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ON *H*-COFINITELY SUPPLEMENTED MODULES

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Abstract. A module M is called H-cofinitely supplemented if for every cofinite submodule E (i.e. M/E is finitely generated) of M there exists a direct summand D of M such that $M = E+X$ holds if and only if $M = D+X$, for every submodule X of M. In this paper we study factors, direct summands and direct sums of H-cofinitely supplemented modules.

Let M be an H-cofinitely supplemented module and let $N \leq M$ be a submodule. Suppose that for every direct summand K of M, $(N + K)/N$ lies above a direct summand of M/N. Then M/N is H-cofinitely supplemented.

Let M be an H-cofinitely supplemented module. Let N be a direct summand of M. Suppose that for every direct summand K of M with $M = N + K$, $N \cap K$ is also a direct summand of M. Then N is H-cofinitely supplemented.

Let $M = M_1 \oplus M_2$. If M_1 is radical M_2-projective (or M_2 is radical M_1-projective) and M_1 and M_2 are H-cofinitely supplemented, then M is H-cofinitely supplemented

1. **Introduction**

Throughout this paper, R will be an associative ring with identity, and all modules are unitary right R-modules. The Jacobson radical of R is denoted by $Jac(R)$.

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A submodule L of a module M is called *small* in M (written $L \ll M$) if $N + L \neq M$ for any proper submodule N of M. A nonzero module H is called *hollow* if every proper submodule is small in H. Let M be a module. Let N be a submodule of M. A submodule K of M is called a *(weak) supplement* of N in M if, $N+K = M$ and $(N \cap K \ll M)$ $N \cap K \ll K$. The module M is called *H-supplemented*, if for every submodule A of M, there exists a direct summand D of M such that $M = A+X$ holds if and only if $M = D + X$ for all $X \leq M$. Note that H-supplemented modules were called *Goldie*-lifting* modules in [4]. A submodule N of M is called *cofinite* in M if the factor module M/N is finitely generated. The module M is called *cofinitely (weak) supplemented* if every cofinite submodule of M has a (weak) supplement in M. A module M is called *H-cofinitely supplemented* if for every cofinite submodule E of M, there exists a direct summand D of M such that $M = E + X$ holds if and only if $M = D + X$ for all $X \leq M$. This notion was introduced by Koşan in [15] and among others, he showed that if M is a module with $Rad(M) \ll M$, then M is H-cofinitely supplemented if and only if every cofinite submodule of $M/Rad(M)$ is a direct summand and each cofinite direct summand of $M/Rad(M)$ lifts to a direct summand of M. Clearly, H-supplemented modules are H-cofinitely supplemented. On the other hand, every finitely generated H-cofinitely supplemented module is H-supplemented. If N is a submodule of a module M, then we say that N lies above a direct summand if there is a direct summand K of M with $K \subseteq N$ and $N/K \ll M/K$.
Let M and N be two modules. Then N is called *radical M-projective*, if for any $K \leq M$ and any homomorphism $f : N \to M/K$ there exists a homomorphism $h : N \to M$ such that $Im(f - \pi h) \ll M/K$, where $\pi : M \to M/K$ is the natural epimorphism (see [12] and [14]).

In section 2, various properties of H-cofinitely supplemented modules are showed. Moreover, we give some examples showing that the concept of H-cofinitely supplemented modules is a proper generalization of the notion of H-supplemented modules.

Section 3 is devoted to the study of factors and direct summands of H-cofinitely supplemented modules. It is unknown if the class of H-cofinitely supplemented modules is closed under direct summands. It is shown that every direct summand of a finite length H-cofinitely supplemented module is again H-cofinitely supplemented (Corollary 3.3). It is also shown that if M is an H-cofinitely supplemented module and N is a direct summand of M such that for every direct summand K of M

with $M = N + K$, $N \cap K$ is also a direct summand of M, then N is H-cofinitely supplemented (Proposition 3.7).
Let M be an H-cofinitely supplemented module and let $N \leq M$ be a submodule. Suppose that for every direct summand K of M, $(N + K)/N$ lies above a direct summand of M/N. Then M/N is H-cofinitely supplemented (Proposition 3.5).

In section 4, we begin by giving an example showing that a direct sum of two H-cofinitely supplemented modules need not be H-cofinitely supplemented (Example 4.1). Then we prove that if $M = M_1 \oplus M_2$ such that M_1 is radical M_2-projective (or M_2 is radical M_1-projective) and M_1 and M_2 are H-cofinitely supplemented, then M is H-cofinitely supplemented (Theorem 4.7).

We conclude the paper by giving some examples of rings whose modules are H-cofinitely supplemented.

2. Some properties of *H*-cofinitely supplemented modules

In this section we investigate some properties of H-cofinitely supplemented modules. We mainly study the relation between the notion of H-cofinitely supplemented modules and some other notions.

A submodule N of a module M has *ample supplements* in M if every submodule L of M such that $M = N + L$ contains a supplement of N in M. The module M is called *amply (cofinitely) supplemented* if every (cofinite) submodule of M has ample supplements in M.

Note that one question still unanswered is whether an H-supplemented module is amply supplemented (see [4] and [16]).

A module M is called *(cofinitely) lifting* if it is amply (cofinitely) supplemented and every supplement of every (cofinite) submodule of M is a direct summand (see [8] and [17]).

It is well-known that the following implications hold: lifting $\Rightarrow$ H-supplemented $\Rightarrow$ H-cofinitely supplemented. It follows from [16, Corollary 4.42] that if R is a semiperfect ring, then the module R_R is H-cofinitely supplemented. On the other hand, it is clear that any module M with $Rad(M) = M$ is H-cofinitely supplemented. This yields that any non-supplemented module M with $Rad(M) = M$ is H-cofinitely supplemented but not H-supplemented. So all injective non-supplemented modules over a Dedekind domain (e.g. the quotient field of a non-local Dedekind domain (see [16, Proposition A.8])) are H-cofinitely supplemented but not H-supplemented.

The following proposition describes the structure of finitely generated *H*-(cofinitely) supplemented modules over commutative local rings.

