g})$, and characterize the properties of $U_{\mathbf{A}}$. Furthermore, we give an equitable presentation for a subalgebra $U_{r,s}^{b-}(\mathfrak{g})$ of $U_{r,s}(\mathfrak{g})$ and show that the equitable generators of $U_{r,s}^{b-}(\mathfrak{g})$ act semisimply on finite dimensional irreducible $U_{r,s}^{b-}(\mathfrak{g})$-modules when $\mathfrak{g}$ is a Kac-Moody algebra.

The remainder of our work is organized as follows. In section 2, we modify the definition of two parameter quantum groups associated with semisimple Lie algebras so as to give the definition of two parameter quantum groups $U_{r,s}(\mathfrak{g})$ associated with generalized Kac-Moody algebra $\mathfrak{g}$. We also give the $\mathbf{A}$-form $U_{\mathbf{A}}$ and the classical limit of $U_{r,s}(\mathfrak{g})$. Moreover, some properties are stated. The equitable presentation for a subalgebra $U_{r,s}^{b-}(\mathfrak{g})$ of $U_{r,s}(\mathfrak{g})$ appears in the final section.

2. The two parameter quantum groups $U_{r,s}(\mathfrak{g})$ and its A-forms $U_{\mathbf{A}}$

In this section, we will modify the Definition 2.1 in [8] to a class of two parameter quantum groups $U_{r,s}(\mathfrak{g})$ associated with a generalized Kac-Moody algebra $\mathfrak{g}$. We also introduce the **A**-form $U_{\mathbf{A}}$ and the classical limit of two parameter groups $U_{r,s}(\mathfrak{g})$.

Let us begin with some preliminaries on the generalized Kac-Moody algebra. Put $I = \{1,\ 2, ...,\ n\}$ or $I = \mathbf{N}$, the natural number set. A real square matrix $A = (a_{ij})_{i,j\in I}$ is called a Borcherds-Cartan matrix if it satisfies:

(a) $a_{ii} = 2$ or $a_{ii} \leq 0$, for all $i \in I$;
(b) $a_{ij} \leq 0$, if $i \neq j$;
(c) $a_{ij} \in \mathbf{Z}$, if $a_{ii} = 2$;
(d) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

A Borcherds-Cartan matrix $A = (a_{ij})_{i,j\in I}$ is called admissible if it satisfies:

(a') $a_{ij} \in \mathbf{Z}$, for all $i, j \in I$;
(b') $a_{ii} \in 2\mathbf{Z} \setminus \{0\}$, for all $i \in I$;
(c') there exists a diagonal matrix $D = \mathrm{diag}(t_i \in \mathbf{N}_{>0} | i \in I)$ such that DA is symmetric and $t_i a_{ii} \in \mathbf{Z} \setminus \{0\}$, for all $i \in I$.

Here, we assume that A is a symmetrizable admissible Borcherds Cartan matrix. Then, we explain some result associated with the generalized Kac-Moody algebra $\mathfrak{g}$. Suppose $P^v = (\oplus_{i\in I}\mathbf{Z}h_i) \oplus (\oplus_{i\in I}\mathbf{Z}d_i)$, and let $\mathcal{H} = \mathbf{C} \otimes_{\mathbf{Z}} P^v$ be the complex vector space with basis $\{h_i,\ d_i\}_{i\in I}$. For $i \in I$, define $\alpha_i \in \mathcal{H}^*$ by setting $\alpha_i(h_j) = a_{ji}$ and $\alpha_i(d_j) = \delta_{ji}$, where $\mathcal{H}^*$ is the dual space of $\mathcal{H}$. Furthermore, the weight lattice is defined to be

$$P = \{\lambda \in \mathcal{H}^* \mid \lambda(P^v) \subset \mathbf{Z}\}.$$

Let $\Pi = \{\alpha_i \mid i \in I\}$ be the set of simple roots, $Q = \bigoplus_{i\in I} \mathbf{Z}\alpha_i$ root lattice, $Q^+ = \oplus_{i\in I}\mathbf{N}\alpha_i$ be the positive root lattice, Λ be the weight lattice, and Λ^+ be the the set of dominant weights. Let Φ be the set of roots and Φ^+ be the set of positive roots.

Suppose $\mathbf{Q}(r, s)$ is the rational functions field in two variables r and s over $\mathbf{Q}$. Set $r_i = r^{t_i},\ s_i = s^{t_i}$, for $i \in I$. Now, let $\mathbf{K} \supseteq \mathbf{Q}(r, s)$ be a field and $(rs^{-1})^{\frac{1}{m}} \in \mathbf{K}$, for some $m \in \mathbf{Z}_+$, such that $m\Lambda \subseteq Q$, for the possibly smallest positive integer m. We always assume that rs^{-1} is not a root of unity. Let $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ be two bilinear forms defined on the root lattice Q by

$$\langle i,j\rangle^{'}=\langle\alpha_i,\alpha_j\rangle^{'}=t_i\delta_{ij}$$

and

$$\langle i,j\rangle=\langle\alpha_i,\alpha_j\rangle=\begin{cases} t_ia_{ij}, & i<j,\\ t_i, & i=j,\\ 0, & i>j.\end{cases}$$

For $\lambda\in\Lambda$, we linearly extend the bilinear forms $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle^{'}$ to $\Lambda\times\Lambda$ such that $\langle\lambda,i\rangle^{'}=\frac{1}{m}\sum_j a_j\langle j,i\rangle^{'}$ and $\langle\lambda,i\rangle=\frac{1}{m}\sum_{j\in I}a_j\langle j,i\rangle$, for $\lambda=\frac{1}{m}\sum_j a_j\alpha_j$ with $a_j\in\mathbf{Z}$.

Definition 2.1. *The two parameter quantum groups $U_{r,s}(\mathfrak{g})$ associated with a generalized Kac-Moody algebra $\mathfrak{g}$ is a unital associative $\boldsymbol{K}$-algebra $U_{r,s}(\mathfrak{g})$ with generators $e_i,\ f_i,\ \omega_i^{\pm1},\ \omega_i^{'\pm1},\ v_i^{\pm1},\ v_i^{'\pm1}$ $(i\in I)$ and the following relations:*

$$\omega_i^{\pm1}\omega_j^{\pm1}=\omega_j^{\pm1}\omega_i^{\pm1},\ \omega_i^{'\pm1}\omega_j^{'\pm1}=\omega_j^{'\pm1}\omega_i^{'\pm1},\tag{2.1}$$

$$\omega_i^{\pm1}\omega_j^{'\pm1}=\omega_j^{'\pm1}\omega_i^{\pm1},\ \omega_i^{\pm1}\omega_i^{\mp1}=\omega_i^{'\pm1}\omega_i^{'\mp1}=1,\tag{2.2}$$

$$v_i^{\pm1}v_j^{\pm1}=v_j^{\pm1}v_i^{\pm1},\ v_i^{\pm1}v_j^{'\pm1}=v_j^{'\pm1}v_i^{\pm1},$$

$$v_i^{'\pm1}v_j^{'\pm1}=v_j^{'\pm1}v_i^{'\pm1},\ v_i^{\pm1}v_i^{\mp1}=v_i^{'\pm1}v_i^{'\mp1}=1,$$

$$\omega_iv_j=v_j\omega_i,\ \omega_iv_j^{'}=v_j^{'}\omega_i,\ \omega_i^{'}v_j^{'}=v_j^{'}\omega_i^{'},\ \omega_i^{'}v_j=v_j\omega_i^{'},$$

$$\omega_ie_j\omega_i^{-1}=r^{\langle j,i\rangle}s^{-\langle i,j\rangle}e_j,\ \omega_i^{'}e_j\omega_i^{'-1}=r^{-\langle i,j\rangle}s^{\langle j,i\rangle}e_j,\tag{2.3}$$

$$v_ie_jv_i^{-1}=r^{\langle j,i\rangle^{'}}s^{-\langle i,j\rangle^{'}}e_j,\ v_i^{'}e_jv_i^{'-1}=r^{-\langle i,j\rangle^{'}}s^{\langle j,i\rangle^{'}}e_j,$$

$$\omega_if_j\omega_i^{-1}=r^{-\langle j,i\rangle}s^{\langle i,j\rangle}f_j,\ \omega_i^{'}f_j\omega_i^{'-1}=r^{\langle i,j\rangle}s^{-\langle j,i\rangle}f_j,\tag{2.4}$$

$$v_if_jv_i^{-1}=r^{-\langle j,i\rangle^{'}}s^{\langle i,j\rangle^{'}}f_j,\ v_i^{'}f_jv_i^{'-1}=r^{\langle i,j\rangle^{'}}s^{-\langle j,i\rangle^{'}}f_j,$$

$$e_if_j-f_je_i=\delta_{ij}\frac{\omega_i-\omega_i^{'}}{r_i-s_i},$$

$$\sum_{n=0}^{1-a_{ij}}(-1)^n\begin{pmatrix}1-a_{i,j}\\ n\end{pmatrix}_{r_is_i^{-1}}(r_is_i^{-1})^{\frac{n(n-1)}{2}}r^{n\langle j,i\rangle}s^{-n\langle i,j\rangle}$$
$$\times e_i^{1-a_{ij}-n}e_je_i^n=0,\ \text{if } a_{ii}=2,\ i\neq j,$$

$$\sum_{n=0}^{1-a_{ij}}(-1)^n \begin{pmatrix} 1-a_{ij} \\ n \end{pmatrix}_{r_is_i^{-1}} (r_is_i^{-1})^{\frac{n(n-1)}{2}} r^{n\langle j,i\rangle} s^{-n\langle i,j\rangle} \tag{2.5}$$
$$\times f_i^n f_j f_i^{1-a_{ij}-n} = 0, \text{ if } a_{ii}=2,\ i\neq j,$$

$$e_ie_j - r^{\langle j,i\rangle} s^{-\langle i,j\rangle} e_je_i = 0, \text{ if } a_{ij}=0,$$

$$f_if_j - r^{\langle i,j\rangle} s^{-\langle j,i\rangle} f_jf_i = 0, \text{ if } a_{ij}=0.$$

