HYERS-ULAM STABILITY OF FIBONACCI FUNCTIONAL EQUATION

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ABSTRACT. We solve the Fibonacci functional equation, $f(x) = f(x-1) + f(x-2)$, and prove its Hyers-Ulam stability in the class of functions $f : \mathbb{R} \to X$, where X is a real Banach space.

1. Introduction

In 1940, Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems (see [19]). Among those was the question concerning the stability of homomorphisms:

> Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot, \cdot)$. Given any $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$, for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$, for all $x \in G_1$?

In the following year, Hyers [8] affirmatively answered the question of Ulam for the case where G_1 and G_2 are Banach spaces. Taking this fact into account, the additive Cauchy functional equation $f(x + y) = f(x) + f(y)$ is said to satisfy the Hyers-Ulam stability. This terminology

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is also applied to the case of other functional equations. Later, the result of Hyers was generalized by Rassias (see [15]). It should be remarked that Aoki [1] proved a particular case of Rassias' theorem regarding the Hyers-Ulam stability of additive functions earlier than Rassias (see [14]). We can find in the books [3, 9, 12] a lot of references concerning the stability of functional equations (see also [2, 4, 5, 6, 7, 10, 11, 16, 17, 18]).

Throughout this paper, we denote by F_n the nth Fibonacci number, for $n \in \mathbb{N}$. In particular, we define $F_0 := 0$. It is well known that $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. From this famous formula, we may derive a functional equation

$$(1.1) \qquad\qquad f(x) = f(x-1) + f(x-2),$$

which may be called the Fibonacci functional equation. A function $f : \mathbb{R} \to X$ will be called a Fibonacci function if it satisfies (1.1), for all $x \in \mathbb{R}$, where X is a real vector space.

By α and β we denote the positive root (respectively the negative root) of the quadratic equation $x^2 - x - 1 = 0$; i.e.,

$$\alpha = \frac{1 + \sqrt{5}}{2} \text{ and } \beta = \frac{1 - \sqrt{5}}{2}.$$

For any $x \in \mathbb{R}$, $[x]$ stands for the largest integer that does not exceed x.

Here, we will solve the Fibonacci functional equation (1.1) and prove its Hyers-Ulam stability for the class of functions $f : \mathbb{R} \to X$.

2. General solution of Fibonacci equation

Here, let X be a real vector space. We investigate the general solution of the Fibonacci functional equation (1.1). As we shall see in the following theorem, the general solution of Fibonacci functional equation is strongly related to the Fibonacci numbers F_n.

Theorem 2.1. *Let X be a real vector space. A function $f : \mathbb{R} \to X$ is a Fibonacci function if and only if there exists a function $g : [-1, 1) \to X$ such that*

(2.1)
$$f(x) = \begin{cases} F_{[x]+1} g(x - [x]) + F_{[x]} g(x - [x] - 1) & (x \geq 0), \\ (-1)^{[x]} \left[F_{-[x]-1} g(x - [x]) - F_{-[x]} g(x - [x] - 1) \right] & (x < 0). \end{cases}$$

Proof. Since $\alpha + \beta = 1$ and $\alpha\beta = -1$, it follows from (1.1) that

$$(2.2) \quad \begin{cases} f(x) - \alpha f(x-1) = \beta \left[f(x-1) - \alpha f(x-2) \right], \\ f(x) - \beta f(x-1) = \alpha \left[f(x-1) - \beta f(x-2) \right]. \end{cases}$$

By mathematical induction, we can easily verify that

$$(2.3) \quad \begin{cases} f(x) - \alpha f(x-1) = \beta^n \left[f(x-n) - \alpha f(x-n-1) \right], \\ f(x) - \beta f(x-1) = \alpha^n \left[f(x-n) - \beta f(x-n-1) \right], \end{cases}$$

for all $x \in \mathbb{R}$ and $n \in \{0, 1, 2, \ldots\}$. If we substitute $x + n$ $(n \geq 0)$ for x in (2.3) and divide the resulting equations by β^n (resp. α^n), and if we then substitute $-m$ for n in the resulting equations, then we obtain the equations in (2.3) with m in place of n, where $m \in \{0, -1, -2, \ldots\}$. Therefore, the equations in (2.3) are true for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

