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HYERS-ULAM-RASSIAS STABILITY OF N-JORDAN $*$-HOMOMORPHISMS ON C^*-ALGEBRAS

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Abstract. In this paper we investigate the Hyers-Ulam-Rassias stability of n-jordan $*$-homomorphisms on C^*-algebras.

1. Introduction

Let $n \in \mathbb{N} - \{1\}$ and let A,B be two algebras (rings). A linear map $h : A \to B$ is called n-jordan homomorphism (n-ring homomorphism) if $h(a^n) = (h(a))^n$ for all $a \in A$, $(h(\sum_{i=1}^n a_i) = \sum_{i=1}^n h(a_i)$ for all $a_1, a_2, ..., a_n \in A)$. The concept of n-jordan homomorphisms was studied for complex algebras by eshaghi et al. [7] (see also [8] and [9]).

The stability of functional equations was first introduced by S. M. Ulam [29] in 1940. More precisely, he proposed the following problem: Given a group G_1, a metric group (G_2, d) and $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \epsilon$ for all $x \in G_1$? As mentioned above, when this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In 1941, Hyers [19] gave a

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partial solution of *Ulam's* problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. In 1950, Aoki [1] generalized Hyers' theorem for approximately additive mappings. In 1978, Rassias [28] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. This phenomenon of stability that was introduced by Rassias [28] is called the Hyers-Ulam-Rassias stability. According to Rassias theorem:

Let $f : E_1 \to E_2$ be a mapping from a normed vector space E_1 into a Banach space E_2 subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E_1$. where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E_1$. if $p < 0$ then inequality (1.1) holds for $x, y \neq 0$, and (1.2) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E_2 is continuous for each fixed $x \in E_1$, then T is linear.

During the last decades several stability problems of functional equations have been investigated by many mathematicians. A large list of references concerning the stability of functional equations can be found in [6, 13, 20, 22]. and see [2–5, 10–12, 15, 21, 24–27].

Miura et al. [23] proved the Hyers-Ulam-Rassias stability of jordan homomorphisms.
in this paper, we consider the Hyers-Ulam-Rassias stability of n-jordan $*$-homomorphisms on C^*-algebras.

2. **Strong convergence**

Let A,B be two algebras. A linear map $h : A \to B$ is called n-jordan homomorphism if

$$h(x^n) = (h(x))^n$$

for all $x \in A$.

Let $n \in \mathbb{N}$ and let A,B be two C^*-algebras. An n-jordan homomorphism $h : A \to B$ is called n-jordan $*$-homomorphism if

$$h(x^*) = (h(x))^*$$

for all $x \in A$.

Theorem 2.1. *Let A, B be two C^*-algebras, let δ, ϵ, p, q be real numbers such that $p, q < 1$ or $p, q > 1$, and that $q > 0$. Assume that $f : A \to B$ satisfies the functional inequalities*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|_B \leq \epsilon(\|x\|_A^p + \|y\|_A^p), \tag{2.1}$$

$$\|f(x^n) - f(x)^n\|_B \leq \delta\|x\|_A^{nq}, \tag{2.2}$$

$$\|f(x^*) - f(x)^*\|_B \leq \delta\|x^*\|_A^q \tag{2.3}$$

for all $x, y \in A$. Then, there exists a unique n-jordan $*$-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \frac{2\epsilon}{|2-2^p|}\|x\|_A^p \tag{2.4}$$

for all $x \in A$.

Proof. Let $s = -sgn(p-1)$, and $h(x) = \lim_m \frac{f(2^{sm}x)}{2^{sm}}$ for all $x \in A$. from [14, 16, 17], we conclude that h is an additive map which satisfies (2.4).
Now, since $\lim_m 2^{smn(q-1)} = 0$, and from (2.2), for all $x \in A$ we have

$$\begin{aligned}\lim_m \tfrac{1}{2^{smn}}[\|f((2^{sm}x)(2^{sm}x) \quad & ...(2^{sm}x)) - (f(2^{sm}x))^n\|_B] \\ & \leq \lim_m \frac{1}{2^{smn}}\delta\|2^{sm}x\|_A^{nq} \\ & = \lim_m (2^{smn(q-1)})\delta\|x\|_A^{nq} \\ & = 0.\end{aligned}$$

Thus

$$\begin{aligned} h(x^n) &= \lim_m \frac{1}{2^{smn}} f(2^{smn}(x^n)) \\ &= \lim_m \frac{1}{2^{smn}} f(2^{sm}x)(2^{sm}x)...(2^{sm}x) \\ &= \lim_m \frac{1}{2^{smn}}[f(2^{sm}x)(2^{sm}x)...(2^{sm}x) - (f(2^{sm}x))^n \\ &\quad +(f(2^{sm}x))^n] \\ &= \lim_m \frac{1}{2^{smn}} f(2^{sm}x)^n \\ &= (h(x))^n. \end{aligned}$$