According to the definition of $U_{r,s}(\mathfrak{g})$, we can verify that $U_{r,s}(\mathfrak{g})$ is a Hopf algebra with the comultiplication, the counit and the antipode as follows:

$$\Delta(\omega_i^{\pm 1}) = \omega_i^{\pm 1}\otimes \omega_i^{\pm 1}, \quad \Delta(v_i^{\pm 1}) = v_i^{\pm 1}\otimes v_i^{\pm 1},$$

$$\Delta(\omega_i'^{\pm 1}) = \omega_i'^{\pm 1}\otimes \omega_i'^{\pm 1}, \quad \Delta(v_i'^{\pm 1}) = v_i'^{\pm 1}\otimes v_i'^{\pm 1},$$

$$\Delta(e_i) = e_i\otimes 1 + \omega_i\otimes e_i, \quad \Delta(f_i) = 1\otimes f_i + f_i\otimes \omega_i',$$

$$\varepsilon(e_i) = \varepsilon(f_i) = 0, \quad \varepsilon(\omega_i^{\pm 1}) = \varepsilon(\omega_i'^{\pm 1}) = \varepsilon(v_i^{\pm 1}) = \varepsilon(v_i'^{\pm 1}) = 1,$$

$$S(e_i) = -\omega_i^{-1}e_i, \quad S(f_i) = -f_i\omega_i'^{-1},$$

$$S(\omega_i^{\pm 1}) = \omega_i^{\mp 1}, \quad S(\omega_i'^{\pm 1}) = \omega_i'^{\mp 1}.$$

Let $U_{r,s}^{+}$ (respectively $U_{r,s}^{-}$) be the subalgebra of $U_{r,s}(\mathfrak{g})$ generated by the elements e_i (respectively f_i), for all $i\in I$, $U_{r,s}^{0}$ the subalgebra of $U_{r,s}(\mathfrak{g})$ with generators $\omega_i^{\mp 1}$, $\omega_i'^{\mp 1}$, $v_i^{\mp 1}$ and $v_i'^{\mp 1}$, for all $i\in I$, and $U_{r,s}^{b+}$ (respectively $U_{r,s}^{b-}$) be the subalgebra of $U_{r,s}(\mathfrak{g})$ generated by the elements e_i, $\omega_i^{\mp 1}$, $\omega_i'^{\mp 1}$ $v_i^{\mp 1}$ and $v_i'^{\mp 1}$ (respectively f_i, $\omega_i^{\pm 1}, \omega_i'^{\pm 1}$ $v_i^{\pm 1}, v_i'^{\pm 1}$), for all $i\in I$.

Remark 2.2. *(i) Let $r=q$, $s=q^{-1}$. Then, $U_{q,q^{-1}}(\mathfrak{g})/\langle \omega_i' - \omega_i^{-1},\ v_i' - v_i^{-1}\rangle$ is isomorphic to the one parameter quantum group $U_q(\mathfrak{g})$ defined in [13].*

(ii) Let $r=q^2$, $s=1$. Then $U_{q^2,1}^{+}$ is isomorphic to the Ringel-Hall algebra of a quiver described in [18].

Similar to the case of one parameter quantum group $U_q(\mathfrak{g})$ in [12] and [13], we have the following results.

Proposition 2.3. $U^{b+}_{r,s} \simeq U^0_{r,s} \otimes U^+_{r,s},\ U^{b-}_{r,s} \simeq U^0_{r,s} \otimes U^-_{r,s}\ U_{r,s}(\mathfrak{g}) \simeq U^-_{r,s} \otimes U^0_{r,s} \otimes U^+_{r,s}$.

For $i \in I,\ c \in \boldsymbol{Z},\ n \in \boldsymbol{Z}_{\geq 0},\ r,s \in \boldsymbol{Q}$, *define*

$$\left\{ \begin{matrix} \omega_i,\ \omega_i',\ c \\ n \end{matrix} \right\}_i = \prod_{k=1}^{n} \frac{\omega_i r_i^{c-k+1} - \omega_i' s_i^{c-k+1}}{r_i^k - s_i^k},$$

$$\left\{ \begin{matrix} v_i,\ v_i',\ c \\ n \end{matrix} \right\}_i = \prod_{k=1}^{n} \frac{v_i r_i^{c-k+1} - v_i' s_i^{c-k+1}}{r_i^k - s_i^k},$$

$$\{n\}_i = \frac{r_i^n - s_i^n}{r_i - s_i},\ \{n\}_i! = \{n\}_i\{n-1\}_i \dots \{2\}_i\{1\}_i,$$

$$\left\{ \begin{matrix} m \\ n \end{matrix} \right\}_i = \frac{\{\ m\ \}_i\ !}{\{\ m-n\ \}_i\ !\ \{\ n\ \}_i\ !}\ (m \geq n \geq 0),$$

With $\{0\}_i! = 1$.

Lemma 2.4. $\{n+m\}_i = r_i^m\{n\}_i + s_i^n\{m\}_i = r_i^n\{m\}_i + s_i^m\{n\}_i$.

Proof. It is a straight forward computation. □

By routine calculations, we have

$$\left\{ \begin{matrix} \omega_i,\ \omega_i',\ \boldsymbol{c} \\ n \end{matrix} \right\}_i = \prod_{k=1}^{n} \frac{1}{\{k\}_i}\Big(r_i^{c-k+1} \left\{ \begin{matrix} \omega_i,\ \omega_i',\ 0 \\ 1 \end{matrix} \right\}_i + \omega_i'\{c-k+1\}_i\Big),$$

$$\left\{ \begin{matrix} v_i,\ v_i',\ \boldsymbol{c} \\ n \end{matrix} \right\}_i = \prod_{k=1}^{n} \frac{1}{\{k\}_i}\Big(r_i^{c-k+1} \left\{ \begin{matrix} v_i,\ v_i',\ 0 \\ 1 \end{matrix} \right\}_i + v_i'\{c-k+1\}_i\Big).$$

Let $\boldsymbol{A} = \boldsymbol{Q}\Big[r,\ s,\ r^{-1},\ s^{-1}, \frac{1}{\{n\}_i},\ i \in I,\ n > 0\Big]$.

Definition 2.5. *The* $\boldsymbol{A}$*-subalgebras* $U_{\boldsymbol{A}}$ *of the two parameter groups* $U_{r,s}(\mathfrak{g})$ *with 1 generated by* $e_i,\ f_i,\ \omega_i^{\pm 1},\ \omega_i'^{\pm 1},\ v_i^{\pm 1},\ v_i'^{\pm 1},\ \left\{ \begin{matrix} v_i,\ v_i',\ 0 \\ 1 \end{matrix} \right\}_i$ *and* $\left\{ \begin{matrix} \omega_i,\ \omega_i',\ 0 \\ 1 \end{matrix} \right\}_i$ $(i \in I)$, *is called the* $\boldsymbol{A}$*-form of* $U_{r,s}(\mathfrak{g})$.

We denote by $U_{\mathbf{A}}^{+}$ (respectively $U_{\mathbf{A}}^{-}$), the $\mathbf{A}$-subalgebra of $U_{r,s}(\mathfrak{g})$ with 1 generated by e_i (respectively f_i), for all $i \in I$, and by $U_{\mathbf{A}}^{0}$, the $\mathbf{A}$-subalgebra of $U_{r,s}(\mathfrak{g})$ with 1 generated by $\omega_i^{\pm 1}, \omega_i'^{\pm 1}, v_i^{\pm 1}, v_i'^{\pm 1}$, $\left\{ \begin{matrix} v_i, v_i', 0 \\ 1 \end{matrix} \right\}_i$ and $\left\{ \begin{matrix} \omega_i,\ \omega_i',\ 0 \\ 1 \end{matrix} \right\}_i$ $(i \in I)$.

Lemma 2.6. For $i,\ j \in I,\ c \in \mathbf{Z}$, and $n \in \mathbf{Z}_{\geq 0}$, we have

$$e_j \left\{ \begin{matrix} \omega_i,\ \omega_i',\ c \\ n \end{matrix} \right\}_i = r_i^{a_{ij}} s_i^{a_{ij}} \left\{ \begin{matrix} \omega_i,\ \omega_i',\ c-a_{ij} \\ n \end{matrix} \right\}_i e_j,\ i<j, \tag{2.6}$$

$$e_j \left\{ \begin{matrix} \omega_i,\ \omega_i',\ c \\ n \end{matrix} \right\}_i = \left\{ \begin{matrix} \omega_i,\ \omega_i',\ c-a_{ij} \\ n \end{matrix} \right\}_i e_j,\ i>j, \tag{2.7}$$

$$e_i \left\{ \begin{matrix} \omega_i,\ \omega_i',\ c \\ n \end{matrix} \right\}_i = r_i s_i \left\{ \begin{matrix} \omega_i,\ \omega_i',\ c-2 \\ n \end{matrix} \right\}_i e_i, \tag{2.8}$$

$$e_j \left\{ \begin{matrix} v_i,\ v_i',\ c \\ n \end{matrix} \right\}_i = \left\{ \begin{matrix} v_i,\ v_i',\ c \\ n \end{matrix} \right\}_i e_j,\ if\ i \neq j,$$

$$e_i \left\{ \begin{matrix} v_i,\ v_i',\ c \\ n \end{matrix} \right\}_i = r_i s_i \left\{ \begin{matrix} v_i,\ v_i',\ c-2 \\ n \end{matrix} \right\}_i e_i,$$

$$\left\{ \begin{matrix} \omega_i,\ \omega_i',\ c \\ n \end{matrix} \right\}_i f_j = r_i^{a_{ij}} s_i^{a_{ij}} f_j \left\{ \begin{matrix} \omega_i,\ \omega_i',\ c-a_{ij} \\ n \end{matrix} \right\}_i,\ i<j,$$

$$\left\{ \begin{matrix} \omega_i,\ \omega_i',\ c \\ n \end{matrix} \right\}_i f_j = f_j \left\{ \begin{matrix} \omega_i,\ \omega_i',\ c-a_{ij} \\ n \end{matrix} \right\}_i,\ i>j,$$

$$\left\{ \begin{matrix} \omega_i,\ \omega_i',\ c \\ n \end{matrix} \right\}_i f_i = r_i s_i f_i \left\{ \begin{matrix} \omega_i,\ \omega_i',\ c-2 \\ n \end{matrix} \right\}_i,$$

$$\left\{ \begin{matrix} v_i,\ v_i',\ c \\ n \end{matrix} \right\}_i f_j = f_j \left\{ \begin{matrix} v_i,\ v_i',\ c \\ n \end{matrix} \right\}_i,\ i \neq j,$$

$$\left\{ \begin{matrix} v_i,\ v_i',\ c \\ n \end{matrix} \right\}_i f_i = r_i s_i f_i \left\{ \begin{matrix} v_i,\ v_i',\ c-2 \\ n \end{matrix} \right\}_i,$$

$$e_i f_j = f_j e_i,\ i \neq j,$$

$$e_i f_i^n = f_i^n e_i + f_i^{n-1} \sum_{t=0}^{n-1} \left\{ \begin{matrix} \omega_i,\ \omega_i',\ -2t \\ 1 \end{matrix} \right\}_i .$$

