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ON THE CONVERGENCE OF THREE-STEP RANDOM ITERATIVE PROCESESS WITH ERRORS OF NONSELF ASYMPTOTICALLY NONEXPANSIVE RANDOM MAPPINGS

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Abstract. In this paper, we prove some strong and weak convergence of three step random iterative scheme with errors to common random fixed points of three asymptotically nonexpansive nonself random mappings in a real uniformly convex separable Banach space.

1. Introduction

Random fixed point theory is playing an increasing role in mathematics and applied sciences. At present, it received considerable attention due to enormous application in many important areas such as nonlinear analysis, probability theory and the study of random equations arising in various applied areas. Random fixed point theorems for random contraction mappings on separable complete metric spaces were first proved by Spacek [29] and Hans [12, 13]. The survey article by Bharucha-Reid [7] in 1976 attracted the attention of several mathematician and gave wings to this theory. Itoh [15] extended Spacek's result and Hans's theorem

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to multivalued contraction mappings. In an attempt to construct iterations for finding fixed points of random operators defined on linear spaces, random Ishikawa scheme was introduced in [10]. This iteration and some other random iterations based on the same ideas have been applied for finding solutions of random operators [10]. Recently, Papageorgiou [22], Xu [35], Beg [2–4], Beg and Shahzad [6] and many other authors have studied the fixed point of random mappings.

The class of asymptotically nonexpansive self-mappings were introduced by Goebel and Kirk [11] in 1972. Iterative techniques for approximating fixed points of nonexpansive self-mappings have been studied by various authors (see, e.g., [8, 14, 17, 20, 21, 26]). For nonself nonexpansive mappings, some authors (see, e.g., [16, 18, 28, 31, 36]) have studied the strong and weak convergence theorems in a Hilbert space or uniformly convex Banach spaces. Suhu [27] introduced a modified Mann iteration process to approximate fixed points of asymptotically nonexpansive self-mappings in a Hilbert space.

The concept of deterministic non-self asymptotically nonexpansive mappings was introduced by Chidume, Ofoedu, Zegeye [9] in 2003 as a generalization of asymptotically nonexpansive self-mappings. They studied the following iteration process

$$x_1 \in C,\ x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \tag{1.1}$$

where $T : C \to E$ is an asymptotically nonexpansive nonself-mapping, $\{\alpha_n\}$ is a real sequence in $(0, 1)$ and P is a nonexpansive retraction from E to C.

Wang [34] generalized the result of Chidume [9] and obtained some new results. He defined and studied the following iteration process:

$$\begin{aligned} x_{n+1} &= P((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n &= P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \quad x_1 \in C, \quad n \geq 1, \end{aligned}$$

where $T_1, T_2 : C \to E$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0, 1)$.

In 2009, Thianwan [33] introduced and studied a new class of iterative scheme. The scheme is defined as follows:

$$\begin{aligned} x_{n+1} &= P((1-\alpha_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n &= P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \quad x_1 \in C, \quad n \geq 1, \end{aligned} \tag{1.2}$$

where $T_1, T_2 : C \to E$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1)$.

Remark 1.1. *If $T_1 = T_2 = T$ and $\beta_n = 0$, for all $n \geq 1$. The iterative scheme (1.2) reduces to (1.1).*

More recently, Rashwan and Altwqi [25] extended the result of Thianwan [33] by introducing the following three-step iteration scheme:

$$\begin{aligned} x_{n+1} &= P((1-\alpha_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n &= P((1-\beta_n)z_n + \beta_n T_2(PT_2)^{n-1}z_n), \\ z_n &= P((1-\gamma_n)x_n + \gamma_n T_3(PT_3)^{n-1}x_n)\ , \quad x_1 \in C, \quad n \geq 1, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1)$. They studied the weak and strong convergence theorems of the above iteration under some conditions.

Remark 1.2. *(i) As $T_2 = T_3$ and $\gamma_n = 0$, for all $n \geq 1$, the iterative scheme (1.3) reduces to (1.2).*
(ii) As $T_1 = T_2 = T_3 = T$ and $\gamma_n = \beta_n = 0$, for all $n \geq 1$, the iterative scheme (1.3) reduces to (1.1).

