*Bulletin of the Iranian Mathematical Society Vol.* 39 *No.* 2 (*2013*), *pp 259-269.*

# OPTIMAL CONVEX COMBINATIONS BOUNDS OF CENTROIDAL AND HARMONIC MEANS FOR LOGARITHMIC AND IDENTRIC MEANS

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Communicated by Behzad Djafari-Rouhani

Abstract. We find the greatest values $\alpha_1$ and $\alpha_2$, and the least values $\beta_1$ and $\beta_2$ such that the inequalities $\alpha_1 C(a,b)+(1-\alpha_1)H(a,b) < L(a,b) < \beta_1 C(a,b)+(1-\beta_1)H(a,b)$ and $\alpha_2 C(a,b)+(1-\alpha_2)H(a,b) < I(a,b) < \beta_2 C(a,b)+(1-\beta_2)H(a,b)$ hold for all $a,b>0$ with $a\neq b$. Here, $C(a,b)$, $H(a,b)$, $L(a,b)$, and $I(a,b)$ are the centroidal, harmonic, logarithmic, and identric means of two positive numbers $a$ and $b$, respectively.

## 1. Introduction

The logarithmic mean $L(a,b)$and identric mean $I(a,b)$ of two positive real numbers $a$ and $b$ with $a \neq b$ are defined by

$$L(a,b)=\frac{a-b}{\log a-\log b} \quad \text{and} \quad I(a,b)=\frac{1}{e}\left(\frac{a^a}{b^b}\right)^{1/(a-b)},$$

respectively. In the recent past, both mean values have been the subject of intensively research. In particular, many remarkable inequalities for $L(a,b)$ and $I(a,b)$ can be found in the literature [1, 3, 4, 6, 8, 14, 15, 17, 19–24, 26–28, 30–34]. In [22, 24, 34] inequalities between the logarithmic mean, identric mean, and classical arithmetic-geometric mean of Gauss

MSC(2010): Primary: 26E60; Secondary: 26D15.
Keywords: Logarithmic mean, identric mean, centroidal mean, harmonic mean.
Received: 11 June 2011, 15 January 2012.
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are proved. The ratio of identric means leads to the weighted geometric mean

$$\frac{I(a^2,b^2)}{I(a,b)} = (a^a b^b)^{1/(a+b)}$$

which has been investigated in [20, 21, 27]. It might be surprising that the logarithmic mean has applications in physics, economics, and even in meteorology [10,16,18]. In [10] the authors study a variant of Jensen's functional equation involving the logarithmic mean, which appears in a heat conduction problem. A reprsentation of $L(a,b)$ as an infinite product and an iterative algorithm for computing the logarithmic mean as the common limit of two sequences of special geometric and arithmetic means are given in [8]. In [7,9] it is shown that $L(a,b)$ can be expressed in terms of Gauss' hypergeometric function ${}_2F_1$. And, in [9] the authors prove that the reciprocal of the logarithmic mean is strictly totally positive, that is, every $n \times n$ determinant with elements $1/L(a_i, b_i)$, where $0 < a_1 < a_2 < \ldots < a_n$ and $0 < b_1 < b_2 < \ldots < b_n$, is positive for all $n \geq 1$.

The one-parameter mean value family

$$L_r(a,b) = \left(\frac{a^r - b^r}{r(a-b)}\right)^{1/(r-1)} \qquad (r \neq 0, 1; a, b > 0, a \neq b)$$

is known as Stolarsky's generalized logarithmic mean [11–13,29,32]. The limit relations

$$\lim_{r\to 0} L_r(a,b) = L(a,b) \qquad \text{and} \qquad \lim_{r\to 1} L_r(a,b) = I(a,b)$$

reveal that $L_r(a,b)$ can be defined for all real parameters $r$, such that the logarithmic and identric means become members of this family. Historical remarks on these and related mean values can be found in [12].

