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GORENSTEIN FLAT AND GORENSTEIN INJECTIVE DIMENSIONS OF SIMPLE MODULES

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Abstract. Let R be a right GF-closed ring with finite left and right Gorenstein global dimension. We prove that if I is an ideal of R such that R/I is a semi-simple ring, then the Gorenstein flat dimension of R/I as a right R-module and the Gorenstein injective dimension of R/I as a left R-module are identical. In particular, we show that for a simple module S over a commutative Gorenstein ring R, the Gorenstein flat dimension of S equals to the Gorenstein injective dimension of S.

1. Introduction

In classical homological algebra, the projective, injective and flat dimensions of modules are important and fundamental research objects. As a generalization of the notion of projective dimension of modules, Auslander and Bridger [1] introduced the G-dimension, denoted by $\text{G-dim}_R(M)$, for every finitely generated R-module M over a two-sided Noetherian ring R. They proved the inequality $\text{G-dim}_R(M) \leq \text{pd}_R(M)$ with equality $\text{G-dim}_R(M) = \text{pd}_R(M)$ when $\text{pd}_R(M)$ is finite.

Several decades later, Enochs and Jenda [8, 9] defined the notion of Gorenstein projective dimension, as an extension of G-dimension of modules that are not necessarily finitely generated, and the Gorenstein

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injective dimension as a dual notion of Gorenstein projective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda and Torrecillas [10] introduced the Gorenstein flat dimension. Some references are [4, 5, 8, 9, 10].

Since then, the so-called Gorenstein homological algebra has been so interesting to many authors. A central topic in this study is to generalize the classical results in homological algebra to their Gorenstein counterpart. For example, the well known Regularity Theorem and Auslander-Buchsbaum Formula have already been proven to be true in the Gorenstein case, see [4]. Another interesting example of this kind, proved by Bennis and Mahdou in [3], is that the left Gorenstein global projective dimension of a ring R equals to the left Gorenstein global injective dimension of R, and the common value of these two invariants is then called left Gorenstein global dimension.

The aim of this paper is to generalize the following results to the Gorenstein case: (1) Over a commutative ring R, a simple R-module is flat if and only if it is injective (see [14, Theorem 1.1]). (2) Over a commutative coherent ring R, the flat dimension of a simple module S and the injective dimension of S are identical (see [6, Lemma 3.1]).

In order to show that the Gorenstein counterpart of the above results also hold true in some cases, we need to assume that the ground ring is right GF-closed and with finite left and right Gorenstein global dimension. Recall that a ring R is called left (resp. right) GF-closed [2] if the class of Gorenstein flat left (resp. right) R-modules is closed under extensions. The class of right GF-closed rings includes strictly the left coherent rings and rings with finite weak dimension by [2, Proposition 2.2]. In fact, with the above assumptions, we can give a more general result:

Theorem 1.1. *If R is a right GF-closed and with finite left and right Gorenstein global dimension, I is an ideal of R such that R/I is a semi-simple ring, then $Gfd_{R^{op}}(R/I) = Gid_R(R/I)$.*

Note that the rings satisfying the conditions in Theorem 1.1 also have finite left and right weak Gorenstein global dimensions. Some familiar examples of rings with these properties are:
(1) Any ring of finite left and right global dimension and any quasi-Frobenius ring.
(2) Any Gorenstein ring R, which is a two-sided Noetherian ring with finite left and right self-injective dimension by [7, Theorem 12.3.1].

(3) Any Noetherian PI Hopf algebra over a field by [13].

As an immediate consequence of Theorem 1.1, we have that if R is a commutative Gorenstein ring, then the Gorenstein flat dimension and Gorenstein injective dimension of a simple R-module S are identical. In particular, S is Gorenstein flat if and only if it is Gorenstein injective. This result generalizes the corresponding results of [14] and [6] in some sense.

Moreover, we prove that if $R \to \Lambda$ is a homomorphism of rings, where R is right GF-closed and with finite left and right Gorenstein global dimension, and ${}_\Lambda E$ is an injective cogenerator for the category of left Λ-modules, then the Gorenstein flat dimension of Λ as a right R-module and the Gorenstein injective dimension of E as a left R-module are identical. As an application, we have that if Λ is an Artinian algebra with the center R whose Gorenstein global dimension is finite, then, as R-modules, the Gorenstein flat dimension of Λ and the Gorenstein injective dimension of $\mathbb{D}(\Lambda)$ are identical, where $\mathbb{D}$ is the usual duality of Λ.