For random operators, Beg and Abbas [5] studied the different random iterative algorithms for weakly contractive and asymptotically nonexpansive random operators on an arbitrary Banach space. They also established convergence of an implicit random iterative process to a common fixed point for a finite family of asymptotically quasi-nonexpansive operators. Plubtieng [23, 24] studied weak and strong convergence theorems established for a modified random Noor iterative scheme with errors for three asymptotically nonexpansive self-mappings in Banach space defined as follows:

$$\begin{aligned} \xi_{n+1}(t) &= \alpha_n T_1^n(t, \eta_n(t)) + \beta_n \xi_n(t) + \gamma_n f_n(t), \\ \eta_n(t) &= \alpha'_n T_2^n(t, \zeta_n(t)) + \beta'_n \xi_n(t) + \gamma'_n f'_n(t), \\ \zeta_n(t) &= \alpha''_n T_3^n(t, \xi_n(t)) + \beta''_n \xi_n(t) + \gamma''_n f''_n(t)\ , n \geq 1, t \in \Omega, \end{aligned} \tag{1.4}$$

where $T_1, T_2, T_3 : \Omega \times C \to C$ are three asymptotically nonexpansive random mappings, $\xi_1(t) : \Omega \to C$ is a measurable mapping from Ω to C, $\{f_n(t)\}$, $\{f'_n(t)\}$, $\{f''_n(t)\}$ are bounded sequences of measurable functions from Ω to C and $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\beta'_n\}$, $\{\beta''_n\}$, $\{\gamma_n\}$, $\{\gamma'_n\}$, $\{\gamma''_n\}$ are sequences of real numbers in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$.

Remark 1.3. *If we take $T_1 = T_2 = T_3 = T$ and $\gamma_n = \gamma_n' = \gamma_n'' = 0$, then (1.4) becomes as the following:*

$$\begin{aligned} \xi_{n+1}(t) &= \alpha_n T^n(t, \eta_n(t)) + \beta_n \xi_n(t), \\ \eta_n(t) &= \alpha_n' T^n(t, \zeta_n(t)) + \beta_n' \xi_n(t), \\ \zeta_n(t) &= \alpha_n'' T^n(t, \xi_n(t)) + \beta_n'' \xi_n(t) \ , n \geq 1, t \in \Omega, \end{aligned}$$

which was studied by Beg and Abbas in [5].

For nonself random mappings, Zhou and Wang [37] studied the approximation of the following iteration process:

$$\begin{aligned} \xi_{n+1}(t) &= P((1-\alpha_n)\xi_n(t) + \alpha_n T(PT)^{n-1}(t, \eta_n(t))), \\ \eta_n(t) &= P((1-\beta_n)\xi_n(t) + \beta_n T(PT)^{n-1}(t, \xi_n(t))) \ , n \geq 1, t \in \Omega, \end{aligned}$$

where $T : \Omega \times C \to E$ is an asymptotically nonexpansive nonself random mapping, $\xi_1(t) : \Omega \to C$ is a measurable mapping from Ω to C, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in [0,1] and P is a nonexpansive retraction from E to C.
In this paper, we construct a projection type random iteration with errors and study its approximation to common random fixed points of three nonself asymptotically nonexpansive random mappings in a real uniformly convex separable Banach space. Our results extend and improve some recent results in [23, 25, 37].

2. Preliminaries

Let (Ω, Σ) be a measurable space, C a nonempty subset of E. A mapping $\xi : \Omega \to C$ is called measurable if $\xi^{-1}(B \bigcap C) \in \Sigma$ for every Borel subset B of a Banach space E.
A mapping $T : \Omega \times C \to C$ is said to be random mapping if for each fixed $x \in C$, the mapping $T(., x) : \Omega \to C$ is measurable.
A measurable mapping $\xi : \Omega \to C$ is called a random fixed point of the random mapping $T : \Omega \times C \to C$ if $T(t, \xi(t)) = \xi(t)$ for each $t \in \Omega$.
Throughout this paper, we denote the set of all random fixed points of random mapping T by $RF(T)$ and the nth iterate $T(t, T(t, T(, ...T(t, x))))$ of T by $T^n(t, x)$. The letter I denotes the identity random mapping $I : \Omega \times C \to C$ defined by $I(t, x) = x$ and $T^0 = I$.

Definition 2.1. *[19] A Banach space E is said to satisfy the Opial's condition if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ weakly as $n \to \infty$ and*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,$$

for all $y \in E$ with $y \neq x$.