We multiply the first and the second equations of (2.3) by β and α, respectively. If we subtract the first resulting equation from the second one, then we obtain:

$$(2.4) \quad f(x) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} f(x-n) + \frac{\alpha^n - \beta^n}{\alpha - \beta} f(x-n-1),$$

for any $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

For any given $x \geq 0$, if we put $n = [x]$ in (2.4), then it follows from Binet's formula (see [13, Theorem 5.6]) that

$$f(x) = F_{[x]+1} f(x - [x]) + F_{[x]} f(x - [x] - 1).$$

If $x < 0$, then we put $n = [x] = -|[x]|$ in (2.4). Since $\alpha\beta = -1$, by Binet's formula, we have,

$$\begin{aligned} f(x) &= \frac{\alpha^{-|[x]|+1} - \beta^{-|[x]|+1}}{\alpha - \beta} f(x - [x]) + \frac{\alpha^{-|[x]|} - \beta^{-|[x]|}}{\alpha - \beta} f(x - [x] - 1) \\ &= \frac{-1}{(\alpha\beta)^{|[x]|-1}} \frac{\alpha^{|[x]|-1} - \beta^{|[x]|-1}}{\alpha - \beta} f(x - [x]) \\ &\quad + \frac{-1}{(\alpha\beta)^{|[x]|}} \frac{\alpha^{|[x]|} - \beta^{|[x]|}}{\alpha - \beta} f(x - [x] - 1) \\ &= (-1)^{[x]} F_{|[x]|-1} f(x - [x]) + (-1)^{[x]+1} F_{|[x]|} f(x - [x] - 1) \\ &= (-1)^{[x]} \left[F_{-[x]-1} f(x - [x]) - F_{-[x]} f(x - [x] - 1) \right]. \end{aligned}$$

Since $0 \leq x - [x] < 1$ and $-1 \leq x - [x] - 1 < 0$, if we define a function $g : [-1, 1) \to X$ by $g := f|_{[-1,1)}$, then we can see that f is a function of the form (2.1).

Now, we assume that f is a function of the form (2.1), where $g : [-1, 1) \to X$ is an arbitrary function. We show that f is a Fibonacci function.

First, we assume that $x \geq 2$. Since $x > x - 1 > x - 2 \geq 0$ and $(x - 1) - [x - 1] = x - [x]$, it follows from (2.1) that

$$
\begin{cases}
f(x) = F_{[x]+1} g(x - [x]) + F_{[x]} g(x - [x] - 1), \\
f(x - 1) = F_{[x]} g(x - [x]) + F_{[x]-1} g(x - [x] - 1), \\
f(x - 2) = F_{[x]-1} g(x - [x]) + F_{[x]-2} g(x - [x] - 1).
\end{cases}
$$

Thus, we get

$$
\begin{aligned}
f(x - 1) &+ f(x - 2) \\
&= \left(F_{[x]} + F_{[x]-1} \right) g(x - [x]) + \left(F_{[x]-1} + F_{[x]-2} \right) g(x - [x] - 1) \\
&= F_{[x]+1} g(x - [x]) + F_{[x]} g(x - [x] - 1) \\
&= f(x).
\end{aligned}
$$

Now, suppose that $1 \leq x < 2$. For this case, we have $0 \leq x - 1 < 1$ and $-1 \leq x - 2 < 0$. Since $[x] = 1$, $[x - 1] = 0$ and $[x - 2] = -1$, it follows from (2.1) that

$$
\begin{cases}
f(x) = F_2 g(x - [x]) + F_1 g(x - [x] - 1), \\
f(x - 1) = F_1 g(x - [x]) + F_0 g(x - [x] - 1), \\
f(x - 2) = - \left[F_0 g(x - [x]) - F_1 g(x - [x] - 1) \right].
\end{cases}
$$

Consequently, since $F_0 = 0$ and $F_1 = F_2 = 1$, we obtain:

$$
f(x - 1) + f(x - 2) = F_2 g(x - [x]) + F_1 g(x - [x] - 1) = f(x).
$$

If $0 \leq x < 1$, then $-1 \leq x - 1 < 0$ and $-2 \leq x - 2 < -1$. In this case, we have $[x] = 0$, $[x - 1] = -1$ and $[x - 2] = -2$. Hence, (2.1) implies that

$$
\begin{cases}
f(x) = F_1 g(x - [x]), \\
f(x - 1) = F_1 g(x - [x] - 1), \\
f(x - 2) = F_1 g(x - [x]) - F_2 g(x - [x] - 1).
\end{cases}
$$