By that h is an additive map and from (2.3) we have

$$\begin{aligned}\|h(x^*)-(h(x))^*\|_B &= \|\frac{1}{2^{sm}}h(2^{sm}x^*)-\frac{1}{2^{sm}}(h(2^{sm}x))^*\|_B\\ &\leq \frac{1}{2^{sm}}\delta(2^{sm})^q\|x^*\|_A^q\\ &= 2^{sm(q-1)}\delta\|x^*\|_A^q.\end{aligned}$$

Since $\lim_m 2^{sm(q-1)}=0$, so

$$h(x^*)=(h(x))^*$$

for all $x\in A$. Therefore h is n-jordan $*$-homomorphism.
The uniqueness property of h follows from [15, 28]. □

Theorem 2.2. *Let A, B be two C^*-algebras, let δ, ϵ, p, q be real numbers such that $p<1$ or $q<0$. If $f: A\to B$ is a mapping with $f(0)=0$, such that the inequalities (2.1) , (2.2) and (2.3) are valid. Then, there exists a unique n-jordan $*$-homomorphism $h: A\to B$ such that*

$$\|f(x)-h(x)\|_B\leq\frac{2\epsilon}{|2-2^p|}\|x\|_A^p \tag{2.5}$$

for all $x\in A$.

Proof. Let $\|0\|^p=\infty$. from [28] we conclude that there exists an additive map $h: A\to B$ satisfies (2.5). Now, it suffices to show that $h(x^n)=(h(x))^n$ and $h(x^*)=(h(x))^*$ for all $x\in A$. Since h is an additive map, we get $h(0)=0$, and so the case $x=0$ is omitted. By assumption $x\in A$ and $x\neq 0$, by the proof of Theorem 2.1 we have $h(x^*)=(h(x))^*$ and if $x^n\neq 0$ then $h(x^n)=(h(x))^n$. thus we need to investigate only the case $x^n=0$. Since $f(0)=0$, Replacing x by $2^m x$ in (2.2) we get

$$\begin{aligned}\|\frac{1}{2^{mn}}(f(2^m x))^n-\frac{1}{2^{mn}}f(2^m x^n)\|_B &= \|\frac{1}{2^{mn}}(f(2^m x))^n\|_B\\ &\leq \frac{1}{2^{mn}}\delta\|2^m x\|_A^{nq}\\ &= 2^{mn(q-1)}\delta\|x\|_A^{nq}.\end{aligned}$$

From the relation above it follows

$$\lim_m\frac{1}{2^{mn}}(f(2^m x))^n=0. \tag{2.6}$$

Thus, we assume

$$h(a)=\lim_m\frac{1}{2^m}(f(2^m x)). \tag{2.7}$$

Combining (2.6) by (2.7) we get

$$h(x)^n = \lim_m [\frac{1}{2^{mn}}(f(2^m x))^n] = 0.$$

and since $h(x^n) = h(0) = 0$ therefore $h(x^n) = (h(x))^n = 0$. hence, $h : A \to B$ is $*$-preserving, in other words, h is n-jordan $*$-homomorphism. □

Corollary 2.3. *Let A, B be two C^*-algebras, let $\delta, \epsilon \geq 0$ and let p, q be real numbers such that $(p-1)(q-1) > 0$, $q < 0$ or that $(p-1)(q-1) > 0$, $q \geq 0$ and $f(0) = 0$. Assume that $f : A \to B$ satisfies functional inequalities*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|_B \leq \epsilon(\|x\|_A^p + \|y\|_A^p),$$

$$\|f(x^n) - f(x)^n\|_B \leq \delta\|x\|_A^{nq},$$

$$\|f(x^*) - f(x)^*\|_B \leq \delta\|x^*\|_A^q$$

for all $x, y \in A$. Then, there exists a unique n-jordan $*$-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \frac{2\epsilon}{|2 - 2^p|}\|x\|_A^p$$

for all $x \in A$.

Proof. It follows from Theorem 2.1 and Theorem 2.2. □

A linear map $h : A \to B$ is an n-homomorphism if $h(\prod_{i=1}^n x_i) = \prod_{i=1}^n h(x_i)$, for all $x_1, x_2, ..., x_n \in A$ (see [18]).

Corollary 2.4. *Let $n \in \{3, 4, 5\}$ be fixed and let A, B be two commutative C^*-algebras. Then every n-jordan $*$-homomorphism $h : A \to B$ is an n-homomorphism—, that is h is $*$-preserving.*

Proof. The proof follows from [7, 23] □

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