Proposition 2.1. *Let R be a commutative local ring with maximal ideal m. The following are equivalent for a finitely generated R-module M:*

(1) *M is* H-*supplemented;*

(2) *Every direct summand of M is* H-*supplemented;*

(3) *$M \cong \frac{R}{I_1} \times \cdots \times \frac{R}{I_n}$ for some ideals $I_1, \ldots, I_n$ of R with $I_1 \subseteq \cdots \subseteq I_n \subsetneq R$.*

If $m^2 = 0$, then (1)-(3) are equivalent to:

(4) *M is supplemented and every supplement submodule of M is a direct summand;*

(5) *M is lifting.*

Proof. (1) $\Rightarrow$ (3) By [22, Satz 3.2].

(3) $\Rightarrow$ (2) By the Krull-Schmidt-Azumaya theorem, every direct summand of M has the same structure as the one given in (3). The result follows from [22, Satz 3.2].

(2) $\Rightarrow$ (1) This is immediate.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) follow from [17, Proposition 2.5] and the fact that $mI_n \subseteq m^2 \subseteq I_1$. $\square$

Proposition 2.2. *Let M be a module. If every cofinite submodule of M lies above a direct summand, then M is* H-*cofinitely supplemented.*

Proof. Let N be a cofinite submodule of M. By assumption, there exists a direct summand K of M such that N lies above K. It is easy to check that $M = N + X$ if and only if $M = K + X$ for all $X \leq M$. $\square$

The converse of Proposition 2.2 is in general false. See the following example.

Example 2.3. *Let I and J be two ideals of a commutative local ring R with maximal ideal m such that $I \subset J \subseteq m$ and $mJ \nsubseteq I$ (e.g. R is a DVR with maximal ideal m, $I = m^3$ and $J = m$). We consider the module $M = R/I \times R/J$. From Proposition 2.1 it follows that M is* H-*supplemented. On the other hand, according to* [22, Folgerung 3.3] *and the Krull-Schmidt-Azumaya theorem, M is not lifting. Hence not every cofinite submodule of M lies above a direct summand (see* [21, 41.12]*).*

Another example of modules showing that the class of *H*-supplemented modules is properly contained in the class of *H*-cofinitely supplemented modules is:

Example 2.4. *Let R be a commutative local ring which is not perfect with maximal ideal $\mathfrak{m}$ (e.g., we can take R to be $K[[x]]$, ring of all power series $\sum_{i=0}^{\infty} k_i x^i$ in an indeterminate x and with coefficients from a field K). Then $Rad(R_R^{(\mathbb{N})})$ is not small in $R_R^{(\mathbb{N})}$ by* [21, 43.9]*. Hence $M = R_R^{(\mathbb{N})}$ is not supplemented by* [21, 42.5]*. So M is not H-supplemented. By* [17, Corollary 2.23 and Proposition 2.33]*, every cofinite submodule of M lies above a direct summand. Hence M is* H*-cofinitely supplemented by Proposition 2.2.*

A submodule N of a module M is called *projection invariant* in M if $f(N) \subseteq N$, for any idempotent $f \in End(M)$.

Theorem 2.5. *Let M be an* H*-cofinitely supplemented module and let N be a cofinite projection invariant submodule of M. Then there exists a direct summand K of M such that $K \subseteq N$ and $N/K \ll M/K$.*

Proof. Since M is H-cofinitely supplemented, there exists a direct summand K of M such that $M = X + N$ if and only if $M = X + K$ for all $X \leq M$. Let $p : M \to K$ be the projection of M onto K and let $i : K \to M$ be the inclusion map. Set $e = ip$. Thus $e(M) = K$. Since $M = K + (1 - e)(M)$, we have $M = (1 - e)(M) + N$. It follows that $K = e(M) = e(N) \subseteq N$ since N is projection invariant. Let Y be a submodule of M with $K \leq Y$ and $N/K + Y/K = M/K$. Then $N + Y = M$. Hence $M = Y + K = Y$. Therefore $N/K \ll M/K$. This proves the theorem. □

Lemma 2.6. *Let M be a cofinitely weak supplemented module and let X be a submodule of M such that $Rad(M) \subseteq X$. Then every cofinite submodule of M/X is a direct summand.*

Proof. Let $N \leq M$ be a submodule such that $X \leq N$ and $\frac{M/X}{N/X}$ is finitely generated. Since M/N is finitely generated, there exists a submodule $K \leq M$ such that $M = N + K$ and $N \cap K \ll M$. Thus $N \cap K \subseteq Rad(M) \subseteq X$. Therefore $N \cap (K + X) = X + N \cap K = X$. So $M/X = [N/X] \oplus [(K + X)/X]$. This completes the proof. □

The following result is a consequence of Lemma 2.6.

Proposition 2.7. *Let M be a cofinitely weak supplemented module. Then for every submodule X of M such that $Rad(M) \subseteq X$, M/X is* H*-cofinitely supplemented.*

A module M is called a *local* module if the sum of all proper submodules of M is also a proper submodule of M.

Proposition 2.8. *The following statements are equivalent for an indecomposable module M:*
(1) *M is* H-*cofinitely supplemented;*
(2) *$Rad(M) = M$ or M is a local module.*

Proof. (1) $\Rightarrow$ (2) Suppose that $Rad(M) \neq M$. Then M has a maximal submodule L. By assumption, there exists a direct summand K of M such that $M = L + X$ if and only if $M = K + X$ for all $X \leq M$. But M is indecomposable. Then $K = 0$ or $K = M$. It is easily seen that $K \neq M$ because $L \neq M$. Thus $K = 0$. So for all $X \leq M$, $M = L + X$ implies that $X = M$. This gives $L \ll M$. Therefore L is the only maximal submodule of M. It follows that L is the sum of all proper submodules of M. Hence M is a local module.
(2) $\Rightarrow$ (1) This is clear. □

Let U be a submodule of a module M and let V be a direct summand of M. We say that V is an H-*supplement of U in M* if, there is a direct summand W of M such that $M = V \oplus W$ and $M = U + X$ if and only if $M = W + X$ for all $X \leq M$. Clearly, every *H*-supplement of U is a supplement of U and it is a direct summand of M.