Proof. We only check the identities (2.6), (2.7) and (2.8), since the other identities can be shown analogously or directly.

$$
\begin{aligned}
(2.9)\qquad & e_j \left\{ \begin{matrix} \omega_i,\ \omega_i',\ c \\ n \end{matrix} \right\}_i \\
= \ & e_j \prod_{k=1}^{n} \frac{1}{\{k\}_i} \left(r_i^{c-k+1} \left\{ \begin{matrix} \omega_i,\ \omega_i',\ 0 \\ 1 \end{matrix} \right\}_i + \omega_i'\{c-k+1\}_i \right) \\
= \ & \prod_{k=1}^{n} \frac{1}{\{k\}_i} \left(r_i^{c-k+1} e_j \frac{\omega_i - \omega_i'}{r_i - s_i} + e_j \omega_i' \{c-k+1\}_i \right) \\
= \ & \prod_{k=1}^{n} \frac{1}{\{k\}_i} \Big(r_i^{c-k+1} \frac{\omega_i r^{-\langle j,i\rangle} s^{\langle i,j\rangle} - \omega_i' r^{\langle i,j\rangle} s^{-\langle j,i\rangle}}{r_i - s_i} e_j \\
& + r^{\langle i,j\rangle} s^{-\langle j,i\rangle} \{c-k+1\}_i \omega_i' e_j \Big).
\end{aligned}
$$

By the definition of $\langle i,j\rangle$, we obtain

$$
r^{-\langle j,i\rangle} s^{\langle i,j\rangle} = \begin{cases} s_i^{a_{ij}}, & i<j, \\ r_i^{-1} s_i, & i=j, \\ r_i^{-a_{ij}}, & i>j, \end{cases}
$$

and

$$
r^{\langle i,j\rangle} s^{-\langle j,i\rangle} = \begin{cases} r_i^{a_{ij}}, & i<j, \\ r_i s_i^{-1}, & i=j, \\ s_i^{-a_{ij}}, & i>j. \end{cases}
$$

Therefore, if $i<j$, then the right part of (2.9) is equal to

$$
\begin{aligned}
(2.10)\qquad & \prod_{k=1}^{n} \frac{1}{\{k\}_i} \Big(r_i^{c-k+1} s_i^{a_{ij}} \frac{\omega_i - \omega_i'}{r_i - s_i} + r_i^{c-k+1} \frac{s_i^{a_{ij}} - r_i^{a_{ij}}}{r_i - s_i} \omega_i' \\
& + \{c-k+1\}_i r_i^{a_{ij}} \omega_i' \Big) e_j \\
& = \prod_{k=1}^{n} \frac{1}{\{k\}_i} \Big(r_i^{c-k+1} s_i^{a_{ij}} \frac{\omega_i - \omega_i'}{r_i - s_i} - r_i^{c-k+1} \{a_{ij}\}_i \omega_i' \\
& + \{c-k+1\}_i r_i^{a_{ij}} \omega_i' \Big) e_j.
\end{aligned}
$$

By Lemma 2.4, the right hand side of (2.10) is equal to

$$\begin{aligned}
&\prod_{k=1}^{n} \frac{1}{\{k\}_i} \left(r_i^{c-k+1} s_i^{a_{ij}} \frac{\omega_i - \omega_i^{'}}{r_i - s_i} + \{c - a_{ij} - k + 1\}_i r_i^{a_{ij}} s_i^{a_{ij}} \omega_i^{'} \right) e_j \\
= \quad & r_i^{a_{ij}} s_i^{a_{ij}} \prod_{k=1}^{n} \frac{1}{\{k\}_i} \left(r_i^{c-k-a_{ij}+1} \frac{\omega_i - \omega_i^{'}}{r_i - s_i} + \{c - a_{ij} - k + 1\}_i \omega_i^{'} \right) e_j \\
= \quad & r_i^{a_{ij}} s_i^{a_{ij}} \left\{ \begin{array}{c} \omega_i, \ \omega_i^{'}, \ c - a_{ij} \\ n \end{array} \right\}_i e_j.
\end{aligned}$$

If $i > j$, then the right hand side of (2.9) is equal to

$$\begin{aligned}
(2.11) \qquad & \prod_{k=1}^{n} \frac{1}{\{k\}_i} \Big(r_i^{c-k+1} r_i^{-a_{ij}} \frac{\omega_i - \omega_i^{'}}{r_i - s_i} + r_i^{c-k+1} \frac{r_i^{-a_{ij}} - s_i^{-a_{ij}}}{r_i - s_i} \omega_i^{'} \\
& + \{c - k + 1\}_i s_i^{-a_{ij}} \omega_i^{'} \Big) e_j \\
= \quad & \prod_{k=1}^{n} \frac{1}{\{k\}_i} \Big(r_i^{c-k-a_{ij}+1} \frac{\omega_i - \omega_i^{'}}{r_i - s_i} - r_i^{c-k+1} \{-a_{ij}\}_i \omega_i^{'} \\
& + \{c - k + 1\}_i s_i^{-a_{ij}} \omega_i^{'} \Big) e_j.
\end{aligned}$$

Using Lemma 2.4, the right hand side of (2.11) is equal to

$$\begin{aligned}
&\prod_{k=1}^{n} \frac{1}{\{k\}_i} \left(r_i^{c-k-a_{ij}+1} \frac{\omega_i - \omega_i^{'}}{r_i - s_i} + \{c - a_{ij} - k + 1\}_i \omega_i^{'} \right) e_j \\
= \quad & \left\{ \begin{array}{c} \omega_i, \ \omega_i^{'}, \ c - a_{ij} \\ n \end{array} \right\}_i e_j.
\end{aligned}$$

If $i = j$, then the right part of (2.9) is equal to

$$\begin{aligned}
(2.12) \qquad & \prod_{k=1}^{n} \frac{1}{\{k\}_i} \Big(r_i^{c-k} s_i \frac{\omega_i - \omega_i^{'}}{r_i - s_i} + r_i^{c-k+1} \frac{r_i^{-1} s_i - r_i s_i^{-1}}{r_i - s_i} \omega_i^{'} \\
& + \{c - k + 1\}_i r_i s_i^{-1} \omega_i^{'} \Big) e_i \\
= \quad & \prod_{k=1}^{n} \frac{1}{\{k\}_i} \Big(r_i^{c-k} s_i \frac{\omega_i - \omega_i^{'}}{r_i - s_i} - r_i^{c-k} s_i^{-1} \{2\}_i + r_i s_i^{-1} \\
& \times \{c - k + 1\}_i \omega_i^{'} \Big) e_i.
\end{aligned}$$

In view of Lemma 2.4, the right hand side of (2.12) is equal to

$$\begin{aligned} & r_is_i\prod_{k=1}^{n}\frac{1}{\{k\}_i}\left(r_i^{c-k-1}\frac{\omega_i-\omega_i^{'}}{r_i-s_i}+\{c-k-1\}_i\omega_i^{'}\right)e_i \\ = \quad & r_is_i\left\{\begin{array}{c}\omega_i,\ \omega_i^{'},\ c-2\\ n\end{array}\right\}_i e_i. \end{aligned}$$

Thus, we have shown that the identities (2.6), (2.7) and (2.8) hold. □

As an immediate consequence of Lemma 2.6, we get the triangular decomposition of the algebra $U_{\mathbf{A}}$.

Theorem 2.7. $U_{\boldsymbol{A}}\simeq U_{\boldsymbol{A}}^{-}\otimes U_{\boldsymbol{A}}^{0}\otimes U_{\boldsymbol{A}}^{+}$.

A $U_{r,s}(\mathfrak{g})$-module $V^{r,s}$ is said to be diagonalizable, if it admits a weight space decomposition $V^{r,s}=\oplus_{\lambda\in P}V_{\lambda}^{r,s}$, where

$$\begin{aligned} V_{\lambda}^{r,s}= \quad & \{v\in V^{r,s}\mid \omega_iv=r^{\langle\lambda,\alpha_i\rangle}s^{-\langle\alpha_i,\lambda\rangle}v,\ \omega_i^{'}v=r^{-\langle\alpha_i,\lambda\rangle}s^{\langle\lambda,\alpha_i\rangle}v, \\ & v_iv=r^{\langle\lambda,\alpha_i\rangle^{'}}s^{-\langle\alpha_i,\lambda\rangle^{'}}v, v_i^{'}v=r^{-\langle\alpha_i,\lambda\rangle^{'}}s^{\langle\lambda,\alpha_i\rangle^{'}}v,\ i\in I\}. \end{aligned}$$

A diagonalizable $U_{r,s}(\mathfrak{g})$-module $V^{r,s}$ is a highest weight module with highest weight $\lambda\in P$, if there is a nonzero vector $v_\lambda\in V^{r,s}$ satisfying (i) $e_iv_\lambda=0$, for all $i\in I$, and (ii) $\omega_iv=r^{\langle\lambda,\alpha_i\rangle}s^{-\langle\alpha_i,\lambda\rangle}v,\ \omega_i^{'}v=r^{-\langle\alpha_i,\lambda\rangle}s^{\langle\lambda,\alpha_i\rangle}v,\ v_iv=r^{\langle\lambda,\alpha_i\rangle^{'}}s^{-\langle\alpha_i,\lambda\rangle^{'}}v,\ v_i^{'}v=r^{-\langle\alpha_i,\lambda\rangle^{'}}s^{\langle\lambda,\alpha_i\rangle^{'}}v\ (i\in I)$, and (iii) $V^{r,s}=U_{r,s}(\mathfrak{g})v_\lambda$. The vector v_λ is called a highest weight vector. Note that by Theorem 2.7, condition (iii) can be replaced by (iv) $V^{r,s}=U_{r,s}^{-}(\mathfrak{g})v_\lambda$.