2. **Proof of main results and applications**

We use R-Mod to denote the category of left R-modules and the category of right R-modules is denoted by R^{op}-Mod. Let $\mathcal{GP}_n$ denotes the class of left R-modules whose Gorenstein projective dimensions are at most n. For any $M \in R$-Mod (orR^{op}-Mod), $Hom_Z(M, Q/Z)$ is denoted by M^+, where Z is the additive group of integers and Q is the additive group of rational numbers. For a left R-module, we denote the Gorenstein injective dimension of M by $Gid_R M$, and for a right R module N, we denote the Gorenstein flat dimension of N by $Gfd_{R^{op}} N$.

Now we prove Theorem 1.1.

Proof. We first show that for any left R-module M and a positive integer n, $Ext^n_R(M, R/I) = 0$ if and only if $Tor^R_n(R/I, M) = 0$. Assume that $Ext^n_R(M, R/I) = 0$. We have an isomorphism $Ext^n_R(M, (R/I)^+) \cong (Tor^R_n(R/I, M))^+$ by [7, Theorem 3.2.1]. Since $I(R/I)^+ = 0$, $(R/I)^+$ is a semi-simple left R-module and $(R/I)^+ \cong \oplus_{i\in\Gamma} S_i$, where each S_i is a simple left R-module and Γ is an index set. Since R/I is semi-simple, each S_i is a direct summand of R/I. So $Ext^n_R(M, \prod_{i\in\Gamma} S_i) \cong$

$\prod_{i\in\Gamma} Ext^n_R(M,S_i)=0$. Note that $\prod_{i\in\Gamma} S_i$ is also a semi-simple module, so $(R/I)^+ \cong \oplus_{i\in\Gamma} S_i$ is isomorphic to a direct summand of $\prod_{i\in\Gamma} S_i$. Thus $Ext^n_R(M,(R/I)^+)=0$ and therefore $Tor^R_n(R/I,M)=0$. Conversely, assume that $Tor^R_n(R/I,M)=0$. we have an isomorphism $Ext^n_R(M,(R/I)^{++}) \cong (Tor^R_n((R/I)^+,M))^+$. By a similar argument as above, we have $Tor^R_n((R/I)^+,M)=0$. So $(Tor^R_n((R/I)^+,M))^+=0$. On the other hand, the canonical evaluation homomorphism $R/I \to (R/I)^{++}$ is a monomorphism. Since $I(R/I)^{++}=0$, $(R/I)^{++}$ is a semi-simple left R-module and R/I is isomorphic to a direct summand of $(R/I)^{++}$. Thus $Ext^n_R(M,R/I)=0$.

We now prove that $Gfd_{R^{op}}(R/I) \geq Gid_R(R/I)$. Without loss of generality, suppose that $Gfd_{R^{op}}(R/I)=m<\infty$. Then, for any $i \geq m+1$ and injective left R-module E, we have that $Tor^R_i(R/I,E)=0$ by [2, Theorem 2.8]. Then we obtain that $Ext^i_R(E,R/I)=0$ as above. It follows from [11, Theorem 2.22] that $Gid_R(R/I) \leq m$. This proves $Gfd_{R^{op}}(R/I) \geq Gid_R(R/I)$.

We next prove the converse inequality. Without loss of generality, suppose that $Gid_R(R/I)=n<\infty$. Then, for any $i \geq n+1$ and any injective left R-module E, we have that $Ext^i_R(E,R/I)=0$ by [11, Theorem 2.22]. We obtain that $Tor^R_i(R/I,E)=0$ as above. It follows from [2, Theorem 2.8] that $Gfd_{R^{op}}(R/I) \leq n$. This proves $Gfd_{R^{op}}(R/I) \leq Gid_R(R/I)$. □

Corollary 2.1. *If R is of finite left and right global dimension or a quasi-Frobenius ring or a Gorenstein ring or a Noetherian PI Hopf algebra over a field, then for any ideal I such that R/I is semi-simple, one has $Gfd_{R^{op}}(R/I)=Gid_R(R/I)$.* □

Recall that R is called a semi-local ring if $R/J(R)$ is a semi-simple ring, where $J(R)$ is the Jacobson radical of R. It is well known that a semiperfect ring (more specially, a left (or right) Artinian ring or a semi-primary ring) is semilocal. By Theorem 1.1, we have the following:

Corollary 2.2. *If R is a semi-local ring which is right GF-closed and of finite left and right Gorenstein global dimension, then $Gfd_{R^{op}}(R/J(R)) = Gid_R(R/J(R))$. Moreover, if R is a semi-primary ring, which is right GF-closed and with finite left and right Gorenstein global dimension, then $Gfd_{R^{op}}(R/J(R))=Gid_R(R/J(R))=l.G.gldim(R)=Gpd_R(R/J(R))$, where $l.G.gldim(R)$ is the left Gorenstein global dimension of R.*

Proof. By [12, p23], we know that every left R-module M over a semi-primary ring has a finite filtration $\{M_i\}_{i=1}^n$ for some fixed positive number n such that M_{i+1}/M_i is semisimple for every $1 \leq i \leq n$. Thus $M \in \mathcal{GP}_n$ if every $M_{i+1}/M_i \in \mathcal{GP}_n$ by [11, Theorem 2.5, Proposition 2.19]. So we have $l.G.gldim(R) = Gpd_R(R/J(R))$ since every simple module is a direct summand of $R/J(R)$. Dually, $Gid_R(R/J(R)) = l.G.gldim(R)$. □

Corollary 2.3. *If R is a commutative ring, which is GF-closed and of finite Gorenstein global dimension, then for any simple R-module S, $Gfd_R(S) = Gid_R(S)$; therefore, S is Gorenstein flat if and only if it is Gorenstein injective. In particular, the assertions hold for a commutative Gorenstein ring.*

Proof. For any simple R-module S, we have $S \cong R/m$ for some maximal ideal m of R. So S is a simple ring, hence the assertion follows from Theorem 1.1. □

As an application of the techniques developed in the proof of Theorem 1.1, we give the following result:

Theorem 2.4. *Let $R \to \Lambda$ be a homomorphism of rings and R is right GF-closed and of finite left and right Gorenstein global dimension. If ${}_\Lambda E$ is an injective cogenerator for Λ-Mod, then $Gfd_{R^{op}}(\Lambda) = Gid_R(E)$.*

Proof. Let Q be any injective left R-module. Then for any $n \geq 1$, we have that $Ext_R^n(Q, E) \cong Ext_R^n(Q, Hom_\Lambda(\Lambda, E)) \cong Hom_\Lambda(Tor_n^R(\Lambda, Q), E)$ by [7, Theorem 3.2.1]. Since ${}_\Lambda E$ is an injective cogenerator for Λ-Mod, $Ext_R^n(Q, E) = 0$ if and only if $Tor_n^R(\Lambda, Q) = 0$. It follows from [2, Theorem 2.8] and [11, Theorem 2.22] that $Gfd_{R^{op}}(\Lambda) = Gid_R(E)$. □

Finally, we give two applications of Theorem 2.4.

Corollary 2.5. *Let $R \to \Lambda$ be a homomorphism of rings with R right GF-closed and of finite left and right Gorenstein global dimension. Then we have*

(1) $Gfd_{R^{op}}(\Lambda) = Gid_R(\Lambda^+)$.
(2) *If Λ is a quasi-Frobenius ring, then* $Gfd_{R^{op}}(\Lambda) = Gid_R(\Lambda)$.

Proof. Note that Λ^+ is an injective cogenerator for Λ-Mod. On the other hand, it is well known that ${}_\Lambda\Lambda$ is an injective cogenerator for Λ-Mod if

Λ is a quasi-Frobenius ring. Thus both assertions follow immediately from Theorem 2.4. □

Recall that an algebra Λ is called an Artinian algebra if Λ is a two-sided Artinian ring and Λ is finitely generated as an R-module, where R is the center of Λ. It is well known that if Λ is an Artinian algebra, then its center R is a commutative Artinian ring. We use $\mathbb{D}$ to denote the usual duality of Λ, that is, $\mathbb{D} = Hom_R(-, R/J(R))$. Note $\mathbb{D}(\Lambda)$ is an injective cogenerator for Λ-Mod if Λ is an Artinian algebra, so we have the following

Corollary 2.6. *Let Λ be an Artinian algebra with center R. If R is of finite Gorenstein global dimension, then $Gfd_R(\Lambda) = Gid_R(\mathbb{D}(\Lambda))$.*

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