Proposition 2.9. *Let M be an* H-*cofinitely supplemented module and let N be a cofinite submodule of M. Then every* H-*supplement of N in M is finitely generated.*

Proof. Let K be an *H*-supplement of N in M. Then $M = N + K$ and $N \cap K \ll K$. Since N is cofinite, $K/(N \cap K)$ is finitely generated. It follows that K is finitely generated since $N \cap K \ll K$. □

The next result gives some new characterizations of *H*-cofinitely supplemented modules. Its proof is similar to that of [14, Theorem 2.1] but we present it for completeness.

Theorem 2.10. *Let M be a module. The following are equivalent:*
(1) *M is* H-*cofinitely supplemented;*
(2) *For each cofinite submodule Y of M there exists a direct summand D of M such that $(Y + D)/D \ll M/D$ and $(Y + D)/Y \ll M/Y$;*
(3) *For each cofinite submodule Y of M there exist a submodule $X \leq M$ and a direct summand D of M with $Y + D \subseteq X$ such that $X/Y \ll M/Y$ and $X/D \ll M/D$;*
(4) *For each cofinite submodule Y of M there exist a supplement L of Y and a supplement K of L such that $(Y + K)/Y \ll M/Y$, $(Y +$*

$K)/K \ll M/K$ and every homomorphism $f : M \to M/(K \cap L)$ can be lifted to a homomorphism $\overline{f} : M \to M$.

Proof. (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (3) Let Y be a cofinite submodule of M. Then there exists a direct summand D of M such that $(Y + D)/Y \ll M/Y$ and $(Y + D)/D \ll M/D$. Now take $X = Y + D$.

(3) $\Rightarrow$ (1) Let Y be a cofinite submodule of M. Then there exist a submodule X of M and a direct summand D of M such that $Y + D \subseteq X$, $X/Y \ll M/Y$ and $X/D \ll M/D$. It is easy to see that $M = A + D$ if and only if $M = A + Y$ for any $A \leq M$. Thus, M is H-cofinitely supplemented.

(2) $\Rightarrow$ (4) Let Y be a cofinite submodule of M. Then there exist submodules D and D' of M such that $M = D \oplus D'$, $(Y + D)/Y \ll M/Y$ and $(Y + D)/D \ll M/D$. It is easy to show that D' is a supplement of Y and D is a supplement of D'. So (4) follows by taking $L = D'$ and $K = D$.

(4) $\Rightarrow$ (2) Put $S = K \cap L$. We have $S \ll K$ and also $S \ll L$. Let $g : M \to M/L$ and $f : M \to M/S$ be the natural maps. Note that there exists an isomorphism $t : M/L \to K/S$. By assumption, there exists $h : M \to M$ such that $fh = tg$. We have $K/S = f(K) = tg(K) = fh(K)$. Hence, $K + Kerf = h(K) + Kerf$, i.e., $K + S = h(K) + S$. Hence, $K = h(K)$ as $S \ll K$. Note that $h(M) = K$. Hence, $K = h(K) = h(M)$. Therefore, $K + Kerh = M$. As $Kerh$ is contained in L and L is a supplement of K, $Kerh = L$. Now $L = Ker(tg) = Ker(fh)$ implies $Kerf = 0$, i.e., $S = 0$. Thus, $M = K \oplus L$. This completes the proof. $\square$

A module M is called *π-projective* if for every two submodules U, V of M such that $U + V = M$, there exists an endomorphism f of M with $Im(f) \subseteq U$ and $Im(1 - f) \subseteq V$ (see [21, p. 359]).

Let M be an R-module. A projective module P together with a small epimorphism $f : P \to M$ is called a *projective cover* of M. Çalişici and Pancar [7] introduced the concept of cofinitely semiperfect modules. A module M is called *cofinitely semiperfect* if every finitely generated factor module of M has a projective cover.

Proposition 2.11. *The following are equivalent for a π-projective module M:*

(1) *M is* H-*cofinitely supplemented;*

(2) *M is cofinitely lifting;*

(3) *M is cofinitely supplemented and every supplement of every cofinite submodule of M is a direct summand;*

(4) *Every cofinite submodule of M lies above a direct summand;*

(5) *Every cofinite submodule of M has a supplement that is a direct summand.*

If M is projective, then (1)-(5) are equivalent to:

(6) *M is cofinitely semiperfect.*

Proof. (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) See [17, Proposition 2.33].

(1) $\Rightarrow$ (4) Let N be a cofinite submodule of M. By hypothesis, there exist submodules D and D' of M such that $M = D \oplus D'$ and $M = D + X$ if and only if $M = N + X$ for all $X \leq M$. Thus D' is a supplement of N in M. But M is π-projective. Then there exists a submodule $N' \leq N$ such that $M = N' \oplus D'$ by [21, 41.14]. Now it suffices to show that N lies above N'. Let A be a submodule of M with $N' \leq A$ and $M/N' = N/N' + A/N'$. Hence $M = N + A$. Since $N = N' \oplus (N \cap D')$, we have $M = N' + (N \cap D') + A$. Therefore $M = (N \cap D') + A$. But $N \cap D' \ll D'$. Then $M = A$. This shows that $N/N' \ll M/N'$.

(4) $\Rightarrow$ (1) By Proposition 2.2.

(5) $\Rightarrow$ (1) Let N be a cofinite submodule of M. By hypothesis, there exist submodules K_1 and K_2 of M such that $M = K_1 \oplus K_2 = N + K_1$ and $N \cap K_1 \ll K_1$. Since M is π-projective, there exists a submodule $K_3 \subseteq N$ such that $M = K_3 \oplus K_1$ by [21, 41.14]. It follows that $M = N + X$ if and only if $M = K_3 + X$ for all $X \leq M$. Hence M is H-cofinitely supplemented.

(1) $\Rightarrow$ (5) This is clear.

If M is projective, then (5) $\Leftrightarrow$ (6) by [7, Theorem 2.1]. □

Now we give an example which shows that an H-cofinitely supplemented module need not be cofinitely lifting.