Therefore, we have

$$
f(x - 1) + f(x - 2) = F_1 g(x - [x]) = f(x).
$$

Finally, assume that $x < 0$. In view of (2.1), we see that

$$
\begin{cases}
f(x) = (-1)^{[x]} \left[F_{-[x]-1} g(x - [x]) - F_{-[x]} g(x - [x] - 1) \right], \\
f(x - 1) = (-1)^{[x]-1} \left[F_{-[x]} g(x - [x]) - F_{-[x]+1} g(x - [x] - 1) \right], \\
f(x - 2) = (-1)^{[x]-2} \left[F_{-[x]+1} g(x - [x]) - F_{-[x]+2} g(x - [x] - 1) \right].
\end{cases}
$$

These equations yield:

$$
\begin{aligned}
f(x &- 1) + f(x - 2) \\
&= (-1)^{[x]} \left[\left(F_{-[x]+1} - F_{-[x]} \right) g(x - [x]) \right. \\
&\qquad \left. - \left(F_{-[x]+2} - F_{-[x]+1} \right) g(x - [x] - 1) \right] \\
&= (-1)^{[x]} \left[F_{-[x]-1} g(x - [x]) - F_{-[x]} g(x - [x] - 1) \right] \\
&= f(x),
\end{aligned}
$$

which completes our proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3. Hyers-Ulam stability of Fibonacci equation

As already stated, α denotes the positive root of the quadratic equation $x^2 - x - 1 = 0$ and β is its negative root. We can prove the Hyers-Ulam stability of the Fibonacci functional equation (1.1) as we see in the following theorem.

Theorem 3.1. *Let $(X, \| \cdot \|)$ be a real Banach space. If a function $f : \mathbb{R} \to X$ satisfies the inequality,*

$$
\tag{3.1} \| f(x) - f(x - 1) - f(x - 2) \| \leq \varepsilon,
$$

for all $x \in \mathbb{R}$ and for some $\varepsilon > 0$, then there exists a Fibonacci function $G : \mathbb{R} \to X$ such that

$$
\tag{3.2} \| f(x) - G(x) \| \leq \left(1 + \frac{2}{\sqrt{5}} \right) \varepsilon,
$$

for all $x \in \mathbb{R}$.

Proof. Analogous to the first equation of (2.2), we get from (3.1):

$$
\| f(x) - \alpha f(x - 1) - \beta [f(x - 1) - \alpha f(x - 2)] \| \leq \varepsilon,
$$

for each $x \in \mathbb{R}$. If we replace x by $x - k$ in the last inequality, then we have,

$$\|f(x-k) - \alpha f(x-k-1) - \beta[f(x-k-1) - \alpha f(x-k-2)]\| \leq \varepsilon$$

and furthermore,

$$
(3.3) \quad
\begin{aligned}
&\|\beta^k[f(x-k) - \alpha f(x-k-1)] \\
&\quad - \beta^{k+1}[f(x-k-1) - \alpha f(x-k-2)]\| \\
&\leq \; |\beta|^k \varepsilon,
\end{aligned}
$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. By (3.3), we obviously have,

$$
(3.4) \quad
\begin{aligned}
&\|f(x) - \alpha f(x-1) - \beta^n[f(x-n) - \alpha f(x-n-1)]\| \\
&\leq \; \sum_{k=0}^{n-1} \|\beta^k[f(x-k) - \alpha f(x-k-1)] \\
&\qquad\qquad - \beta^{k+1}[f(x-k-1) - \alpha f(x-k-2)]\| \\
&\leq \; \sum_{k=0}^{n-1} |\beta|^k \varepsilon,
\end{aligned}
$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