Definition 2.2. *A map $T : C \to E$ is called demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \to x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.*

Definition 2.3. *A mapping $T : C \to C$ is completely continuous if and only if $\{Tx_n\}$ has a convergent subsequence for every bounded sequence $\{x_n\}$ in C.*

Definition 2.4. *A subset C of E is said to be a retract of E if there exists a continuous map $P : E \to C$ such that $Px = x$ for all $x \in C$. Note that every closed convex subset of uniformly convex Banach space is retract. A map $P : E \to E$ is a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P.*

Definition 2.5. *A mapping $T : C \to E$ is said to be semicompact if for any sequence $\{x_n\}$ in C with $\lim\limits_{n\to\infty} \| x_n - Tx_n \| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to q \in C$.*

Definition 2.6. *[1] A finite family $\{T_i : i \in I\}$ of N continuous random operators from $\Omega \times C$ to E with $F = \bigcap\limits_{i=1}^{N} RF(T_i) \neq \emptyset$, is said to satisfy condition B if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) \geq 0$ for all $r \in (0,\infty)$ such that for all $t \in \Omega$ $f(d(\xi(t), F)) \leq \max\limits_{1\leq i\leq N}\{\|\xi(t) - T_i\ (t, \xi(t))\|\}$ for all $\xi(t)$, where $d(\xi(t), F) = \inf\{\|\xi(t) - q(t)\| : q(t) \in F = \bigcap\limits_{i=1}^{N} RF(T_i)\}$.*

Definition 2.7. *[9, 37] Let C be a nonempty closed convex subset of a real uniformly convex separable Banach space and let $T : \Omega \times C \to E$ be a nonself random mapping. Then T is said to be*

(1) *Nonexpansive random operator if for arbitrary $x, y \in C$, $\|T(t, x) - T(t, y)\| \leq \|x - y\|$ for all $t \in \Omega$.*
(2) *Non-self asymptotically nonexpansive random mapping if there exists a measurable mapping sequence $r_n(t) : \Omega \to [1,\infty)$ with $\lim_{n\to\infty} r_n(t) = 1$ for each $t \in \Omega$ such that for arbitrary $x, y \in C$ and $t \in \Omega$*

$$\|T(PT)^{n-1}(t, x) - T(PT)^{n-1}(t, y)\| \leq r_n(t)\|x - y\|, n = 1, 2, \cdots .$$

(3) *Uniformly L-Lipschitzian random mapping if there exists a constant $L > 0$ such that for arbitrary $x, y \in C$ and $t \in \Omega$*

$$\|T(PT)^{n-1}(t,x) - T(PT)^{n-1}(t,y)\| \leq L\|x-y\|, \quad n = 1, 2, \cdots .$$

(4) *Semicompact random mapping if for any sequence of measurable mappings $\{\xi_n\}$ from Ω to C with $\lim_{n\to\infty} \|\xi_n(t) - T(t, \xi_n(t))\| = 0$ for all $t \in \Omega$ there exists a subsequence $\{\xi_{n_j}(t)\}$ of $\{\xi_n(t)\}$ such that $\{\xi_{n_j}(t)\} \to \{\xi_n(t)\}$ as $j \to \infty$ for each $t \in \Omega$, where $\{\xi(t)\}$ is a measurable mapping from Ω to C.*

(5) *Completely continuous random mapping if and only if $\{T(t, \xi_n(t))\}$ has a convergent subsequence for every bounded sequence $\{\xi_n(t)\}$ in C.*

Remark 2.8. *Every nonself asymptotically nonexpansive random mapping is uniformly L-Lipschitzian, where $L = \sup\limits_{t\in\Omega, n\geq 1} r_n(t)$.*

Definition 2.9. *Let $T_1, T_2, T_3 : \Omega \times C \to E$ be three nonself random mappings, where C is a nonempty convex subset of a separable Banach space E. Let $\xi_1 : \Omega \to C$ be a measurable mapping from Ω to C. The projection random iteration scheme with errors is defined for $t \in \Omega$ as follows:*

$$\begin{aligned} \xi_{n+1}(t) &= P((1-\alpha_n-\sigma_n)\eta_n(t)+\alpha_n T_1(PT_1)^{n-1}(t,\eta_n(t))+\sigma_n f_n(t)), \\ \eta_n(t) &= P((1-\beta_n-\delta_n)\zeta_n(t)+\beta_n T_2(PT_2)^{n-1}(t,\zeta_n(t))+\delta_n g_n(t)), \\ \zeta_n(t) &= P((1-\gamma_n-\lambda_n)\xi_n(t)+\gamma_n T_3(PT_3)^{n-1}(t,\xi_n(t))+\lambda_n h_n(t)) \ , n\geq 1, \end{aligned} \tag{2.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\sigma_n\}$, $\{\delta_n\}$ and $\{\lambda_n\}$ are sequences in $[0, 1]$, $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ are bounded sequence of measurable functions from Ω to C, and P is a nonexpansive retraction from E to C.