Example 2.12. *Let p be any prime number. Let M denote the $\mathbb{Z}$-module $\mathbb{Q} \oplus (\mathbb{Z}/\mathbb{Z}p)$, where $\mathbb{Q}$ is the field of rational numbers. By* [1, Corollary 4.9]*, M is not amply cofinitely supplemented. It follows that M is not cofinitely lifting. Let L be any cofinite submodule of M. Then M/L is a noetherian $\mathbb{Z}$-module. Hence $\mathbb{Q}/(\mathbb{Q} \cap L)$ is finitely generated. So $\mathbb{Q} \subseteq L$. It follows that $L = \mathbb{Q} \oplus [L \cap (\mathbb{Z}/\mathbb{Z}p)]$. Then $L = \mathbb{Q}$ or $L = M$. Therefore M is* H*-cofinitely supplemented.*

Following [5], a module M is called *w-local* if it has a unique maximal submodule.

Remark 2.13. *Note that it is easy to check that the following conditions are equivalent for a module M:*

(1) *For every cofinite submodule N of M, there is a maximal submodule U of M such that $N \leq U$ and $U/N \ll M/N$;*

(2) *For every cofinite submodule N of M, M/N is a w-local module.*

Proposition 2.14. *Let M be a module. Suppose that for every cofinite submodule N of M, M/N is a w-local module. Then the following statements are equivalent:*

(1) *M is* H-*cofinitely supplemented;*

(2) *Every maximal submodule of M has an* H-*supplement in M.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) Let $N \leq M$ be a cofinite submodule. By assumption, there is a maximal submodule U of M such that $N \leq U$ and $U/N \ll M/N$. It is easy to check that for any $X \leq M$, $M = N + X$ if and only if $M = U + X$. By (2), there exists a direct summand D of M such that $M = U + X$ if and only if $M = D + X$ for every $X \leq M$. Consequently, M is H-cofinitely supplemented. □

Next we give an example showing that in Proposition 2.11 the implication (1) $\Rightarrow$ (3) does not hold, in general, if the module M is not π-projective.

Example 2.15. *(1) Let $M = N \oplus L$ such that $Rad(N) = N$ and L is a local module with maximal submodule K. It is clear that $Rad(M) = N \oplus K$ is the unique maximal submodule of M. Thus M is a w-local module. It is easily seen that $M = N + X$ if and only if $M = Rad(M) + X$ for every $X \leq M$. Proposition 2.14 shows that M is* H-*cofinitely supplemented.*

(2) Let R be a discrete valuation ring with maximal ideal m and quotient field Q. It is proved in [17, Example 2.30] *that the R-module $M = Q \oplus (R/m)$ does not satisfy the condition (3) of Proposition 2.11. On the other hand, M is* H-*cofinitely supplemented by (1).*

Recall that a module M is called *coatomic* if every proper submodule is contained in a maximal submodule.

Let $M = \sum_{\lambda \in \Lambda} M_\lambda$ be a sum of submodules M_λ ($\lambda \in \Lambda$) of a module M. Then, this sum is called *irredundant* if, for every $\lambda_0 \in \Lambda$, $\sum_{\lambda \neq \lambda_0} M_\lambda \neq M$.

Proposition 2.16. *Let M be an* H-*cofinitely supplemented module. If M is coatomic, then $M = \sum_{i \in I} L_i$ is an irredundant sum of local submodules $L_i (i \in I)$ which are direct summands of M.*

Proof. This is a consequence of [9, Proposition 2.18]. □

Proposition 2.17. *Let R be a right perfect ring and let M be a nonsingular* H-*cofinitely supplemented injective R-module. Then $M = \oplus_{i \in I} M_i$ is a direct sum of local submodules $M_i (i \in I)$.*

Proof. The proof is exactly the same as that of [13, Lemma 2.19] since it uses only the fact that every maximal submodule of M has a supplement that is a direct summand. □

A module M is said to be *refinable* if for any submodules U, V of M with $U + V = M$, there exists a direct summand U' of M with $U' \subseteq U$ and $U' + V = M$ (see [8, 11.26]). Clearly, semisimple modules, hollow modules and lifting modules are refinable.

Note that we have the following hierarchy for a module M:

M is *H*-cofinitely supplemented $\Rightarrow$ M is cofinitely supplemented $\Rightarrow$ M is cofinitely weak supplemented.

Proposition 2.18. *Let M be a refinable module. Then the following conditions are equivalent:*

(1) *M is* H-*cofinitely supplemented;*
(2) *M is cofinitely supplemented;*
(3) *M is cofinitely weak supplemented.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1) Suppose that M is cofinitely weak supplemented. Let N be any cofinite submodule of M. Then there exists a submodule K of M such that $M = N + K$ and $N \cap K \ll M$. Since M is refinable, there is a direct summand L of M such that $L \subseteq N$ and $M = L + K$. But $N = L + (N \cap K)$ and $N \cap K \ll M$. Then $M = N + X$ if and only if $M = L + X$ for all $X \leq M$. This completes the proof. □

3. Factors and direct summands of *H*-cofinitely supplemented modules

It is unknown if the properties *H*-cofinitely supplemented and *H*-supplemented are inherited by direct summands. A module M is said to be *completely* H-*(cofinitely) supplemented* if every direct summand of M is *H*-(cofinitely) supplemented. On the other hand, Example 3.1 shows that a factor module of an H-(cofinitely) supplemented module is not in general H-(cofinitely) supplemented.

A commutative ring R is a *valuation* ring if it satisfies one of the following three equivalent conditions:

(i) For any two elements a and b, either a divides b or b divides a;

(ii) The ideals of R are linearly ordered by inclusion;

(iii) R is a local ring and every finitely generated ideal is principal.

Example 3.1. *Let R be a commutative noetherian local ring which is not a principal ideal ring (e.g. $R = k[x^2, x^3]/(x^4)$ where k is any field or we can take $R = F[[x, y]]$ the ring of formal power series over a field F in the indeterminates x and y). Then R is not a valuation ring. Let $n \geq 2$. By* [19, Theorem 2]*, there exists a submodule L of the R-module $M = R^{(n)}$ such that the R-module $N = M/L$ is indecomposable and N cannot be generated by fewer than n elements. Thus N is not a local R-module. So N is not* H-*(cofinitely) supplemented by Proposition 2.8. However, Proposition 2.1 shows that M is* H-*(cofinitely) supplemented.*

The following result may be proved in much the same way as [22, Lemma 1.1(a)].