Assume $\lambda\in P$ and let $V^{r,s}$ be a highest weight module over $U_{r,s}(\mathfrak{g})$ with highest weight λ and highest weight vector v_λ. Define the $\mathbf{A}$-form $V^{\mathbf{A}}$ of $V^{r,s}$ to be the $U_{\mathbf{A}}$-submodule of $V^{r,s}$, generated by v_λ, that is, $V^{\mathbf{A}}=U_{\mathbf{A}}v_\lambda$.

Proposition 2.8. $V^{\boldsymbol{A}}=U_{\boldsymbol{A}}^{-}v_\lambda$.

Proof. According to Theorem 2.7, every element μ of $U_{\mathbf{A}}$ can be expressed as a sum of monomials of the form $\mu^{-}\mu^{0}\mu^{+}$, where $\mu^0\in U_{\mathbf{A}}^0,\ \mu^{\pm}\in U_{\mathbf{A}}^{\pm}$. By definition, $\mu^{+}v_\lambda=0$, unless $\mu^{+}\in\mathbf{A}$. For $i\in I,\ c\in\mathbf{Z},\ n\in\mathbf{Z}_{\geq 0}$, we have

$$\left\{\begin{array}{c}\omega_i,\ \omega_i^{'},\ c\\ n\end{array}\right\}_i v_\lambda=\prod_{k=1}^{n}\frac{\omega_ir_i^{c-k+1}-\omega_i^{'}s_i^{c-k+1}}{r_i^k-s_i^k}v_\lambda$$

$$= \prod_{k=1}^{n} \frac{r_i^{c-k+1} r^{\langle \lambda, \alpha_i \rangle} s^{-\langle \alpha_i, \lambda \rangle} - s_i^{c-k+1} s^{\langle \lambda, \alpha_i \rangle} r^{-\langle \alpha_i, \lambda \rangle}}{r_i^k - s_i^k} v_\lambda$$
$$= (r_i s_i)^{-a} \prod_{k=1}^{n} \frac{r_i^{c-k+1+a+b} - s_i^{c-k+1+a+b}}{r_i^k - s_i^k} v_\lambda,$$

where $a = \frac{\langle \alpha_i, \lambda \rangle}{t_i}$, $b = \frac{\langle \lambda, \alpha_i \rangle}{t_i}$, which implies that $\left\{ \begin{matrix} \omega_i, \ \omega_i^{'}, \ c \\ n \end{matrix} \right\}_i v_\lambda \in \mathbf{A} v_\lambda$. Similarly, $\left\{ \begin{matrix} v_i, \ v_i^{'}, \ c \\ n \end{matrix} \right\}_i v_\lambda \in \mathbf{A} v_\lambda$. Thus, $\mu^- \mu^0 \mu^+ v_\lambda \in \mathbf{A} \mu^- v_\lambda \subset U_{\mathbf{A}}^- v_\lambda$. It follows that $V^{\mathbf{A}} = U_{\mathbf{A}}^- v_\lambda$. □

Let J be the ideal of $\mathbf{A} = \mathbf{Q}\left[r, \ s, \ r^{-1}, \ s^{-1}, \frac{1}{\{n\}_i}, \ i \in I, \ n > 0\right]$, generated by $r-1, \ s-1$. Then there is an isomorphism of fields $\mathbf{A}/J \cong Q$, given by $f + J \mapsto f(1,1)$, for $f \in \mathbf{A}$. Define $U = Q \otimes_{\mathbf{A}} U_{\mathbf{A}}$. Then, $U \cong U_{\mathbf{A}}/JU_{\mathbf{A}}$.

Consider the natural maps $U_{\mathbf{A}} \longrightarrow U_{\mathbf{A}}/JU_{\mathbf{A}} \cong U$. We note that $r \to 1, \ s \to 1$. The passage from $U_{\mathbf{A}}$ to U under these maps is referred to as taking the classical limit. We denote by $\tilde{u}$ the images of the elements $u \in U_{\mathbf{A}}$. We also denote by $\tilde{h}_i$ and $\tilde{d}_i$ for the images of $\left\{ \begin{matrix} \omega_i, \ \omega_i^{'}, 0 \\ 1 \end{matrix} \right\}_i$ and $\left\{ \begin{matrix} v_i, \ v_i^{'}, 0 \\ 1 \end{matrix} \right\}_i$, respectively.

Lemma 2.9. *For the algebra U, we have $\tilde{\omega}_i = \tilde{\omega_i^{'}}, \ \tilde{v}_i = \tilde{v_i^{'}}$, for all $i \in I$.*

Proof. For $U_{\mathbf{A}}$, we have $\omega_i - \omega_i^{'} = (r_i - s_i) \left\{ \begin{matrix} \omega_i, \ \omega_i^{'}, 0 \\ 1 \end{matrix} \right\}_i$. Letting $r \to 1, \ s \to 1$, we get $\tilde{\omega}_i = \tilde{\omega_i^{'}}$ in U. Analogously, $\tilde{v}_i = \tilde{v_i^{'}}$ in U. □

Let R be the ideal of U, generated by the elements $\tilde{\omega}_i - 1, \ \tilde{v}_i - 1 \ (i \in I)$, and set $U_1 = U/R$. We call that U_1 is the classical limit of $U_{r,s}(\mathfrak{g})$. By abuse of notation, we will also use $\tilde{u} \in U_1$ for the image of the element $u \in U_{\mathbf{A}}$ in U_1, $\tilde{h}_i$ and $\tilde{d}_i$ for the images of $\left\{ \begin{matrix} \omega_i, \ \omega_i^{'}, 0 \\ 1 \end{matrix} \right\}_i$ and $\left\{ \begin{matrix} v_i, \ v_i^{'}, 0 \\ 1 \end{matrix} \right\}_i$ in U_1, respectively. Then, $\tilde{\omega}_i = \tilde{\omega_i^{'}} = \tilde{\omega}_i = \tilde{\omega_i^{'}} = 1$ in U_1 by Lemma 2.9. Hence, U_1 is generated by the elements $\tilde{e}_i, \ \tilde{f}_i, \ \tilde{h}_i, \ \tilde{d}_i$.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of the Kac-Moody algebra $\mathfrak{g}$ with the generators $e_i, \ f_i, \ h_i$ and d_i $(i \in I)$ (see [13]).

Theorem 2.10. $U_1 \cong U(\mathfrak{g})$ as Hopf algebras.

Proof. Since $\left[\frac{\omega_i-\omega_i^{'}}{r_i-s_i}, \frac{\omega_j-\omega_j^{'}}{r_j-s_j}\right] = 0$, for $i,\ j \in I$, we have $[\tilde{h_i}, \tilde{h_j}] = 0$. Similarly, $[\tilde{h_i}, \tilde{d_j}] = [\tilde{d_i}, \tilde{d_j}] = 0$. Due to (2.3),

$$\begin{aligned} & \frac{\omega_i - \omega_i^{'}}{r_i - s_i} e_j - e_j \frac{\omega_i - \omega_i^{'}}{r_i - s_i} \\ = & \frac{(1 - r^{-\langle j,i\rangle} s^{\langle i,j\rangle})\omega_i e_j - (1 - r^{\langle i,j\rangle} s^{-\langle j,i\rangle})\omega_i^{'} e_j}{r_i - s_i} \\ = & \frac{r_i^{1+a_{ij}} - s_i^{1+a_{ij}}}{(r_i - s_i) r_i s_i}\omega_i e_j + \frac{(r_i s_i - r_i^{1+a_{ij}})(\omega_i - \omega_i^{'})}{(r_i - s_i) r_i s_i} e_j. \end{aligned}$$

Letting $r \to 1,\ s \to 1$, we have $\tilde{h_i}\tilde{e_j} - \tilde{e_j}\tilde{h_i} = a_{ij}\tilde{e_j}$ in U_1. Analogously, in U_1 we have

$$\tilde{h_i}\tilde{f_j} - \tilde{f_j}\tilde{h_i} = -a_{ij}\tilde{f_j},\ \tilde{d_i}\tilde{e_j} - \tilde{e_j}\tilde{d_i} = \delta_{ij}\tilde{e_j},\ \tilde{d_i}\tilde{f_j} - \tilde{f_j}\tilde{d_i} = -\delta_{ij}\tilde{f_j}.$$

Hence, we have

$$\sum_{m+n=1-a_{ij}} (-1)^m \frac{\tilde{e_i}^m}{m!}\tilde{e_j}\frac{\tilde{e_i}^n}{n!} = 0, \text{ if } a_{ii} = 2 \text{ and } i \neq j,$$

$$[\tilde{e_i}, \tilde{e_j}] = [\tilde{f_i}, \tilde{f_j}] = 0, \text{ if } a_{ij} = 0,$$

for all $i,\ j \in I$. That is, the generators of U_1 satisfy the defining relations of $U(\mathfrak{g})$. Put $\varphi : U_1 \longrightarrow U(\mathfrak{g})$, where $\varphi(\tilde{e_i}) = e_i,\ \varphi(\tilde{f_i}) = f_i,\ \varphi(\tilde{d_i}) = d_i$ and $\varphi(\tilde{h_i}) = h_i$, for all $i \in I$. It is easy to check that φ is an isomorphism of algebras. According to the comultiplication, counit and antipode of $U_{r,s}(\mathfrak{g})$, we have in $U_{\mathbf{A}}$,

$$\Delta(\left\{ \begin{matrix} \omega_i,\ \omega_i^{'}, 0 \\ 1 \end{matrix} \right\}_i) = \left\{ \begin{matrix} \omega_i,\ \omega_i^{'}, 0 \\ 1 \end{matrix} \right\}_i \otimes \omega_i + \omega_i^{'} \otimes \left\{ \begin{matrix} \omega_i,\ \omega_i^{'}, 0 \\ 1 \end{matrix} \right\}_i,$$

$$\Delta(\left\{ \begin{matrix} v_i,\ v_i^{'}, 0 \\ 1 \end{matrix} \right\}_i) = \left\{ \begin{matrix} v_i,\ v_i^{'}, 0 \\ 1 \end{matrix} \right\}_i \otimes v_i + v_i^{'} \otimes \left\{ \begin{matrix} v_i,\ v_i^{'}, 0 \\ 1 \end{matrix} \right\}_i,$$

$$\varepsilon(\left\{ \begin{matrix} \omega_i,\ \omega_i^{'}, 0 \\ 1 \end{matrix} \right\}_i) = \varepsilon(\left\{ \begin{matrix} v_i,\ v_i^{'}, 0 \\ 1 \end{matrix} \right\}_i) = 0,$$

$$S(\left\{ \begin{matrix} \omega_i,\ \omega_i^{'}, 0 \\ 1 \end{matrix} \right\}_i) = -\left\{ \begin{matrix} \omega_i,\ \omega_i^{'}, 0 \\ 1 \end{matrix} \right\}_i,$$

$$S(\left\{ \begin{matrix} v_i,\ v_i^{'}, 0 \\ 1 \end{matrix} \right\}_i) = -\left\{ \begin{matrix} v_i,\ v_i^{'}, 0 \\ 1 \end{matrix} \right\}_i.$$