Clearly, ξ_n, η_n and ζ_n are measurable sequences from Ω to C.
The following lemmas are useful for proving our main results.

Lemma 2.10. *[32] Let $\{a_n\}$, $\{b_n\}$ and $\{m_n\}$ be nonnegative real sequences satisfying*

$$a_{n+1} \leq (1 + m_n)a_n + b_n,$$

for all $n \geq 1$. If $\sum\limits_{n=1}^{\infty} m_n < \infty$ and $\sum\limits_{n=1}^{\infty} b_n < \infty$, then

(1) $\lim\limits_{n\to\infty} a_n$ *exists.*
(2) $\lim_{n\to\infty} a_n = 0$ *whenever* $\liminf_{n\to\infty} a_n = 0$.

Lemma 2.11. *[27] Let E be a uniformly convex Banach space, and $0 \leq p \leq t_n \leq q < 1$ for all positive integer $n \geq 1$. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.12. *[9] Let E be a uniformly convex Banach space, C a nonempty closed convex subset of E and $T : C \to E$ a nonself asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed at zero, i.e., if $x_n \to x$ weakly and $\|x_n - Tx_n\| \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed points of T.*

Lemma 2.13. *[30] Let E be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in E. Let $u, v \in E$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exists. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

3. Main Results

In this section, we will prove the strong and weak convergence of the iteration scheme (2.1) to a common random fixed point for three asymptotically nonexpansive nonself random mappings in a uniformly convex separable Banach space.

Lemma 3.1. *Let E be a real uniformly convex separable Banach space and let C be a nonempty closed convex subset of E with P as a nonexpansive retraction. Let $T_i : \Omega \times C \to E$, $i = 1, 2, 3$ be three asymptotically nonexpansive nonself random mappings with sequences of measurable mappings $\{r_{i_n}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (r_{i_n}(t) - 1) < \infty$, $r_{i_n}(t) \to 1$ as $n \to \infty$, for all $t \in \Omega$ and $i = 1, 2, 3$. Suppose that $\bigcap_{i=1}^{3} RF(T_i) \neq \emptyset$ and let $\{\xi_n(t)\}$ be the sequence defined in (2.1) with the additional assumption $\sum_{n=1}^{\infty} \sigma_n < \infty, \sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$. Then $\lim_{n\to\infty} \|\xi_n(t) - \xi(t)\|$ exists for all $\xi(t) \in \bigcap_{i=1}^{3} F(T_i)$.*

Proof. Let $\xi(t) \in \bigcap_{i=1}^{3} F(T_i)$. Since $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ are bounded sequences of measurable function from Ω to C, for each $t \in \Omega$ we can put

$$M_n(t) = \sup_{n\geq 1} \|f_n(t) - \xi(t)\| \vee \sup_{n\geq 1} \|g_n(t) - \xi(t)\| \vee \sup_{n\geq 1} \|h_n(t) - \xi(t)\|. \tag{3.1}$$

Then $M_n(t) < \infty$ for each $t \in \Omega$ and $n \geq 1$. Also, for each $n \geq 1$ let $r_n(t) = \max\{r_{i_n}(t) : i = 1, 2, 3\}$. Setting $r_n(t) = 1 + u_n(t)$. Thus by the hypothesis of the theorem we have $\sum_{n=1}^{\infty} u_n(t) = \sum_{n=1}^{\infty} (r_n(t) - 1) < \infty$.