Lemma 3.2. *Let M_0 be a direct summand of a module M such that for every decomposition $M = N \oplus K$ of M, there exist submodules $N' \leq N$ and $K' \leq K$ such that $M = M_0 \oplus N' \oplus K'$. If M is* H-*(cofinitely) supplemented, then M/M_0 is* H-*(cofinitely) supplemented.*

Corollary 3.3. *Let M be an* H-*cofinitely supplemented module and let M_0 be a direct summand of M. Assume*

(i) *R is commutative or right noetherian and M_0 is a finite direct sum of local R-modules, or*

(ii) *M_0 is a semisimple finitely generated module, or*

(iii) *M_0 is a finite direct sum of local projective modules, or*

(iv) *M_0 is a finite direct sum of indecomposable injective modules, or*

(v) *M_0 is a module of finite length.*

Then M/M_0 is H-*cofinitely supplemented.*

Proof. (i) By [10, Theorems 4.1, 4.2], [3, Lemma 26.4] and Lemma 3.2.

(ii) It is well-known that the endomorphism ring of a simple module is a division ring. The result follows from [3, Lemma 26.4] and Lemma 3.2.

(iii) By [10, Theorem 4.4], [3, Lemma 26.4] and Lemma 3.2.

(iv) By [3, Lemmas 25.4, 26.4] and Lemma 3.2.

(v) By [3, Proposition 10.14, Lemmas 12.8, 26.4] and Lemma 3.2. □

A module M has *finite hollow dimension* if for some $n \in \mathbb{N}$, there exists a small epimorphism from M to a direct sum of n hollow modules (see [8, 5.2]).

Proposition 3.4. *Suppose that R is commutative or right noetherian. Let M be a finitely generated R-module. If M is* H*-supplemented, then M is a direct sum of local submodules.*

Proof. Suppose that M is *H*-supplemented. Note that M has finite hollow dimension by [8, 18.6]. Therefore M has the ascending chain on direct summands by [8, 5.3]. On the other hand, M has a direct summand M_0 that is local by Proposition 2.16. Let M_0' be a submodule of M such that $M = M_0 \oplus M_0'$. By Corollary 3.3, M_0' is *H*-supplemented. If M_0' is not local, then M_0' has a direct summand M_1 that is local and M_0'/M_1 is *H*-supplemented by Corollary 3.3. We continue in this fashion to obtain an ascending chain of direct summands of M ($M_0 \subseteq M_0 \oplus M_1 \subseteq \cdots$). It follows that M is a finite direct sum of local modules. □

Proposition 3.5. *Let M be an* H*-cofinitely supplemented module and let $N \leq M$ be a submodule. Suppose that for every direct summand K of M, $(K+N)/N$ lies above a direct summand of M/N. Then M/N is* H*-cofinitely supplemented.*

Proof. Let $Y/N \leq M/N$ be a cofinite submodule. Since M is *H*-cofinitely supplemented, there exists a direct summand K of M such that $M = X + Y$ if and only if $M = X + K$ for all $X \leq M$. By assumption, there is a submodule L of M such that $N \subseteq L \subseteq K + N$, L/N is a direct summand of M/N and $\frac{(K+N)/N}{L/N} \ll \frac{M/N}{L/N}$. Let $X \leq M$ be a submodule such that $N \subseteq X$. If $M/N = X/N + L/N$, then $M = X + L$. Thus $M = X + K + N = X + K$. Hence $M = X + Y$. Therefore $M/N = X/N + Y/N$. On the other hand, if $M/N = X/N + Y/N$, then $M = X + Y$. So $M = X + K$. Thus $M/N = [(X+L)/N] + [(K+N)/N]$. It follows that $\frac{M/N}{L/N} = \frac{(X+L)/N}{L/N} + \frac{(K+N)/N}{L/N}$. This gives $M/N = X/N + L/N$. This completes the proof. □

Proposition 3.6. *Suppose that R is commutative or right noetherian. Let M be an* H*-cofinitely supplemented module. Let N be a cofinite direct summand of M such that for every direct summand K of M, $(K+N)/N$ is a direct summand of M/N. Then N is* H*-cofinitely supplemented.*

Proof. Let K be a submodule of M such that $M = N \oplus K$. By Proposition 3.5, K is *H*-cofinitely supplemented. By hypothesis, K is finitely generated. It follows that $K = \oplus_{i=1}^{n} L_i$ is a direct sum of local submodules $L_i (1 \leq i \leq n)$ by Proposition 3.4. Now Corollary 3.3 shows that $N \cong M/K$ is *H*-cofinitely supplemented. $\square$

Proposition 3.7. *Let M be an* H-*cofinitely supplemented module. Let N be a direct summand of M. Suppose that for every direct summand K of M with $M = N + K$, $N \cap K$ is also a direct summand of M. Then N is* H-*cofinitely supplemented.*

Proof. Let N' be a submodule of M such that $M = N \oplus N'$. Let A be a cofinite submodule of N. Then $A \oplus N'$ is a cofinite submodule of M. By assumption, there is a submodule D of M such that $M = Y + D$ if and only if $M = Y + A + N'$ for all $Y \leq M$. Since $M = N + A + N'$, we have $M = N + D$. So $D \cap N$ is a direct summand of N. Let $X \leq N$ be a submodule. If $N = X + A$, then $M = X + A + N'$. Thus $M = X + D$. Hence $N = X + (D \cap N)$. On the other hand, if $N = X + (D \cap N)$, then $M = X + (D \cap N) + D = X + D$ since $M = N + D$. So $M = X + A + N'$. As $X + A \leq N$, we have $N = X + A$. Consequently, N is *H*-cofinitely supplemented. $\square$

We say that a module M has (D_3) if for any direct summands M_1 and M_2 of M with $M = M_1 + M_2$, $M_1 \cap M_2$ is a direct summand of M.

Recall that a module M is said to have the SIP (*Summand Intersection Property*) if the intersection of two direct summands of M is again a direct summand of M (see [11] or [20]).