Hence, by tensoring these mappings with the identity map on $\mathbf{A}/J$, we get mapping on U, which we denote by $\tilde{\Delta},\ \tilde{\varepsilon},\ \tilde{S}$, giving U a Hopf algebra structure. In particular, in the algebra U_1, we have

$$\tilde{\Delta}(\tilde{h_i}) = \tilde{h_i} \otimes 1 + 1 \otimes \tilde{h_i},\ \tilde{\Delta}(\tilde{d_i}) = \tilde{d_i} \otimes 1 + 1 \otimes \tilde{d_i},$$
$$\tilde{\Delta}(\tilde{e_i}) = \tilde{e_i} \otimes 1 + 1 \otimes \tilde{e_i},\ \tilde{\Delta}(\tilde{f_i}) = \tilde{f_i} \otimes 1 + 1 \otimes \tilde{f_i},$$

and $\tilde{\varepsilon}(\tilde{X}) = 0,\ \tilde{S}(\tilde{X}) = -\tilde{X}$, for any $\tilde{X} = \tilde{e_i},\ \tilde{f_i},\ \tilde{h_i},\ \tilde{d_i}\ (i \in I)$. Therefore, the algebra U_1 has a Hopf algebra structure $(\tilde{\Delta},\ \tilde{\varepsilon},\ \tilde{S})$. It follows that $\varphi : U_1 \longrightarrow U(\mathfrak{g})$ is an isomorphism of Hopf algebras. □

3. The equitable presentation for the subalgebra of two parameter quantum groups $U_{r,s}(\mathfrak{g})$ associated with the Kac-Moody algebra $\mathfrak{g}$

We now concentrate on the subalgebra $U_{r,s}^{b^-}$ of $U_{r,s}(\mathfrak{g})$ generated by the elements $f_i,\ \omega_i^{\pm 1},\ \omega_i'^{\pm 1},\ v_i^{\pm 1}$ and $v_i'^{\pm 1}$, for all $i \in I$. From the Definition 2.1, we can see that the generators $f_i,\ \omega_i^{\pm 1},\ \omega_i'^{\pm 1},\ v_i^{\pm 1}$ and $v_i'^{\pm 1}$ of $U_{r,s}^{b^-}$ play very different roles. In the following, we will introduce a presentation for $U_{r,s}^{b^-}$ whose generators are on a more equal footing. The presentation has the attractive feature that all of its generators act semisimply on finite dimensional irreducible $U_{r,s}^{b^-}$-modules associated with a Kac-Moody algebras $\mathfrak{g}$. This result for the case of one parameter quantum group has been proved in [2].

Theorem 3.1. The $\mathbf{K}$-algebra $U_{r,s}^{b^-}$ is isomorphic to the unital associative $\mathbf{K}$-algebra $U^{b^-}$with generators $X_i^{\pm 1},\ X_i'^{\pm 1},\ Y_i^{\pm 1},\ Y_i'^{\pm 1},\ Z_i\ (i \in I)$ and the following relations:

$$X_i^{\pm 1},\ X_j'^{\pm 1},\ Y_l^{\pm 1} \text{ and } Y_k'^{\pm 1} \text{ are commutative with each other },$$
$$X_i^{\pm 1} X_i^{\mp 1} = X_i'^{\pm 1} X_i'^{\mp 1} = 1,\ Y_i^{\pm 1} Y_i^{\mp 1} = Y_i'^{\pm 1} Y_i'^{\mp 1} = 1,$$
$$X_i Z_j - r^{-\langle j,i\rangle} s^{\langle i,j\rangle} Z_j X_i = (1 - r^{-\langle j,i\rangle} s^{\langle i,j\rangle}) X_i X_j',$$
$$X_i' Z_j - r^{\langle i,j\rangle} s^{-\langle j,i\rangle} Z_j X_i' = (1 - r^{\langle i,j\rangle} s^{-\langle j,i\rangle}) X_i' X_j', \tag{3.1}$$
$$Y_i Z_j - r^{-\langle j,i\rangle'} s^{\langle i,j\rangle'} Z_j Y_i = (1 - r^{-\langle j,i\rangle'} s^{\langle i,j\rangle'}) Y_i Y_j',$$
$$Y_i' Z_j - r^{\langle i,j\rangle'} s^{-\langle j,i\rangle'} Z_j Y_i' = (1 - r^{\langle i,j\rangle'} s^{-\langle j,i\rangle'}) Y_i' Y_j',$$

$$(3.2)\qquad \sum_{k=0}^{1-a_{ij}}(-1)^k \left(\begin{array}{c}1-a_{ij}\\ k\end{array}\right)_p c_{ij}^{(k)} Z_i^k Z_j Z_i^{1-a_{ij}-k}$$

$$= X_i^{'1-a_{ij}} X_j' \prod_{l=0}^{-a_{ij}} \left(1-p^l r^{\langle j,i\rangle} s^{-\langle i,j\rangle}\right),\ if\ i\neq j,$$

where $c_{ij}^{(k)}=(r_is_i^{-1})^{\frac{k(k-1)}{2}} r^{k\langle j,i\rangle}s^{-k\langle i,j\rangle}$ for $i\neq j$, and $p=r_is_i^{-1}$.
An isomorphism $\varphi: U^{b^-}\longrightarrow U_{r,s}^{b^-}$ *is defined as follows:*

$$X_i^{\pm 1}\to \omega_i^{\pm 1},$$

$$X_i^{'\pm 1}\to \omega_i^{'\pm 1},$$

$$Y_i^{\pm 1}\to v_i^{\pm 1},$$

$$Y_i^{'\pm 1}\to v_i^{'\pm 1},$$

$$Z_i\to \omega_i^{'}+f_i(r_i-s_i).$$

The inverse of φ *is* $\psi: U_{r,s}^{b^-}\longrightarrow U^{b^-}$*:*

$$\omega_i^{\pm 1}\to X_i^{\pm 1},$$

$$\omega_i^{'\pm 1}\to X_i^{'\pm 1},$$

$$v_i^{\pm 1}\to Y_i^{\pm 1},$$

$$v_i^{'\pm 1}\to Y_i^{'\pm 1},$$

$$f_i\to (Z_i-X_i^{'})(r_i-s_i)^{-1}.$$

Before we give the proof of Theorem 3.1, we first give some useful identities.

Lemma 3.2 (11)**.** For integer $m\geq k\geq 1$,

$$\left[\begin{array}{c}m\\ k\end{array}\right]_{q_i}+q_i^{m+1}\left[\begin{array}{c}m\\ k-1\end{array}\right]_{q_i}=q_i^k\left[\begin{array}{c}m+1\\ k\end{array}\right]_{q_i}\quad (1\leq i\leq n).$$

Lemma 3.3 (7)**.** For integer $m \geq 0$, and indeterminate λ,

$$\sum_{k=0}^{m}(-1)^k \left[\begin{array}{c} m \\ k \end{array}\right]_{q_i} \lambda^k = \sum_{k=0}^{m}(-1)^k \left(\begin{array}{c} m \\ k \end{array}\right)_{q_i^2} q_i^{k(k-m)} \lambda^k = \prod_{s=0}^{m-1}(1-\lambda q_i^{1-m+2s}).$$

By induction and relation (2.4), we have

$$\left(\omega_i' + (r_i - s_i)f_i\right)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right)_{r_i s_i^{-1}} (r_i - s_i)^k f_i^k \omega_i'^{n-k}. \tag{3.3}$$

Now, we give the proof of Theorem 3.1.