For any $x \in \mathbb{R}$, (3.3) implies that $\{\beta^n[f(x-n) - \alpha f(x-n-1)]\}$ is a Cauchy sequence (note that $|\beta| < 1$). Therefore, we can define a function $G_1 : \mathbb{R} \to X$ by

$$G_1(x) = \lim_{n \to \infty} \beta^n[f(x-n) - \alpha f(x-n-1)],$$

since X is complete. In view of the above definition of G_1, we obtain:

$$
\begin{aligned}
&G_1(x-1) + G_1(x-2) \\
&= \; \beta^{-1} \lim_{n \to \infty} \beta^{n+1}[f(x-(n+1)) - \alpha f(x-(n+1)-1)] \\
&\qquad + \beta^{-2} \lim_{n \to \infty} \beta^{n+2}[f(x-(n+2)) - \alpha f(x-(n+2)-1)] \\
&= \; \beta^{-1} G_1(x) + \beta^{-2} G_1(x) \\
&= \; G_1(x),
\end{aligned}
$$

for all $x \in \mathbb{R}$. Hence, G_1 is a Fibonacci function. If n goes to infinity, then (3.4) yields:

$$(3.5) \qquad \|f(x) - \alpha f(x-1) - G_1(x)\| \leq \frac{3 + \sqrt{5}}{2} \varepsilon,$$

for every $x \in \mathbb{R}$.

On the other hand, it also follows from (3.1) that

$$\|f(x) - \beta f(x-1) - \alpha[f(x-1) - \beta f(x-2)]\| \leq \varepsilon$$

(see the second equation in (2.2)). Analogous to (3.3), replacing x by $x+k$ in the above inequality, we have,

(3.6)
$$\|\alpha^{-k}[f(x+k) - \beta f(x+k-1)]$$
$$- \alpha^{-k+1}[f(x+k-1) - \beta f(x+k-2)]\|$$
$$\leq \quad \alpha^{-k}\varepsilon,$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. By using (3.6), we further obtain:

$$\|\alpha^{-n}[f(x+n) - \beta f(x+n-1)] - [f(x) - \beta f(x-1)]\|$$

$$\leq \quad \sum_{k=1}^{n} \|\alpha^{-k}[f(x+k) - \beta f(x+k-1)]$$

(3.7)
$$- \alpha^{-k+1}[f(x+k-1) - \beta f(x+k-2)]\|$$

$$\leq \quad \sum_{k=1}^{n} \alpha^{-k}\varepsilon,$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

On account of (3.6), we see that $\{\alpha^{-n}[f(x+n) - \beta f(x+n-1)]\}$ is a Cauchy sequence, for any fixed $x \in \mathbb{R}$. Hence, we can define a function $G_2 : \mathbb{R} \to X$ by

$$G_2(x) = \lim_{n \to \infty} \alpha^{-n}[f(x+n) - \beta f(x+n-1)].$$

Using the above definition of G_2, we get:

$$G_2(x-1) + G_2(x-2)$$
$$= \quad \alpha^{-1} \lim_{n \to \infty} \alpha^{-(n-1)}[f(x+n-1) - \beta f(x+(n-1)-1)]$$
$$+ \alpha^{-2} \lim_{n \to \infty} \alpha^{-(n-2)}[f(x+n-2) - \beta f(x+(n-2)-1)]$$
$$= \quad \alpha^{-1} G_2(x) + \alpha^{-2} G_2(x)$$
$$= \quad G_2(x),$$

for any $x \in \mathbb{R}$. So, G_2 is also a Fibonacci function. If we let n go to infinity, then it follows from (3.7) that

(3.8)
$$\|G_2(x) - f(x) + \beta f(x-1)\| \leq \frac{\sqrt{5}+1}{2}\varepsilon,$$

for $x \in \mathbb{R}$.

By (3.5) and (3.8), we have,

$$\left\| f(x) - \left[\frac{\beta}{\beta - \alpha} G_1(x) - \frac{\alpha}{\beta - \alpha} G_2(x) \right] \right\|$$

$$= \frac{1}{|\beta - \alpha|} \| (\beta - \alpha) f(x) - [\beta G_1(x) - \alpha G_2(x)] \|$$

$$\leq \frac{1}{\alpha - \beta} \| \beta f(x) - \alpha\beta f(x-1) - \beta G_1(x) \|$$

$$+ \frac{1}{\alpha - \beta} \| \alpha G_2(x) - \alpha f(x) + \alpha\beta f(x-1) \|$$

$$\leq \left(1 + \frac{2}{\sqrt{5}} \right) \varepsilon,$$

for all $x \in \mathbb{R}$. We now set:

$$G(x) = \frac{\beta}{\beta - \alpha} G_1(x) - \frac{\alpha}{\beta - \alpha} G_2(x),$$

and it is not difficult to show that G is a Fibonacci function.

Before we prove the uniqueness of the Fibonacci function G of Theorem 3.1, we show the following result.