Theorem 3.8. *Let M be an* H-*cofinitely supplemented module with (D_3) or having the SIP. Then M is a completely* H-*cofinitely supplemented module.*

Proof. It follows immediately from Proposition 3.7. $\square$

The condition (D_3) is not necessary in Theorem 3.8 as the following example shows.

Example 3.9. *Let I and J be two ideals of a commutative local ring R with maximal ideal m such that $I \subset J \subseteq m$ (e.g. R is a DVR with maximal ideal m, $I = m^3$ and $J = m^2$). We consider the module $M = \frac{R}{I} \times \frac{R}{J}$ and its submodules $A = R(\bar{1}, \bar{0})$, $B = R(\bar{1}, \bar{1})$ and $C = R(\bar{0}, \bar{1})$. Note that $M = A + B = A \oplus C = B \oplus C$. On the other hand, we have $A \cap B = J/I \times 0$. Hence $A \cap B \subseteq Rad(M)$ and $A \cap B \ll M$. Therefore*

$0 \neq A \cap B$ is not a direct summand of M. So M does not satisfy (D_3). Moreover, every direct summand of M is H*-supplemented by Proposition 2.1.*

Proposition 3.10. *Let M be a module and let $N \leq M$ be a submodule such that for each decomposition $M = M_1 \oplus M_2$, we have $N = (N \cap M_1) \oplus (N \cap M_2)$. If M is* H*-cofinitely supplemented, then M/N is* H*-cofinitely supplemented. If, moreover, N is a direct summand of M, then N is also* H*-cofinitely supplemented.*

Proof. Let D and D' be submodules of M such that $M = D \oplus D'$. By assumption, we have $N = (D \cap N) \oplus (D' \cap N)$. Then $(D+N) \cap (D'+N) = [D \oplus (D' \cap N)] \cap [(D \cap N) \oplus D'] = (D \cap N) \oplus (D' \cap N) = N$. So $M/N = [(D+N)/N] \oplus [(D'+N)/N]$. Proposition 3.5 shows that M/N is an *H*-cofinitely supplemented module.

Now assume that N is a direct summand of M. Let D and D' be submodules of M such that $M = D \oplus D' = N + D$. Since $N = (D \cap N) \oplus (D' \cap N)$, we have $M = (D \cap N) + (D' \cap N) + D = D \oplus (D' \cap N)$. This implies that $D' \cap N = D'$. Therefore $D' \subseteq N$. It follows that $N = (D \cap N) \oplus D'$. The result follows from Proposition 3.7. □

Corollary 3.11. *Let N be a projection invariant submodule of a module M.*

(i) *If M is* H*-cofinitely supplemented, then M/N is* H*-cofinitely supplemented.*

(ii) *If M is* H*-cofinitely supplemented and N is a direct summand of M, then N is also* H*-cofinitely supplemented.*

Proof. It is well-known that for each decomposition $M = M_1 \oplus M_2$, we have $N = (N \cap M_1) \oplus (N \cap M_2)$. The result follows from Proposition 3.10. □

Theorem 3.12. *Let $M = M_1 \oplus M_2$. If M is* H*-cofinitely supplemented and M_2 is M_1-projective, then M_1 is* H*-cofinitely supplemented.*

Proof. Let D be a direct summand of M such that $M = M_1 + D$. Since M_2 is M_1-projective, we have $M = M_1 \oplus D'$ for some submodule $D' \leq D$ by [8, 4.12]. Thus $D = (M_1 \cap D) \oplus D'$. So $M_1 \cap D$ is a direct summand of M. Consequently, M_1 is *H*-cofinitely supplemented by Proposition 3.7. □

Corollary 3.13. *Every π-projective (in particular, every projective)* H*-cofinitely supplemented module is completely* H*-cofinitely supplemented.*

Proof. By [8, 4.14(4)] and Theorem 3.8. □

Recall that a module M is called *radical* if M has no maximal submodules. Let M be any module. We denote by $P(M)$ the sum of all radical submodules of M. Clearly, if $M = K \oplus L$, then $P(M) = P(K) \oplus P(L) = [K \cap P(M)] \oplus [L \cap P(M)]$.

Proposition 3.14. *Suppose that R is a right noetherian ring. Let M be an* H-*cofinitely supplemented module such that $P(M)$ is a cofinite submodule of M. Then $M = P(M) \oplus K$ such that $P(M)$ and K are* H-*cofinitely supplemented and $K = \oplus_{i=1}^{n} K_i$ is a finite direct sum of local submodules $K_i (1 \leq i \leq n)$.*

Proof. Since $P(M)$ is a cofinite submodule of M, there exist submodules D and D' of M such that $M = D \oplus D'$ and $M = P(M) + X$ if and only if $M = D + X$ for all $X \leq M$. Then D' is an H-supplement of $P(M)$ in M. Thus $M = P(M) + D'$ and $P(M) \cap D' \ll D'$. Since $P(M) = P(D) \oplus P(D')$, we have $M = P(D) \oplus D'$. So $P(D) = D$ and $P(M) = D \oplus [D' \cap P(M)]$. Hence $P(D') = D' \cap P(M) \ll D'$. But $D'/P(D') \cong M/P(M)$ is a finitely generated module. Then D' is finitely generated and $P(D') = 0$. So $P(M) = D$. Therefore, $M = P(M) \oplus D'$. Moreover, by Proposition 3.10, $P(M)$ and D' are both *H*-cofinitely supplemented. Now Proposition 3.4 shows that D' is a finite direct sum of local submodules. □

Lemma 3.15. *Any direct summand of a refinable module is again refinable.*

Proof. Let M be a refinable module. Let N and K be submodules of M such that $M = N \oplus K$. Let U, V be two submodules of N with $U + V = N$. Then $M = U + V + K$. Since M is refinable, there is a direct summand U' of M with $U' \subseteq U$ and $M = U' + V + K$. It is easy to see that $U' + V = N$ and U' is a direct summand of N. It follows that N is a refinable module. □

Proposition 3.16. *Let M be a refinable* H-*cofinitely supplemented module. Then M is completely* H-*cofinitely supplemented.*

Proof. By [2, Proposition 2.5], Lemma 3.15 and Proposition 2.18. □

4. Direct sums of *H*-cofinitely supplemented modules

The following example shows that the class of *H*-cofinitely supplemented modules is not closed under finite direct sums.