The integral formula

$$L_r(a,b) = \exp\left(\frac{1}{r-1}\int_1^r \frac{1}{t}\log I(a^t, b^t)dt\right)$$

(see [32]), shows that the identric mean plays a "central role" [12, p. 209] in $L_r(a,b)$. The arithmetic and geometric means of $a$ and $b$, $A(a,b) = (a+b)/2$ and $G(a,b) = \sqrt{ab}$ also belong to $L_r(a,b)$. Indeed, we have $L_{-1}(a,b) = G(a,b)$ and $L_2(a,b) = A(a,b)$. Since $r \to L_r(a,b)(a \neq b)$ is strictly increasing on $\mathbb{R}$ (see [32]), we obtain

$$G(a,b) < L(a,b) < I(a,b) < A(a,b) \quad (a,b > 0, a \neq b). \tag{1.1}$$

It is shown that these inequalities can be applied to get some interesting results on Euler's number $e$. In [8, 12, 19] the authors present the bounds for logarithmic and identric means in terms of geometric and arithmetic means as follows:

$$\sqrt[3]{G^2(a,b)A(a,b)} < L(a,b) < \frac{2G(a,b)+A(a,b)}{3}$$

and

$$I(a,b) > \frac{G(a,b)+2A(a,b)}{3}$$

for all $a,b>0$ with $a \neq b$.

The following companion of (1.1) provides inequalities for the geometric and arithmetic means of $L$ and $I$. A proof can be found in [4].

$$\sqrt{G(a,b)A(a,b)} < \sqrt{L(a,b)I(a,b)} < \frac{1}{2}(L(a,b)+I(a,b)) < \frac{1}{2}(G(a,b)+A(a,b))$$

for all $a,b>0$ with $a \neq b$.

The power mean of order $r$ of the positive real numbers $a$ and $b$ is defined by

$$M_r(a,b) = \left(\frac{a^r+b^r}{2}\right)^{1/r} \ (r \neq 0) \quad \text{and} \quad M_0(a,b) = \sqrt{ab}.$$

The main properties of these means are given in [5]. In particular, the function $r \to M_r(a,b)$ $(a \neq b)$ is continuous and strictly increasing on $\mathbb{R}$. Many authors discussed the relationship of certain means to $M_r$. The following sharp bounds for $L$, $I$, $(LI)^{1/2}$, and $(L+I)/2$ in terms of power means are proved in [3, 4, 6, 14, 15, 17, 33]:

$$M_0(a,b) < L(a,b) < M_{1/3}(a,b), \quad M_{2/3}(a,b) < I(a,b) < M_{\log 2}(a,b),$$

$$M_0(a,b) < \sqrt{L(a,b)I(a,b)} < M_{1/2}(a,b) \quad \text{and} \quad \frac{1}{2}(L(a,b)+I(a,b)) < M_{1/2}(a,b)$$

for all $a,b>0$ with $a \neq b$.

Alzer and Qiu [2] prove that inequalities

$$\alpha A(a,b) + (1-\alpha)G(a,b) < I(a,b) < \beta A(a,b) + (1-\beta)G(a,b)$$

and

$$M_c(a,b) < \frac{1}{2}(L(a,b)+I(a,b))$$

hold for all $a,b>0$ with $a \neq b$ if and only if $\alpha \leq 2/3, \beta \geq 2/e = 0.73575\cdots$ and $c \leq \log 2/(1+\log 2) = 0.40938\cdots$.

Let $H(a,b) = 2ab/(a+b)$, and $C(a,b) = 2(a^2+ab+b^2)/[3(a+b)]$ be the harmonic, and centroidal means of two positive real numbers $a$ and $b$. Then it is well-known that

$$H(a,b) < G(a,b) < L(a,b) < I(a,b) < A(a,b) < C(a,b) \tag{1.2}$$

for $a \neq b$.

It is the aim of this paper to answer the questions: what are the greatest values $\alpha_1$ and $\alpha_2$, and the least values $\beta_1$ and $\beta_2$ such that the inequalities

$$\alpha_1 C(a,b) + (1-\alpha_1)H(a,b) < L(a,b) < \beta_1 C(a,b) + (1-\beta_1)H(a,b)$$

and

$$\alpha_2 C(a,b) + (1-\alpha_2)H(a,b) < I(a,b) < \beta_2 C(a,b) + (1-\beta_2)H(a,b)$$

hold for all $a, b > 0$ with $a \neq b$.

## 2. **Lemmas**

In order to establish our main results we need three lemmas, which we present them in this section.