Proof. We only prove that φ keeps the equality (3.2) (it is easy to check that φ preserves the other identities). We denote $p = r_i s_i^{-1}$, $h = r^{\langle j,i\rangle} s^{-\langle i,j\rangle}$ and $g = r^{\langle i,j\rangle} s^{-\langle j,i\rangle}$. By the definition of $\langle i,j\rangle$, we obtain

$$gh = p^{a_{ij}} \ (i \neq j). \tag{3.4}$$

Applying φ to the left hand side of (3.2), we obtain

$$\varphi\left(\sum_{k=0}^{1-a_{i,j}} (-1)^k \left(\begin{array}{c} 1-a_{ij} \\ k \end{array}\right)_p c_{ij}^{(k)} Z_i^k Z_j Z_i^{1-a_{ij}-k}\right)$$

to be equal to

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left(\begin{array}{c} 1-a_{i,j} \\ k \end{array}\right)_p p^{\frac{k(k-1)}{2}} h^k (\omega_i' + f_i(r_i - s_i))^k \times(\omega_j' + f_j(r_j - s_j))(\omega_i' + f_i(r_i - s_i))^{1-a_{ij}-k}. \tag{3.5}$$

Observe that (3.5) is equal to

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left(\begin{array}{c} 1-a_{i,j} \\ k \end{array}\right)_p p^{\frac{k(k-1)}{2}} h^k (\omega_i' + f_i(r_i - s_i))^k \omega_j' \times(\omega_i' + f_i(r_i - s_i))^{1-a_{ij}-k} \tag{3.6}$$

plus $r_j - s_j$ times

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1-a_{i,j} \\ k \end{pmatrix}_p p^{\frac{k(k-1)}{2}} h^k (\omega_i^{'} + f_i(r_i - s_i))^k \times f_j(\omega_i^{'} + f_i(r_i - s_i))^{1-a_{ij}-k}. \tag{3.7}$$

Applying φ to the right hand side of (3.2), we get

$$\begin{aligned} &\varphi\left(X_i^{'1-a_{ij}} X_j^{'} \prod_{l=0}^{-a_{ij}} \left(1 - p^l r^{\langle j,i\rangle} s^{-\langle i,j\rangle}\right)\right) \\ = \;& \omega_i^{'1-a_{ij}} \omega_j^{'} \prod_{l=0}^{-a_{ij}} \left(1 - p^l r^{\langle j,i\rangle} s^{-\langle i,j\rangle}\right). \end{aligned} \tag{3.8}$$

For $i \neq j$, we prove that the expressions (3.6), (3.8) are equal and (3.7) is equal to 0. Explicitly, by identity (3.3), (3.6) is equal to

$$\sum_{k=0}^{1-a_{ij}} \sum_{\eta=0}^{k} \sum_{t=0}^{1-a_{ij}-k} (-1)^k \begin{pmatrix} 1-a_{ij} \\ k \end{pmatrix}_p \begin{pmatrix} k \\ \eta \end{pmatrix}_p \begin{pmatrix} 1-a_{ij}-k \\ t \end{pmatrix}_p \times (r_i - s_i)^{\eta+t} p^{\frac{k(k-1)}{2}} h^k f_i^{\eta} \omega_i^{'k-\eta} \omega_j^{'} f_i^t \omega_i^{'1-a_{ij}-k-t}. \tag{3.9}$$

Taking the account of relation (2.4), (3.8) is equal to

$$\sum_{k=0}^{1-a_{ij}} \sum_{\eta=0}^{k} \sum_{t=0}^{1-a_{ij}-k} (-1)^k \begin{pmatrix} 1-a_{ij} \\ k \end{pmatrix}_p \begin{pmatrix} k \\ \eta \end{pmatrix}_p \begin{pmatrix} 1-a_{ij}-k \\ t \end{pmatrix}_p \times p^{\frac{k(k-1)}{2}} h^{t+k} p^{t(k-\eta)} (r_i - s_i)^{\eta+t} f_i^{\eta+t} \omega_i^{'1-a_{ij}-t-\eta} \omega_j^{'}. \tag{3.10}$$

In the above equality, let $u = k + t$ and $v = \eta + t$. We find that for $0 \leq v \leq 1 - a_{ij}$, the coefficient of $f_i^v \omega_i^{'1-a_{ij}-v} \omega_j^{'}$ in (3.10) is equal to

$$\begin{pmatrix} 1-a_{ij} \\ v \end{pmatrix}_p (r_i - s_i)^v \tag{3.11}$$

times

$$\sum_{t=0}^{v} (-1)^t \begin{pmatrix} v \\ t \end{pmatrix}_p p^{\frac{t(1-2v+t)}{2}} \tag{3.12}$$

times

$$\sum_{u=v}^{1-a_{ij}}(-1)^u \begin{pmatrix} 1-a_{ij} \\ u \end{pmatrix}_p p^{\frac{u^2-u}{2}} h^u. \tag{3.13}$$

For $v=0$, the expression (3.12) is equal to 1; for $v\neq 0$, by Lemma 3.3, the expression (3.12) is equal to

$$\prod_{l=0}^{v-1}(1-p^{1-v+l})=0.$$

When $v=0$, (3.11) is equal to 1 and (3.13) is equal to

$$\sum_{u=0}^{1-a_{ij}}(-1)^u \begin{pmatrix} 1-a_{ij} \\ u \end{pmatrix}_p p^{\frac{u^2-u}{2}} h^u. \tag{3.14}$$

According to Lemma 3.3, (3.14) is equal to

$$\prod_{l=0}^{-a_{ij}}(1-p^l h).$$

Hence, (3.8) is equal to (3.9). Subsequently, we show that (3.7) is equal to zero. By identity (3.3), (3.7) is equal to

(3.15)

$$\sum_{k=0}^{1-a_{ij}}\sum_{\eta=0}^{k}\sum_{t=0}^{1-a_{ij}-k}(-1)^k \begin{pmatrix} 1-a_{ij} \\ k \end{pmatrix}_p \begin{pmatrix} k \\ \eta \end{pmatrix}_p \begin{pmatrix} 1-a_{ij}-k \\ t \end{pmatrix}_p$$
$$\times(r_i-s_i)^{\eta+t}p^{\frac{k(k-1)}{2}}h^k f_i^\eta \omega_i'^{k-\eta} f_j f_i^t \omega_i'^{1-a_{ij}-k-t}.$$

Using (2.4), (3.15) is equal to

(3.16)

$$\sum_{k=0}^{1-a_{ij}}\sum_{\eta=0}^{k}\sum_{t=0}^{1-a_{ij}-k}(-1)^k \begin{pmatrix} 1-a_{ij} \\ k \end{pmatrix}_p \begin{pmatrix} k \\ \eta \end{pmatrix}_p \begin{pmatrix} 1-a_{ij}-k \\ t \end{pmatrix}_p$$
$$p^{\frac{k(k-1)}{2}}h^k g^{k-\eta} p^{t(k-\eta)}(r_i-s_i)^{\eta+t} f_i^\eta f_j f_i^t \omega_i'^{1-a_{ij}-t-\eta}.$$

Then, by identity (3.4), (3.16) is equal to

(3.17)

$$\sum_{k=0}^{1-a_{ij}}\sum_{\eta=0}^{k}\sum_{t=0}^{1-a_{ij}-k}(-1)^k \begin{pmatrix} 1-a_{ij} \\ k \end{pmatrix}_p \begin{pmatrix} k \\ \eta \end{pmatrix}_p \begin{pmatrix} 1-a_{ij}-k \\ t \end{pmatrix}_p$$
$$\times p^{\frac{k(k-1)}{2}}h^\eta p^{(t+a_{ij})(k-\eta)}(r_i-s_i)^{\eta+t} f_i^\eta f_j f_i^t \omega_i'^{1-a_{ij}-t-\eta}.$$

In the expression (3.17), let $\eta+t=v, k-\eta=u$. Then, $0\le v\le 1-a_{ij}$ and the coefficient of $f_i^{\eta}f_jf_i^{v-\eta}{\omega_i'}^{1-a_{ij}-v}$ in (3.17) is equal to

$$(-1)^{\eta}\begin{pmatrix}1-a_{ij}\\ v\end{pmatrix}_p\begin{pmatrix}v\\ \eta\end{pmatrix}_p(r_i-s_i)^vp^{\frac{\eta^2-\eta}{2}}h^{\eta} \tag{3.18}$$

times

$$\sum_{u=0}^{1-a_{ij}-v}(-1)^u\begin{pmatrix}1-a_{ij}-v\\ u\end{pmatrix}_p p^{\frac{2uv+2a_{ij}u+u^2-u}{2}}. \tag{3.19}$$

For $v=1-a_{ij}$, the expression (3.19) is equal to 1 and for $v\neq 1-a_{ij}$, by Lemma 3.3, the expression (3.19) is equal to

$$\prod_{l=0}^{-v-a_{ij}}(1-p^{a_{ij}+v+l})=0.$$

When $v=1-a_{ij}$, (3.18) is equal to

$$(-1)^{\eta}\begin{pmatrix}1-a_{ij}\\ \eta\end{pmatrix}_p(r_i-s_i)^{1-a_{ij}}p^{\frac{\eta^2-\eta}{2}}h^{\eta}.$$

Therefore, (3.17) is equal to

$$(r_i-s_i)^{1-a_{ij}}\sum_{\eta=0}^{1-a_{ij}}(-1)^{\eta}\begin{pmatrix}1-a_{ij}\\ \eta\end{pmatrix}_p p^{\frac{\eta(\eta-1)}{2}}h^{\eta}f_i^{\eta}f_jf_i^{1-a_{ij}-\eta}. \tag{3.20}$$

By identity (2.5), (3.3) is equal to zero, that is, (3.7) is equal to zero. Hence, we have proved that φ preserves the equality (3.2). The proof of ψ being a homomorphism from $U^{b^-}(\mathfrak{g})$ to U^{b^-} is similar to the proof of φ. One routinely verifies that these maps are inverses. □

Definition 3.4. *The presentation given in the above theorem is called the equitable presentation for $U_{r,s}^{b^-}$. We call $X_i^{\pm1}$, ${X_i'}^{\pm1}$, $Y_i^{\pm1}$, ${Y_i'}^{\pm1}$ and Z_i $(i\in I)$ the equitable generators.*

For notational convenience, we identify the copy of $U_{r,s}^{b^-}$ given in Definition 2.1 with the copy of U^{b^-} given in Theorem 3.1, via the isomorphism given in Theorem 3.1.

The Hopf algebra structure of $U_{r,s}^{b^-}$ looks as follows in terms of the equitable generators.