Example 4.1. *Let R be a commutative noetherian local ring which is not a principal ideal ring (see Example 3.1). Then R is not a valuation ring. Therefore R contains two ideals I_1 and I_2 such that $I_1 \nsubseteq I_2$ and $I_2 \nsubseteq I_1$. It follows from Proposition 2.1 and the Krull-Schmidt-Azumaya theorem that the module $R/I_1 \oplus R/I_2$ is not* H-*cofinitely supplemented. However, R/I_1 and R/I_2 are local modules.*

Proposition 4.2. *Let R be a commutative local ring. The following statements are equivalent:*

(1) *Every direct sum of two local R-modules is* H-*cofinitely supplemented;* (2) *R is a valuation ring.*

Proof. By Proposition 2.1 and the Krull-Schmidt-Azumaya theorem. □

Definition 4.3. (See [12] and [14]) *Let M and N be two modules. Then N is called radical M-projective, if for any $K \leq M$ and any homomorphism $f : N \to M/K$, there exists a homomorphism $h : N \to M$ such that $Im(f - \pi h) \ll M/K$, where $\pi : M \to M/K$ is the natural epimorphism.*

Lemma 4.4. *Let $M = M_1 \oplus M_2$. Consider the following conditions:*

(i) *M_1 is radical M_2-projective.*

(ii) *For every $K \leq M$ such that $K + M_2 = M$, there exists $M_3 \leq M$ such that $M = M_2 \oplus M_3$ and $(K + M_3)/K \ll M/K$.*

Then (i) $\Rightarrow$ (ii).

Proof. See [14, Theorem 3.5]. □

Lemma 4.5. *Let $M = M_1 \oplus M_2$ be the direct sum of two* H-*cofinitely supplemented modules M_1 and M_2. Let N be a cofinite submodule of M with $M_1 \subseteq N$. Then there exists a direct summand D_2 of M_2 such that $M = X + N$ if and only if $M = X + M_1 + D_2$ for all $X \leq M$.*

Proof. Since M/M_1 is *H*-cofinitely supplemented and N/M_1 is a cofinite submodule of M/M_1, there exists a submodule D of M containing M_1 such that D/M_1 is a direct summand of M/M_1 and $M/M_1 = X/M_1 + N/M_1$ if and only if $M/M_1 = X/M_1 + D/M_1$ for all $X \leq M$ with $M_1 \subseteq X$. Let D' be a submodule of M such that $M_1 \subseteq D'$ and $(D/M_1) \oplus (D'/M_1) = M/M_1$. Since $D = M_1 \oplus (M_2 \cap D)$, we have $D' + (M_2 \cap D) = M$. But $D' \cap (M_2 \cap D) \leq M_1 \cap M_2 = 0$. Then $D' \oplus (M_2 \cap D) = M$. Let $D_2 = M_2 \cap D$. It is clear that D_2 is a direct summand of M_2. Now let X be a submodule of M. If $M = X + N$, then $M = X + M_1 + N$. So $M = X + M_1 + D = X + M_1 + D_2$. On the other hand, if $M = X + M_1 + D_2$, then $M = X + M_1 + D$. So $M = X + M_1 + N = X + N$. This completes the proof. □

Lemma 4.6. *Let K, L and N be submodules of a module M. Assume that $K+L=M$ and $(K\cap L)+N=M$. Then $K+(L\cap N)=L+(K\cap N)=M$.*

Proof. See [8, Lemma 1.24]. □

Theorem 4.7. *Let $M=M_1\oplus M_2$. If M_1 is radical M_2-projective (or M_2 is radical M_1-projective) and M_1 and M_2 are* H*-cofinitely supplemented, then M is* H*-cofinitely supplemented.*

Proof. Let Y be a cofinite submodule of M. Then M/Y is cofinitely weak supplemented by [2, Propositions 2.5 and 2.12]. Therefore there exists a submodule L of M such that $Y\subseteq L$, $M/Y=(L/Y)+[(Y+M_2)/Y]$ and $[L\cap(Y+M_2)]/Y\ll M/Y$. Then $M=L+M_2$. Lemma 4.4 shows that there is no loss of generality in assuming that $(L+M_1)/L\ll M/L$. By Lemma 4.5, there is a direct summand D_1 of M_1 such that $X+Y+M_2=M$ if and only if $X+D_1+M_2=M$ for all $X\leq M$. Again by Lemma 4.5, there is a direct summand D_2 of M_2 such that $X+L+M_1=M$ if and only if $X+M_1+D_2=M$ for all $X\leq M$. We put $D=D_1\oplus D_2=(D_1\oplus M_2)\cap(M_1\oplus D_2)$. Clearly, D is a direct summand of M. Let $X\leq M$ be a submodule. Thus,

$$
\begin{aligned}
M=X+D \quad &\Leftrightarrow \quad M=X+[(D_1\oplus M_2)\cap(M_1\oplus D_2)].\\
&\Leftrightarrow \quad \begin{cases} M=(D_1\oplus M_2)+[X\cap(M_1\oplus D_2)], \text{ and}\\ M=X+(M_1\oplus D_2) \text{ by Lemma 4.6.}\end{cases}\\
&\Leftrightarrow \quad \begin{cases} M=(Y+M_2)+[X\cap(M_1\oplus D_2)], \text{ and}\\ M=X+(M_1\oplus D_2).\end{cases}\\
&\Leftrightarrow \quad \begin{cases} M=(M_1\oplus D_2)+[X\cap(Y+M_2)], \text{ and}\\ M=X+Y+M_2 \text{ by Lemma 4.6.}\end{cases}\\
&\Leftrightarrow \quad \begin{cases} M=(L+M_1)+[X\cap(Y+M_2)], \text{ and}\\ M=X+Y+M_2.\end{cases}\\
&\Leftrightarrow \quad \begin{cases} M=L+[X\cap(Y+M_2)], \text{ and}\\ M=X+Y+M_2 \text{ since } (L+M_1)/L\ll M/L.\end{cases}\\
&\Leftrightarrow \quad \begin{cases} M=X+[L\cap(Y+M_2)] \text{ by Lemma 4.6}\\ \text{and using the fact that } M=L+M_2.\end{cases}\\
&\Leftrightarrow \quad M=X+Y \text{ since } [L\cap(Y+M_2)]/Y\ll M/Y.
\end{aligned}
$$