Using (2.1) and (3.1) for $\xi(t) \in \bigcap_{i=1}^{3} F(T_i)$ and $t \in \Omega$ we have

$$\begin{aligned}
\|\zeta_n(t)-\xi(t)\| &= \|P((1-\gamma_n-\lambda_n)\xi_n(t)+\gamma_n T_3(PT_3)^{n-1}(t,\xi_n(t))+\lambda_n h_n(t))-\xi(t)\| \\
&\leq \|(1-\gamma_n-\lambda_n)\xi_n(t)+\gamma_n T_3(PT_3)^{n-1}(t,\xi_n(t))+\lambda_n h_n(t)-\xi(t)\| \\
&= \|(1-\gamma_n-\lambda_n)(\xi_n(t)-\xi(t))+\gamma_n(T_3(PT_3)^{n-1}(t,\xi_n(t))-\xi(t)) \\
&+ \lambda_n(h_n(t)-\xi(t))\| \\
&\leq (1-\gamma_n-\lambda_n)\|\xi_n(t)-\xi(t)\|+\gamma_n\|T_3(PT_3)^{n-1}(t,\xi_n(t))-\xi(t)\| \\
&+ \lambda_n\|h_n(t)-\xi(t)\| \\
&\leq (1-\gamma_n-\lambda_n)\|\xi_n(t)-\xi(t)\|+\gamma_n r_{3_n}(t)\|\xi_n(t)-\xi(t)\|+\lambda_n M_n(t) \\
&\leq (1-\gamma_n-\lambda_n)\|\xi_n(t)-\xi(t)\|+\gamma_n r_n(t)\|\xi_n(t)-\xi(t)\|+\lambda_n M_n(t) \\
&\leq (1-\gamma_n-\lambda_n)\|\xi_n(t)-\xi(t)\|+\gamma_n(1+u_n(t))\|\xi_n(t)-\xi(t)\|+\lambda_n M_n(t) \\
&\leq (1-\lambda_n+\gamma_n u_n(t))\|\xi_n(t)-\xi(t)\|+\lambda_n M_n(t) \\
&\leq (1+u_n(t))\|\xi_n(t)-\xi(t)\|+\lambda_n M_n(t). \qquad (3.2)
\end{aligned}$$

Again using (2.1) and (3.2) we obtain

$$\begin{aligned}
\|\eta_n(t)-\xi(t)\| &= \|P((1-\beta_n-\delta_n)\zeta_n(t)+\beta_n T_2(PT_2)^{n-1}(t,\zeta_n(t))+\delta_n g_n(t))-\xi(t)\| \\
&\leq (1-\beta_n-\delta_n)\|\zeta_n(t)-\xi(t)\|+\beta_n\|T_2(PT_2)^{n-1}(t,\zeta_n(t))-\xi(t)\| \\
&+ \delta_n\|g_n(t)-\xi(t)\| \\
&\leq (1-\beta_n-\delta_n)\|\zeta_n(t)-\xi(t)\|+\beta_n r_{2_n}(t)\|\zeta_n(t)-\xi(t))\|+\delta_n M_n(t) \\
&\leq (1-\beta_n-\delta_n)\|\zeta_n(t)-\xi(t)\|+\beta_n(1+u_n(t))\|\zeta_n(t)-\xi(t))\|+\delta_n M_n(t) \\
&\leq (1+u_n(t))\|\zeta_n(t)-\xi(t))\|+\delta_n M_n(t) \\
&\leq (1+u_n(t))[(1+u_n(t))\|\xi_n(t)-\xi(t)\|+\lambda_n M_n(t)]+\delta_n M_n(t) \\
&= (1+u_n(t))^2\|\xi_n(t)-\xi(t)\|+(1+u_n(t))\lambda_n M_n(t)+\delta_n M_n(t) \\
&= (1+2u_n(t)+u_n^2(t))\|\xi_n(t)-\xi(t)\|+A_n(t), \qquad (3.3)
\end{aligned}$$

where $A_n(t) = (1+u_n(t))\lambda_n M_n(t) + \delta_n M_n(t)$. Note that $\sum_{n=1}^{\infty} A_n(t) < \infty$. Thus by using (3.3) we get