**Lemma 2.1.** *Let $f(t) = (t^2+4t+1)\log t - 3t^2 + 3$. Then $f(t) > 0$ for $t > 1$.*

*Proof.* Simple computation leads to

$$f(1) = 0, \tag{2.1}$$

$$f'(t) = 2(t+2)\log t - 5t + \frac{1}{t} + 4,$$

$$f'(1) = 0, \tag{2.2}$$

$$f''(t) = 2\log t + \frac{4}{t} - \frac{1}{t^2} - 3,$$

$$f''(1) = 0, \tag{2.3}$$

$$f'''(t) = \frac{2(t-1)^2}{t^3} > 0 \tag{2.4}$$

for $t > 1$

Therefore, Lemma 2.1 follows from inequality (2.4) and equations (2.1)-(2.3). □

**Lemma 2.2.** *Let $g(t) = -[t^3 + (2e-1)t^2 + (2e-1)t + 1]\log t + 2(e-1)t^3 + 2(3-e)t^2 - 2(3-e)t - 2(e-1)$. Then there exists $\lambda > 1$ such that $g(t) > 0$ for $t \in (1, \lambda)$ and $g(t) < 0$ for $t \in (\lambda, +\infty)$.*

*Proof.* Simple computation leads to

$$g(1)=0, \tag{2.5}$$

$$\lim_{t\to+\infty} g(t)=-\infty, \tag{2.6}$$

$$g'(t)=-[3t^2+2(2e-1)t+(2e-1)]\log t+(6e-7)t^2-(6e-13)t-\frac{1}{t}-5,$$

$$g'(1)=0, \tag{2.7}$$

$$\lim_{t\to+\infty} g'(t)=-\infty, \tag{2.8}$$

$$g''(t)=-2[3t+(2e-1)]\log t+(12e-17)t-\frac{2e-1}{t}+\frac{1}{t^2}-5(2e-3),$$

$$g''(1)=0, \tag{2.9}$$

$$\lim_{t\to+\infty} g''(t)=-\infty, \tag{2.10}$$

$$g'''(t)=-6\log t-\frac{2(2e-1)}{t}+\frac{2e-1}{t^2}-\frac{2}{t^3}+12e-23,$$

$$g'''(1)=2(5e-12)>0, \tag{2.11}$$

$$\lim_{t\to+\infty} g'''(t)=-\infty \tag{2.12}$$

and

$$g^{(4)}(t)=-\frac{6}{t^4}(t-1)\left[t^2\left(1-\frac{2(e-2)}{3t}\right)+1\right]<0 \tag{2.13}$$

for $t>1$.

From inequality (2.13) we know that $g'''(t)$ is strictly decreasing in $[1,+\infty)$, then inequality (2.11) and equation (2.12) lead to the conclusion that there exists $\lambda_1>1$ such that $g'''(t)>0$ for $t\in[1,\lambda_1)$ and $g'''(t)<0$ for $t\in(\lambda_1,+\infty)$. Therefore, $g''(t)$ is strictly increasing in $[1,\lambda_1]$ and strictly decreasing in $[\lambda_1,+\infty)$.

It follows from equations (2.9) and (2.10) together with the piecewise monotonicity of $g''(t)$ that there exists $\lambda_2>\lambda_1>1$ such that $g'(t)$ is strictly increasing in $[1,\lambda_2]$ and strictly decreasing in $[\lambda_2,+\infty)$.

From equations (2.7) and (2.8) together with the piecewise monotonicity of $g'(t)$ we clearly see that there exists $\lambda_3>\lambda_2>1$ such that $g(t)$ is strictly increasing in $[1,\lambda_3]$ and strictly decreasing in $[\lambda_3,+\infty)$.

Therefore, Lemma 2.2 follows from equations (2.5) and (2.6) together with the piecewise monotonicity of $g(t)$. $\square$

**Lemma 2.3.** *Let* $h(t) = (5t^3+19t^2+19t+5)\log t - 14t^3 - 6t^2 + 6t + 14$. *Then* $h(t) > 0$ *for* $t > 1$.

*Proof.* Simple computation leads to

$$h(1) = 0, \tag{2.14}$$

$$h'(t) = (15t^2 + 38t + 19)\log t - 37t^2 + 7t + \frac{5}{t} + 25,$$

$$h'(1) = 0, \tag{2.15}$$

$$h''(t) = 2(15t + 19)\log t - 59t + \frac{19}{t} - \frac{5}{t^2} + 45,$$

$$h''(1) = 0, \tag{2.16}$$

$$h'''(t) = 30\log t + \frac{38}{t} - \frac{19}{t^2} + \frac{10}{t^3} - 29,$$

$$h'''(1) = 0 \tag{2.17}$$

and

$$h^{(4)}(t) = \frac{2}{t^4}(t-1)(15t^2 - 4t + 15) > 0 \tag{2.18}$$

for $t > 1$.