Theorem 3.5. The comultiplication Δ satisfies

$$\Delta(X_i) = X_i \otimes X_i,\ \Delta(X_i^{'}) = X_i^{'} \otimes X_i^{'},$$
$$\Delta(Y_i) = Y_i \otimes Y_i,\ \Delta(Y_i^{'}) = Y_i^{'} \otimes Y_i^{'},$$
$$\Delta(Z_i) = (Z_i - 1) \otimes X_i^{'} + 1 \otimes Z_i.$$

The counit ε satisfies

$$\varepsilon(X_i) = 1,\ \varepsilon(X_i^{'}) = 1,\ \varepsilon(Y_i) = 1,\ \varepsilon(Y_i^{'}) = 1,\ \varepsilon(Z_i) = 1.$$

The antipode S satisfies

$$S(X_i) = X_i^{-1},\ S(X_i^{'}) = X_i^{'-1},\ S(Y_i) = Y_i^{-1},$$
$$S(Y_i^{'}) = Y_i^{'-1},\ S(Z_i) = 1 + X_i^{'-1} - Z_i X_i^{'-1}.$$

Proof. One readily checks that the theorem holds. □

Corollary 3.6. The following holds in $U_{r,s}^{b-}$, for all $i \neq j$:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1-a_{i,j} \\ k \end{pmatrix}_{r_i s_i^{-1}} (r_i s_i^{-1})^{\frac{k(k-1)}{2}} (r^{\langle j,i\rangle} s^{-\langle i,j\rangle})^k \times Z_i^k f_j Z_i^{1-a_{ij}-k} = 0.$$

Proof. This is proved in Theorem 3.1 for identity (3.7). □

When $\mathfrak{g}$ is a Kac-Moody algebra, the corresponding generators of two parameter quantum groups $U_{r,s}(\mathfrak{g})$ are only $e_i,\ f_i,\ \omega_i^{\pm 1},\ \omega_i^{'\pm 1}$ $(i = 1,\ 2, \ldots,\ n)$ and the equitable generators of $U_{r,s}^{b-}$ are only $X_i^{\pm 1},\ X_i^{'\pm 1},\ Z_i$. In what follows, we assume that $\mathfrak{g}$ is a Kac-Moody algebra. We will show that the equitable generators $X_i^{\pm 1},\ X_i^{'\pm 1}$ and Z_i $(i = 1,\ 2, \ldots,\ n)$ of $U_{r,s}^{b-}$ act semisimply on finite dimensional irreducible $U_{r,s}^{b-}$-module V when $\mathfrak{g}$ is a Kac-Moody algebra. In fact, this also holds, when $\mathfrak{g}$ is a generalized Kac-Moody algebra. For convenience, we only consider the case when $\mathfrak{g}$ is a Kac-Moody algebra. In the following, we set $I_0 = \{i = 1,\ 2, \ldots,\ n\}$.

Definition 3.7. *Let V be a finite dimensional irreducible $U_{r,s}^{b-}$-module. We say $v \in V$ is a weight vector, if v is a common eigenvector, for $\omega_i^{\pm 1},\ \omega_i^{'\pm 1}$ $(i \in I_0)$.*

Lemma 3.8. *Let V be a finite dimensional irreducible $U_{r,s}^{b-}$-module. Then, V has a basis consisting of weight vectors.*

Proof. Since $\omega_i,\ \omega_i^{'}$ $(i \in I_0)$ commuting with each other on V, there exists $v \in V$ such that v is a common eigenvector for $\omega_i,\ \omega_i^{'}$ $(i \in I_0)$. Observing identity (2.4), we know that $f_j v$ $(1 \leq j \leq n)$ are weight vectors. Clearly, $U_{r,s}^{b^-}v$ is a nonzero $U_{r,s}^{b^-}$-submodule of V with a basis consisting of weight vectors. Since V is an irreducible $U_{r,s}^{b^-}$-module, we obtain $U_{r,s}^{b^-}v = V$, which implies that V has a basis consisting of weight vectors. □

Lemma 3.9. *Let V be a finite dimensional irreducible $U_{r,s}^{b^-}$-module. Then, the action of $\omega_i,\ \omega_i^{'}$ $(i \in I_0)$ on V is semisimple. Moreover, the eigenvalues of ω_i on V are contained in the set $\{b_i r^{\langle \alpha,i\rangle} s^{-\langle i,\alpha\rangle} | \alpha \in Q\}$, while eigenvalues of $\omega_i^{'}$ are contained in $\{b_i^{'} r^{-\langle i,\alpha\rangle} s^{\langle \alpha,i\rangle} | \alpha \in Q\}$, for some $b_i,\ b_i^{'} \in \boldsymbol{K}^{\times}$.*

Proof. Using Lemma 3.8 and (2.4), the results follow easily. □

Let V be a finite dimensional irreducible $U_{r,s}^{b^-}$-module. Choose $b_i,\ b_i^{'} \in \mathbf{K}^{\times}$ such that the eigenvalues of ω_i(respectively $\omega_i^{'}$) are contained in the set $\{b_i r^{\langle \alpha,i\rangle} s^{-\langle i,\alpha\rangle} | \alpha \in Q\}$ (respectively $\{b_i^{'} r^{-\langle i,\alpha\rangle} s^{\langle \alpha,i\rangle} | \alpha \in Q\}$). Since V is finite dimensional, there exist integers m_i M_i $(i = 1, 2)$, with $m_i < M_i$, $(i = 1,\ 2)$, such that the set of distinct eigenvalues of ω_i on V is contained in

$$\{b_i r_i^{m_1} s_i^{-m_2},\ b_i r_i^{m_1+1} s_i^{-m_2-1}, \cdots, b_i r_i^{M_1} s_i^{-M_2}\},$$

and the set of distinct eigenvalues of $\omega_i^{'}$ on V is contained in

$$\{b_i^{'} r_i^{-m_2} s_i^{m_1},\ b_i^{'} r_i^{-m_2-1} s_i^{m_1+1}, \cdots, b_i^{'} r_i^{-M_2} s_i^{M_1}\}.$$

Choose $m_i,\ M_i,\ (i = 1, 2)$, so that $M_1 - m_1$ and $M_2 - m_2$ are minimal. Let

$$\theta_i = \frac{M_1 - m_1}{2},\ \gamma_i = \frac{M_1 - m_1}{2},$$

$$a_i = b_i r_i^{\frac{M_1+m_1}{2}} s_i^{-\frac{M_2+m_2}{2}},\ a_i^{'} = b_i^{'} r_i^{-\frac{M_2+m_2}{2}} s_i^{\frac{M_1+m_1}{2}}.$$

Then, the eigenvalues of ω_i are contained in the set

$$\{a_i r_i^{-\theta_i} s_i^{\gamma_i},\ a_i r_i^{-\theta_i+1} s_i^{\gamma_i-1},\ \cdots, a_i r_i^{\theta_i} s_i^{-\gamma_i}\},$$

while the eigenvalues of $\omega_i^{'}$ are contained in the set

$$\{a_i^{'} r_i^{\gamma_i} s_i^{-\theta_i},\ a_i^{'} r_i^{\gamma_i-1} s_i^{-\theta_i+1},\ \cdots, a_i^{'} r_i^{-\gamma_i} s_i^{\theta_i}\}.$$

Let $a=(a_1,a_2,\cdots,a_n),\ a^{'}=(a^{'}_1,a^{'}_2,\cdots,a^{'}_n),\ \theta=(\theta_1,\theta_2,\cdots,\theta_n)$ and $\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_n)$. Then, the sequence $a,a^{'}\in\mathbf{K}^n$ are called the types of V, and $\theta,\gamma\in(\frac{1}{2}\mathbf{Z})^n$ are called the shapes of V. We can change the variables and choose $\varepsilon=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)\in\{-1,1\}^n,\ \varepsilon^{'}=(\varepsilon^{'}_1,\varepsilon^{'}_2,\cdots,\varepsilon^{'}_n)\in\{-1,1\}^n$ such that the types of V are $\varepsilon\in\{-1,1\}^n,\varepsilon^{'}\in\{-1,1\}^n$.

In what follows, we fix a finite dimensional irreducible $U^{b-}_{r,s}$-module V of types $\varepsilon\in\{-1,1\}^n,\ \varepsilon^{'}\in\{-1,1\}^n$ and shapes of $\theta,\gamma\in(\frac{1}{2}\mathbf{Z})^n$. By a decomposition of V we mean a sequence of subspaces of V whose direct sum is V.

Lemma 3.10. For $1\le i\le n$, there exists a decomposition $\{U_i(l,\eta)\}$ $(0\le l\le 2\theta_i,\ 0\le\eta\le 2\gamma_i)$ of V satisfying:

(*i*) $U_i(l,\eta)=\{v\in V\mid \omega_i v=\varepsilon_i r_i^{l-\theta_i}s_i^{\gamma_i-\eta}v,\ \omega^{'}_i v=\varepsilon^{'}_i r_i^{\gamma_i-\eta}s_i^{l-\theta_i}v\}$, for $0\le l\le 2\theta_i,\ 0\le\eta\le 2\gamma_i$,

(*ii*) $U_i(0,0)\neq 0$ and $U_i(2\theta_i,2\gamma_i)\neq 0$.

Moreover, $X_i,\ X^{'}_i$ are semisimple on V.

Proof. Clearly, $X_i,\ X^{'}_i$ act semisimple on V. According to the definition of type and shape in the above, the eigenvalues of X_i are contained in the set

$$\{\varepsilon_i r_i^{-\theta_i}s_i^{\gamma_i},\ \varepsilon_i r_i^{-\theta_i+1}s_i^{\gamma_i-1},\ \cdots,\varepsilon_i r_i^{\theta_i}s_i^{-\gamma_i}\},$$

while the eigenvalues of $X^{'}_i$ are contained in the set

$$\{\varepsilon^{'}_i r_i^{\gamma_i}s_i^{-\theta_i},\ \varepsilon^{'}_i r_i^{\gamma_i-1}s_i^{-\theta_i+1},\ \cdots,\varepsilon^{'}_i r_i^{-\gamma_i}s_i^{\theta_i}\}.$$

For any $0\le l\le 2\theta_i$ and $0\le\eta\le 2\gamma_i$, if $\varepsilon_i r_i^{l-\theta_i}s_i^{\gamma_i-\eta}$ is an eigenvalue of X_i, and $\varepsilon^{'}_i r_i^{\gamma_i-\eta}s_i^{l-\theta_i}$ is an eigenvalue of $X^{'}_i$, let $U_i(l,\eta)$ be the eigenspace associated with these eigenvalues. For other cases, let $U_i(l,\eta)=0$. Then, (*i*) holds. Since θ_i,γ_i are choosen to be minimal, both $r_i^{-\theta_i}s_i^{\gamma_i}$ and $r_i^{\theta_i}s_i^{-\gamma_i}$ are eigenvalues of X_i, while $r_i^{\gamma_i}s_i^{-\theta_i}$ and $r_i^{-\gamma_i}s_i^{\theta_i}$ are eigenvalues of $X^{'}_i$. Therefore, we get (*ii*). □

For convenience, we define $U_i(l,\eta)\neq 0$, for $0\le l\le 2\theta_i,\ 0\le\eta\le 2\gamma_i$, and otherwise, $U_i(l,\eta)=0$.