$$\begin{aligned}
\|\xi_{n+1}(t)-\xi(t)\| &= \|P((1-\alpha_n-\sigma_n)\eta_n(t)+\alpha_n T_1(PT_1)^{n-1}(t,\eta_n(t))+\sigma_n f_n(t))-\xi(t)\| \\
&\leq (1-\alpha_n-\sigma_n)\|\eta_n(t)-\xi(t)\|+\alpha_n\|T_1(PT_1)^{n-1}(t,\eta_n(t))-\xi(t)\| \\
&+ \sigma_n\|f_n(t))-\xi(t)\| \\
&\leq (1-\alpha_n-\sigma_n)\|\eta_n(t)-\xi(t)\|+\alpha_n r_{1_n}(t)\|\eta_n(t)-\xi(t)\|+\sigma_n M_n(t) \\
&\leq (1-\alpha_n-\sigma_n)\|\eta_n(t)-\xi(t)\|+\alpha_n(1+u_n(t))\|\eta_n(t)-\xi(t)\|+\sigma_n M_n(t) \\
&\leq (1+u_n(t))\|\eta_n(t)-\xi(t)\|+\sigma_n M_n(t) \\
&\leq (1+u_n(t))[(1+2u_n(t)+u_n^2(t))\|\xi_n(t)-\xi(t)\|+A_n(t)]+\sigma_n M_n(t) \\
&= (1+3u_n(t)+3u_n^2(t)+u_n^3(t))\|\xi_n(t)-\xi(t)\|+(1+u_n(t))A_n(t)+\sigma_n M_n(t) \\
&= (1+B_n(t))\|\xi_n(t)-\xi(t)\|+D_n(t),
\end{aligned}$$

where $B_n(t) = 3u_n(t) + 3u_n^2(t) + u_n^3(t)$ and $D_n(t) = (1 + u_n(t))A_n(t) + \sigma_n M_n(t)$. Since $\sum_{n=1}^{\infty} B_n(t) < \infty$ and $\sum_{n=1}^{\infty} D_n(t) < \infty$, it follows from lemma 2.10 that $\lim_{n\to\infty} \|\xi_n(t) - \xi(t)\|$ exists for all $\xi(t) \in \bigcap_{i=1}^{3} F(T_i)$ and $t \in \Omega$. □

Lemma 3.2. *Let E be a real uniformly convex separable Banach space and let C be a nonempty closed convex subset of E with P as a nonexpansive retraction. Let $T_i : \Omega \times C \to E$, $i = 1, 2, 3$ be three asymptotically nonexpansive nonself random mappings with sequences of measurable mappings $\{r_{i_n}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(r_{i_n}(t) - 1) < \infty$, $r_{i_n}(t) \to 1$ as $n \to \infty$, for all $t \in \Omega$ and $i = 1, 2, 3$. Suppose that $\bigcap_{i=1}^{3} RF(T_i) \neq \emptyset$ and $\{\xi_n(t)\}$ is the sequence defined as in (2.1) with the additional assumptions $\sum_{n=1}^{\infty}\sigma_n < \infty$, $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty}\lambda_n < \infty$. Then $\lim_{n\to\infty} \|\xi_n(t) - T_i(t, \xi_n(t))\| = 0$, for each $t \in \Omega$ and $i = 1, 2, 3$.*

Proof. Let $\xi(t) \in \bigcap_{i=1}^{3} RF(T_i)$. Since $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ are bounded sequences of measurable functions from Ω to C, for each $t \in \Omega$ we can put

$$M_n(t) = \sup_{n\geq 1} \|f_n(t) - \xi(t)\| \vee \sup_{n\geq 1} \|g_n(t) - \xi(t)\| \vee \sup_{n\geq 1} \|h_n(t) - \xi(t)\|.$$

Then $M_n(t) < \infty$ for each $t \in \Omega$. Also, for each $n \geq 1$ let $r_n(t) = \max\{r_{i_n}(t) : i = 1, 2, 3\}$. Let $r_n(t) = 1 + u_n(t)$. Then by the hypothesises of the theorem we have $\sum_{n=1}^{\infty} u_n(t) = \sum_{n=1}^{\infty} (r_n(t) - 1) < \infty$ for each $t \in \Omega$. By Lemma 3.1, we see that $\lim_{n\to\infty} \|\xi_n(t) - \xi(t)\|$ exists for each $t \in \Omega$. Assume that $\lim_{n\to\infty} \|\xi_n(t) - \xi(t)\| = c$. From (3.2), we have

$$\|\zeta_n(t) - \xi(t)\| \leq (1 + u_n(t))\|\xi_n(t) - \xi(t)\| + \lambda_n M_n(t).$$

Taking lim sup on both sides of the above inequality, (where $\lim_{n\to\infty} \lambda_n = 0$), we have

$$\limsup_{n\to\infty} \|\zeta_n(t) - \xi(t)\| \leq c. \tag{3.4}$$

In addition we see that $\|T_2(PT_2)^{n-1}(t, \zeta_n(t)) - \xi(t))\| \leq r_{2_n}(t)\|\zeta_n(t) - \xi(t)\|$. Taking the lim sup on both sides in this inequality, we have

$$\limsup_{n\to\infty} \|T_2(PT_2)^{n-1}(t, \zeta_n(t)) - \xi(t)\| \leq c. \tag{3.5}$$