Therefore, Lemma 2.3 follows from equations (2.14)-(2.17) and inequality (2.18). □

## 3. Main results

**Theorem 3.1.** *The double inequality*

$$\alpha_1 C(a,b) + (1-\alpha_1)H(a,b) < L(a,b) < \beta_1 C(a,b) + (1-\beta_1)H(a,b) \tag{3.1}$$

*holds for all* $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_1 \leq 0$ *and* $\beta_1 \geq 1/2$.

*Proof.* We first prove that inequality

$$L(a,b) < \frac{1}{2}C(a,b) + \frac{1}{2}H(a,b) \tag{3.2}$$

hold for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = a/b > 1$, then

$$\frac{1}{2}C(a,b) + \frac{1}{2}H(a,b) - L(a,b) = \frac{bf(t)}{3(t+1)\log t}, \tag{3.3}$$

where $f(t)$ is defined as in Lemma 2.1.

Therefore, inequality (3.2) follows from Lemma 2.1 and equation (3.3).

From inequalities (1.2) and (3.2) we clearly see that inequality (3.1) holds for all $a, b > 0$ with $a \neq b$ if $\alpha_1 \leq 0$ and $\beta_1 \geq 1/2$.

Next, we prove that $\alpha_1 = 0$ and $\beta_1 = 1/2$ are the best possible parameters such that inequality (3.1) holds for all $a, b > 0$ with $a \neq b$.

For any $\alpha_1 > 0$, $\beta_1 < 1/2$ and $x > 0$ one has

$$\lim_{x\to+\infty} \frac{\alpha_1 C(1,x) + (1-\alpha_1)H(1,x)}{L(1,x)} = +\infty \tag{3.4}$$

and

$$\begin{aligned} &L(1,1+x) - \beta_1 C(1,1+x) - (1-\beta_1)H(1,1+x) \\ &= \frac{x}{\log(x+1)} - \frac{\beta_1(1+x+x^2/3)}{1+x/2} - (1-\beta_1)\frac{1+x}{1+x/2}. \end{aligned} \tag{3.5}$$

Letting $x \to 0$ and making use of Taylor expansion we get

$$\begin{aligned} &\frac{x}{\log(x+1)} - \frac{\beta_1(1+x+x^2/3)}{1+x/2} - (1-\beta_1)\frac{1+x}{1+x/2} \\ &= [1 + x/2 - x^2/12 + o(x^2)] - \beta_1[1 + x/2 + x^2/12 + o(x^2)] \\ &\quad - (1-\beta_1)[1 + x/2 - x^2/4 + o(x^2)] \\ &= \frac{1}{3}\left(\frac{1}{2} - \beta_1\right)x^2 + o(x^2). \end{aligned} \tag{3.6}$$

Equations (3.4)-(3.6) imply that for any $\alpha_1 > 0$ and $\beta_1 < 1/2$ there exist $X_1 = X_1(\alpha_1) > 1$ and $\delta_1 = \delta_1(\beta_1) > 0$, such that $\alpha_1 C(1,x) + (1-\alpha_1)H(1,x) > L(1,x)$ for $x \in (X_1, +\infty)$ and $L(1,1+x) > \beta_1 C(1,1+x) + (1-\beta_1)H(1,1+x)$ for $x \in (0,\delta_1)$. $\square$

**Theorem 3.2.** *The double inequality*

$$\alpha_2 C(a,b) + (1-\alpha_2)H(a,b) < I(a,b) < \beta_2 C(a,b) + (1-\beta_2)H(a,b) \tag{3.7}$$

*holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq 3/(2e) = 0.551819\cdots$ and $\beta_2 \geq 5/8$.*

*Proof.* We first prove that inequalities

$$I(a,b) > \frac{3}{2e}C(a,b) + (1 - \frac{3}{2e})H(a,b) \tag{3.8}$$

and

$$I(a,b) < \frac{5}{8}C(a,b) + \frac{3}{8}H(a,b) \tag{3.9}$$

hold for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = a/b > 1$, then

$$I(a,b) - \left[\frac{3}{2e}C(a,b) + (1-\frac{3}{2e})H(a,b)\right] = \frac{b}{e}\left[t^{\frac{t}{t-1}} - \frac{t^2+2(e-1)t+1}{t+1}\right] \tag{3.10}$$

and

$$\frac{5}{8}C(a,b) + \frac{3}{8}H(a,b) - I(a,b) = b\left[\frac{5t^2+14t+5}{12(t+1)} - \frac{1}{e}t^{\frac{t}{t-1}}\right]. \tag{3.11}$$