Lemma 3.11. For $1\le i\le n$, let the decomposition $\{U_i(l,\eta)\}$ $(0\le l\le 2\theta_i,\ 0\le\eta\le 2\gamma_i)$ be as in Lemma 3.10. Then, for $1\le j\le n,\ 0\le l\le 2\theta_i$ and $0\le\eta\le 2\gamma_i$, we have

(*i*) $\omega_j U_i(l,\eta) = U_i(l,\eta),\ \omega_j^{'} U_i(l,\eta) = U_i(l,\eta)$.
(*ii*) $f_j U_i(l,\eta) \subseteq U_i(l - \sum_{i>j} a_{ij} - 1,\ \eta - \sum_{i<j} a_{ij} - 1)$.

Proof. According to the identies (2.1) and (2.2), we have $\omega_j U_i(l,\eta) = U_i(l,\eta),\ \omega_j^{'} U_i(l,\eta) = U_i(l,\eta)$. For each $v \in U_i(l,\eta)$, by (2.4), we have

$$\begin{aligned}\omega_i f_j v &= r^{-\langle j,i\rangle} s^{\langle i,j\rangle} f_j \omega_i v = r^{-\langle j,i\rangle} s^{\langle i,j\rangle} f_j \varepsilon_i r_i^{l-\theta_i} s_i^{\gamma_i-\eta} v\\ &= \varepsilon_i r^{-\sum_{i>j} t_i a_{ij} - t_i} s^{\sum_{i<j} t_i a_{ij} + t_i} r_i^{l-\theta_i} s_i^{\gamma_i-\eta} f_j v\\ &= r_i^{l-\theta_i-(\sum_{i>j} t_i a_{ij}+1)} s_i^{\gamma_i-\eta+(\sum_{i<j} t_i a_{ij}+1)} f_j v,\end{aligned}$$

and

$$\begin{aligned}\omega_i^{'} f_j v &= r^{\langle i,j\rangle} s^{-\langle j,i\rangle} f_j \omega_i^{'} v = r^{\langle i,j\rangle} s^{-\langle j,i\rangle} \varepsilon_i^{'} r_i^{\gamma_i-\eta} s_i^{l-\theta_i} f_j v\\ &= \varepsilon_i^{'} r^{\sum_{i<j} t_i a_{ij} + t_i} s^{-\sum_{i>j} t_i a_{ij} - t_i} r_i^{\gamma_i-\eta} s_i^{l-\theta_i} f_j v\\ &= \varepsilon_i^{'} r_i^{\gamma_i-\eta+(\sum_{i<j} t_i a_{ij}+1)} s_i^{l-\theta_i-(\sum_{i>j} t_i a_{ij}+1)} f_j v.\end{aligned}$$

Therefore, $f_j U_i(l,\eta) \subseteq U_i(l - \sum_{i>j} a_{ij} - 1,\ \eta - \sum_{i<j} a_{ij} - 1)$. □

Lemma 3.12. For $1 \le i \le n$, let the decomposition $\{U_i(l,\eta)\}$ $(0 \le l \le 2\theta_i,\ 0 \le \eta \le 2\gamma_i)$ be as in Lemma 3.10. Then, for $1 \le j \le n,\ 0 \le l \le 2\theta_i$ and $0 \le \eta \le 2\gamma_i$, we have

$$(Z_i - \varepsilon_i^{'} r_i^{\gamma_i-\eta} s_i^{l-\theta_i} I) U_i(l,\eta) \subseteq U_i(l-1,\eta-1).$$

Proof. Using $Z_i = \omega_i^{'} + f_i(r_i - s_i)$, for any $v \in U_i(l,\eta)$,

$$(\omega_i^{'} + f_i(r_i - s_i) - \varepsilon_i^{'} r_i^{\gamma_i-\eta} s_i^{l-\theta_i} I) v = (r_i - s_i) f_i v \in U_i(l-1,\eta-1).$$

Thus, $(Z_i - \varepsilon_i^{'} r_i^{\gamma_i-\eta} s_i^{l-\theta_i} I) U_i(l,\eta) \subseteq U_i(l-1,\eta-1)$. □

Theorem 3.13. For $1 \le i \le n$, there exists a decomposition $\{V_i(l,\eta)\}$ $(0 \le l \le 2\theta_i,\ 0 \le \eta \le 2\gamma_i)$ of V such that

$$(3.21)\qquad (Z_i - \varepsilon_i^{'} r_i^{\gamma_i-\eta} s_i^{l-\theta_i} I) V_i(l,\eta) = 0\ (0 \le l \le 2\theta_i,\ 0 \le \eta \le 2\gamma_i).$$

Moreover, Z_i acts semisimple on V.

Proof. By Lemma 3.12, we have

$$\Pi_{0 \le l \le 2\theta_i,\ 0 \le \eta \le 2\gamma_i} (Z_i - \varepsilon_i^{'} r_i^{\gamma_i-\eta} s_i^{l-\theta_i} I) = 0$$

on V. Since $\varepsilon_i^{'} r_i^{\gamma_i-\eta} s_i^{l-\theta_i}$ are mutually distinct, we obtain that Z_i acts semisimple on V with eigenvalues contained in the set $\{\varepsilon_i^{'} r_i^{\gamma_i-\eta} s_i^{l-\theta_i} | 0 \le l \le 2\theta_i,\ 0 \le \eta \le 2\gamma_i\}$. Set $V_i(l,\eta) = \{v \in V | Z_i v = \varepsilon_i^{'} r_i^{\gamma_i-\eta} s_i^{l-\theta_i} v\}$. Hence, the result follows. □

For convenience, we define $V_i(l,\eta)\neq 0$, for $0\leq l\leq 2\theta_i,\ 0\leq\eta\leq 2\gamma_i$, and otherwise, $V_i(l,\eta)=0$.

Proposition 3.14. For $1\leq i\leq n$, let the decomposition $\{V_i(l,\eta)\}$ $(0\leq l\leq 2\theta_i,\ 0\leq\eta\leq 2\gamma_i)$ be as in Theorem 3.13. Then, for $0\leq l\leq 2\theta_i,\ 0\leq\eta\leq 2\gamma_i$, we have

(*i*) $(X_i^{'-1}-\varepsilon_i^{'}r_i^{\eta-\gamma_i}s_i^{\theta_i-l}I)V_i(l,\eta)\subseteq V_i(l-1,\eta-1)$,

(*ii*) $f_jV_i(l,\eta)\subseteq\oplus_{m=0}^{-a_{ij}}V_i(l-m+1+\sum_{i<j}a_{ij},\ \eta-m+1+\sum_{i>j}a_{ij})$.

Proof. (*i*) Using (3.1), we obtain

$$(Z_iX_i^{'-1}-\varepsilon_i^{'}r_is_i^{-1}X_i^{'-1}Z_i-I+r_is_i^{-1}I)V_i(l,\eta)=0. \tag{3.22}$$

According to (3.21),

$$Z_iV_i(l,\eta)=\varepsilon_i^{'}r_i^{\gamma_i-\eta}s_i^{l-\theta_i}V_i(l,\eta). \tag{3.23}$$

Combining (3.22) and (3.23), the following holds:

$$\begin{aligned}0&=(Z_iX_i^{'-1}-\varepsilon_i^{'}r_i^{\gamma_i+1-\eta}s_i^{l-\theta_i-1}X_i^{'-1}-\varepsilon_i^{'}r_i^{\eta-\gamma_i}s_i^{\theta_i-l}Z_i+r_is_i^{-1}I)V_i(l,\eta)\\&=(Z_i-\varepsilon_i^{'}r_i^{\gamma_i+1-\eta}s_i^{l-\theta_i-1}I)(X_i^{'-1}-\varepsilon_i^{'}r_i^{\eta-\gamma_i}s_i^{\theta_i-l}I)V_i(l,\eta).\end{aligned}$$

Therefore,

$$(X_i^{'-1}-\varepsilon_i^{'}r_i^{\eta-\gamma_i}s_i^{\theta_i-l}I)V_i(l,\eta)\subseteq V_i(l-1,\eta-1).$$

(*ii*) Choosing any $v\in V_i(l,\eta)$, by Corollary 3.6, we obtain

$$\begin{aligned}&\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{pmatrix}1-a_{i,j}\\k\end{pmatrix}_{r_is_i^{-1}}(r_is_i^{-1})^{\frac{k(k-1)}{2}}(r^{\langle j,i\rangle}s^{-\langle i,j\rangle})^k\\&\times Z_i^kf_jZ_i^{1-a_{ij}-k}v\\=&\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{pmatrix}1-a_{i,j}\\k\end{pmatrix}_{r_is_i^{-1}}(r_is_i^{-1})^{\frac{k(k-1)}{2}}(r^{\langle j,i\rangle}s^{-\langle i,j\rangle})^k\\&\times Z_i^kf_j(\varepsilon_i^{'}r_i^{\gamma_i-\eta}s_i^{l-\theta_i})^{1-a_{ij}-k}v\\=&(-1)^{1-a_{ij}}(r^{\langle j,i\rangle}s^{-\langle i,j\rangle})^{1-a_{ij}}(r_is_i^{-1})^{\frac{(1-a_{ij})a_{ij}}{2}}\\&\times\prod_{m=0}^{-a_{ij}}(Z_i-\varepsilon_i^{'}r_i^{\gamma_i-\eta+m-1-\sum_{i>j}a_{ij}}s_i^{l-\theta_i-m+1+\sum_{i<j}a_{ij}})f_jv\\=&\ 0.\end{aligned}$$

Then, the result follows by using (3.21). □

Remark 3.15. *Let* $U_{r,s}^{b+}$ *be the subalgebra of* $U_{r,s}(\mathfrak{g})$ *generated by the elements* e_i, $\omega_i^{\pm1}$, $\omega_i'^{\pm1}$, $v_i^{\mp1}$ *and* $v_i'^{\pm1}$, *for all* $i \in I_0$. *We can also give an equitable presentation for* $U_{r,s}^{b+}$.

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