Lemma 3.2. *Let $(X, \| \cdot \|)$ be a real normed space and let $u, v \in X$ be given. If $\|F_{n+1}u + F_n v\| \leq C$, for all $n \in \mathbb{N}$ and for some $C \geq 0$, then $\alpha u + v = 0$.*

Proof. We have,

$$\begin{aligned} F_n \| \alpha u + v \| &= \| F_{n+1} u + F_n v - F_{n+1} u + \alpha F_n u \| \\ &\leq \| F_{n+1} u + F_n v \| + |F_{n+1} - \alpha F_n| \, \|u\| \\ &\leq C + \left| \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} - \alpha \frac{\alpha^n - \beta^n}{\alpha - \beta} \right| \|u\| \\ &= C + |\beta|^n \|u\|, \end{aligned}$$

for all $n \in \mathbb{N}$. Since $|\beta| < 1$ and $F_n \to \infty$ as $n \to \infty$, we obtain $\alpha u + v = 0$. $\qquad\square$

In the following theorem, we prove the uniqueness of the Fibonacci function G of Theorem 3.1.

Theorem 3.3. *The Fibonacci function $G : \mathbb{R} \to X$ of Theorem 3.1 is unique.*

Proof. For $i \in \{1, 2\}$, let $G_i : \mathbb{R} \to X$ be a Fibonacci function satisfying

(3.9) $$\|f(x) - G_i(x)\| \leq \left(1 + \frac{2}{\sqrt{5}}\right)\varepsilon,$$

for all $x \in \mathbb{R}$. Due to Theorem 2.1, there exist functions $g_i : [-1, 1) \to X$ ($i \in \{1, 2\}$) such that
(3.10)
$$G_i(x) = \begin{cases} F_{[x]+1}g_i(x - [x]) + F_{[x]}g_i(x - [x] - 1) & (x \geq 0), \\ \\ (-1)^{[x]}\left[F_{-[x]-1}g_i(x - [x]) - F_{-[x]}g_i(x - [x] - 1)\right] & (x < 0), \end{cases}$$

for $i \in \{1, 2\}$.

Fix a t with $0 \leq t < 1$. It then follows from (3.9) that

$$\begin{aligned} &\|G_1(n + t) - G_2(n + t)\| \\ &\leq \quad \|G_1(n + t) - f(n + t)\| + \|f(n + t) - G_2(n + t)\| \\ &\leq \quad 2\left(1 + \frac{2}{\sqrt{5}}\right)\varepsilon, \end{aligned}$$

for any $n \in \mathbb{Z}$. Furthermore, by (3.10) and the last inequality, we obtain:

$$\begin{aligned} &\|F_{n+1}[g_1(t) - g_2(t)] + F_n[g_1(t - 1) - g_2(t - 1)]\| \\ &= \quad \|G_1(n + t) - G_2(n + t)\| \\ &\leq \quad 2\left(1 + \frac{2}{\sqrt{5}}\right)\varepsilon \end{aligned}$$

and

$$\begin{aligned} &\|F_{n-1}[g_1(t) - g_2(t)] - F_n[g_1(t - 1) - g_2(t - 1)]\| \\ &= \quad \|G_1(-n + t) - G_2(-n + t)\| \\ &\leq \quad 2\left(1 + \frac{2}{\sqrt{5}}\right)\varepsilon, \end{aligned}$$

for each $n \in \mathbb{N}$.

According to Lemma 3.2, we have,

$$\begin{cases} \alpha[g_1(t) - g_2(t)] + [g_1(t - 1) - g_2(t - 1)] = 0, \\ \\ -\alpha[g_1(t - 1) - g_2(t - 1)] + [g_1(t) - g_2(t)] = 0 \end{cases}$$

or

$$\begin{pmatrix} \alpha & 1 \\ 1 & -\alpha \end{pmatrix} \begin{pmatrix} g_1(t) - g_2(t) \\ g_1(t-1) - g_2(t-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Because $-\alpha^2 - 1 \neq 0$, we conclude:

$$g_1(t) - g_2(t) = g_1(t-1) - g_2(t-1) = 0.$$

Since $0 \leq t < 1$ is arbitrary, we have $g_1(t) = g_2(t)$, for any $-1 \leq t < 1$; i.e., it follows from (3.10) that $G_1(x) = G_2(x)$, for all $x \in \mathbb{R}$. □

Acknowledgments

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