Let

$$G(t) = \frac{t\log t}{t-1} - \log[t^2+2(e-1)t+1] + \log(t+1) \tag{3.12}$$

and

$$H(t) = \log(5t^2+14t+5) - \log(t+1) - \frac{t}{t-1}\log t + 1 - \log 12. \tag{3.13}$$

Then simple computations lead to

$$\lim_{t\to 1} G(t) = \lim_{t\to +\infty} G(t) = 0, \tag{3.14}$$

$$G'(t) = \frac{g(t)}{(t+1)(t-1)^2[t^2+2(e-1)t+1])}, \tag{3.15}$$

$$\lim_{t\to 1} H(t) = 0 \tag{3.16}$$

and

$$H'(t) = \frac{h(t)}{(t+1)(t-1)^2[5t^2+14t+5]}, \tag{3.17}$$

where $g(t)$ and $h(t)$ are defined as in Lemmas 2.2 and 2.3, respectively.

From Lemma 2.2 and equation (3.15) we know that there exists $\lambda > 1$ such that $G(t)$ is strictly increasing in $(1, \lambda]$ and strictly decreasing in $[\lambda, +\infty)$. Then equation (3.14) leads to the conclusion that

$$G(t) > 0 \tag{3.18}$$

for $t > 1$.

Therefore, inequality (3.8) follows from equations (3.10) and (3.12) together with inequality (3.18).

It follows from equations (3.16) and (3.17) together with Lemma 2.3 that

$$H(t) > 0 \tag{3.19}$$

for $t > 1$.

Therefore, inequality (3.9) follows from equations (3.11) and (3.13) together with inequality (3.19).

Next, we prove that $\alpha_2 = 3/(2e)$ and $\beta_2 = 5/8$ are the best possible parameters such that inequality (3.7) holds for all $a, b > 0$ with $a \neq b$.

For any $\alpha_2 > 3/(2e)$, $\beta_2 < 5/8$ and $x > 0$ one has

$$\lim_{x\to+\infty} \frac{\alpha_2 C(1,x) + (1-\alpha_2)H(1,x)}{I(1,x)} = \frac{2e}{3}\alpha_2 > 1 \tag{3.20}$$

and

$$\begin{aligned} &I(1,1+x) - [\beta_2 C(1,1+x) + (1-\beta_2)H(1,1+x)] \\ &= \frac{1}{e}(1+x)^{\,(1+x)/x} - \frac{\beta_2(1+x+x^2/3)}{1+x/2} - (1-\beta_2)\frac{1+x}{1+x/2}. \end{aligned} \tag{3.21}$$

Letting $x \to 0$ and making use of Taylor expansion we get

$$\begin{aligned} &\frac{1}{e}(1+x)^{\,(1+x)/x} - \frac{\beta_2(1+x+x^2/3)}{1+x/2} - (1-\beta_2)\frac{1+x}{1+x/2} \\ &= [1 + x/2 - x^2/24 + o(x^2)] - \beta_2[1 + x/2 + x^2/12 + o(x^2)] \\ &\quad - (1-\beta_2)[1 + x/2 - x^2/4 + o(x^2)] \\ &= \frac{1}{3}(5/8 - \beta_2)x^2 + o(x^2). \end{aligned} \tag{3.22}$$

Inequality (3.20) and equations (3.21) and (3.22) imply that for any $\alpha_2 > 3/(2e)$ and $\beta_2 < 5/8$ there exists $X_2 = X_2(\alpha_2) > 1$ and $\delta_2 = \delta_2(\beta_2) > 0$, such that $\alpha_2 C(1,x) + (1-\alpha_2)H(1,x) > I(1,x)$ for $x \in (X_2, +\infty)$ and $I(1,1+x) > \beta_2 C(1,1+x) + (1-\beta_2)H(1,1+x)$ for $x \in (0, \delta_2)$. □

## Acknowledgments

This work was supported by the Natural Science Foundation of China under Grant 11071069 and the Innovation Team Foundation of the Department of Education of Zhejiang Province under Grant T200924.

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