Therefore M is *H*-cofinitely supplemented. □

Corollary 4.8. *Let $M=M_1\oplus M_2$.*

(i) *If M_1 is M_2-projective (or M_2 is M_1-projective) and M_1 and M_2 are* H-*cofinitely supplemented, then M is* H-*cofinitely supplemented.*

(ii) *If M_1 is* H-*cofinitely supplemented and M_2 is a projective* H-*cofinitely supplemented module, then M is* H-*cofinitely supplemented.*

(iii) *If M_1 is* H-*cofinitely supplemented and M_2 is semisimple, then M is* H-*cofinitely supplemented.*

Proof. These are consequences of Theorem 4.7. □

Theorem 4.9. *If $P = \bigoplus_{i \in I} P_i$ is a direct sum of projective* H-*cofinitely supplemented modules P_i $(i \in I)$, then P is* H-*cofinitely supplemented.*

Proof. By [7, Theorem 2.9], Proposition 2.11 and the fact that the class of projective modules is closed under direct sums. □

Proposition 4.10. *Let $M = \sum_{i \in I} M_i$ be a refinable module. If each M_i $(i \in I)$ is* H-*cofinitely supplemented, then M is* H-*cofinitely supplemented.*

Proof. Assume that each M_i $(i \in I)$ is *H*-cofinitely supplemented. Then each M_i is cofinitely weak supplemented. By [2, Proposition 2.12], M is a cofinitely weak supplemented module. Since M is refinable, M is *H*-cofinitely supplemented by Proposition 2.18. □

Proposition 4.11. *Let a module $M = \oplus_{i \in I} M_i$ be a direct sum of submodules $M_i (i \in I)$. If for every submodule N of M, we have $N = \oplus_{i \in I}(N \cap M_i)$, then M is (completely)* H-*cofinitely supplemented if and only if all $M_i (i \in I)$ are (completely)* H-*cofinitely supplemented.*

Proof. Clear. □

5. Rings whose modules are *H*-cofinitely supplemented

We conclude this paper by studying some examples of rings whose modules are *H*-cofinitely supplemented.

Theorem 5.1. *The following are equivalent for a ring R:*

(1) *R is semiperfect;*
(2) *Every finitely generated free R-module is* H-*cofinitely supplemented;*
(3) *R_R is* H-*supplemented;*
(4) *R_R is* H-*cofinitely supplemented;*
(5) *Every free R-module is* H-*cofinitely supplemented.*

Proof. By [6, Theorem 2.9] and Proposition 2.11. □

We conclude from Theorem 5.1 that if R is a ring over which all R-modules are H-cofinitely supplemented, then R is semiperfect. But there is a semiperfect ring having an R-module which is not H-cofinitely supplemented as shows the following example.

Example 5.2. Let $R = F[[x, y]]$ be the ring of formal power series over a field F in the indeterminates x and y. Then R is a commutative noetherian local domain with maximal ideal $J = Rx + Ry$. Therefore the ring R is semiperfect and the ideal J is finitely generated. Since R is a domain, J_R is a uniform R-module. So J_R is not a direct sum of cyclic R-modules. By Proposition 2.1, the module J_R is not H-cofinitely supplemented.

Recall that a ring R is called *semilocal* provided $R/Jac(R)$ is a right semisimple ring.

Proposition 5.3. *Let R be a semilocal ring such that every R-module is refinable. Then every R-module is* H-*cofinitely supplemented.*

Proof. By [2, Corollary 2.22], every R-module is cofinitely weak supplemented. So every R-module is H-cofinitely supplemented by Proposition 2.18. □

A module M is *uniserial* if its submodules are linearly ordered by inclusion and it is *serial* if it is a direct sum of uniserial submodules. The ring R is *right (left) serial* if the right (left) R-module R_R $({}_RR)$ is serial and it is *serial* if it is both right and left serial.

In the next example we give a ring which satisfies the conditions of Proposition 5.3.

Example 5.4. Let R be an artinian serial ring with $(Jac(R))^2 = 0$. It is well-known that R is a semilocal ring. By [13, Theorem 3.15], every R-module is lifting. Therefore every R-module is a refinable H-supplemented module.

Recall that a module M is called *coseparable* if for every cofinite submodule U of M, there exists a cofinite submodule U' of M such that $U' \subseteq U$ and U' is a direct summand of M (see [22] and [23]) .

Proposition 5.5. *Let R be a complete discrete valuation ring. Then every R-module is* H-*cofinitely supplemented.*

Proof. By [23, Lemma 1.2 and Satz 1.8], every R-module is coseparable. Therefore every R-module is H-cofinitely supplemented by [22, Folgerung 2.8]. □

In [18, Proposition 3.1], it is shown that a commutative ring R is an artinian principal ideal ring if and only if every R-module is H-supplemented. Proposition 5.5 shows that the implication "every module is H-cofinitely supplemented $\Rightarrow$ every module is H-supplemented" does not hold.

Proposition 5.6. *Let $R = R_1 \oplus \cdots \oplus R_n$ be a commutative ring such that each R_i is either a complete discrete valuation ring or an artinian principal ideal ring. Then every R-module is* H-*cofinitely supplemented.*

Proof. We can write $1_R = e_1 + e_2 + \cdots + e_n$, where e_i is the identity element of the ring R_i and 1_R is the identity element of the ring R. Let M be any R-module. Then $M = e_1M \oplus e_2M \oplus \cdots \oplus e_nM$. Let $1 \leq i \leq n$. Note that e_iM can be regarded as an R_i-module as well as an R-module, and its submodules are the same in both cases, because $(r_1 + r_2 + \cdots + r_n)e_ix = r_ie_ix$, where $r_j \in R_j$ for $1 \leq j \leq n$ and $x \in M$. Following [18, Proposition 3.1], every module over a commutative artinian principal ideal ring is H-supplemented. The result follows from Propositions 4.11 and 5.5. $\square$

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