Similarly, using (3.3), we have

$$\|\eta_n(t) - \xi(t)\| \leq (1 + 2u_n(t) + u_n^2(t))\|\xi_n(t) - \xi(t)\| + A_n(t).$$

Taking lim sup on both sides in the above inequality, (where $\lim_{n\to\infty} A_n(t) = 0$), we have

$$\limsup_{n\to\infty} \|\eta_n(t) - \xi(t)\| \leq c. \tag{3.6}$$

In addition we have $\|T_1(PT_1)^{n-1}(t, \eta_n(t)) - \xi(t)\| \leq r_{1_n}\|\eta_n(t) - \xi(t)\|$. Taking lim sup on both sides in the above inequality, we have

$$\limsup_{n\to\infty} \|T_1(PT_1)^{n-1}(t, \eta_n(t)) - \xi(t)\| \leq c. \tag{3.7}$$

Since $\lim_{n\to\infty}(\sigma_n) = 0$, it follows from (3.7) that

$$\limsup_{n\to\infty} \|T_1(PT_1)^{n-1}(t, \eta_n(t)) - \xi(t) + \sigma_n(f_n(t) - \eta_n(t))\|$$
$$\leq \limsup_{n\to\infty} \|T_1(PT_1)^{n-1}(t, \eta_n(t)) - \xi(t)\| \leq c. \tag{3.8}$$

In addition by (3.6) we have

$$\limsup_{n\to\infty} \|\eta_n(t) - \xi(t) + \sigma_n(f_n(t) - \eta_n(t))\|$$
$$\leq \limsup_{n\to\infty} \|\eta_n(t) - \xi(t)\| \leq c. \tag{3.9}$$

Now, using (2.1) we have

$$\begin{aligned}\|\xi_{n+1}(t)-\xi(t)\| &= \|P((1-\alpha_n-\sigma_n)\eta_n(t)+\alpha_n T_1(PT_1)^{n-1}(t,\eta_n(t))+\sigma_n f_n(t))-\xi(t)\|\\ &\leq \|(1-\alpha_n-\sigma_n)\eta_n(t)+\alpha_n T_1(PT_1)^{n-1}(t,\eta_n(t))+\sigma_n f_n(t))-\xi(t)\|\\ &= \|(1-\alpha_n)\eta_n(t)-(1-\alpha_n)\xi(t)-\sigma_n\eta_n(t)+\sigma_n f_n(t)-\alpha_n\sigma_n f_n(t)\\ &- \alpha_n\sigma_n\eta_n(t)+\alpha_n T_1(PT_1)^{n-1}(t,\eta_n(t))-\alpha_n\xi(t)+\alpha_n\sigma_n f_n(t)-\alpha_n\sigma_n\eta_n(t)\|\\ &= \|(1-\alpha_n)(\eta_n(t)-\xi(t)+\sigma_n(f_n(t)-\eta_n(t)))\\ &+ \alpha_n(T_1(PT_1)^{n-1}(t,\eta_n(t))-\xi(t)+\sigma_n(f_n(t)-\eta_n(t)))\big\|.\end{aligned}$$

Taking lim inf on both sides of the above inequality, we obtain

$$\begin{aligned} c &\leq \liminf_{n\to\infty}\|(1-\alpha_n)(\eta_n(t)-\xi(t)+\sigma_n(f_n(t)-\eta_n(t)))\\ &+ \alpha_n(T_1(PT_1)^{n-1}(t,\eta_n(t))-\xi(t)+\sigma_n(f_n(t)-\eta_n(t)))\|. \end{aligned} \tag{3.10}$$

On the other hand, using (3.8) and (3.9) we get

$$\begin{aligned}\limsup_{n\to\infty} & \|(1-\alpha_n)(\eta_n(t)-\xi(t)+\sigma_n(f_n(t)-\eta_n(t)))\\ + & \alpha_n(T_1(PT_1)^{n-1}(t,\eta_n(t))-\xi(t)+\sigma_n(f_n(t)-\eta_n(t)))\|\\ \leq & (1-\alpha_n)\limsup_{n\to\infty}\|\eta_n(t)-\xi(t)+\sigma_n(f_n(t)-\eta_n(t))\|\\ + & \alpha_n\limsup_{n\to\infty}\|T_1(PT_1)^{n-1}(t,\eta_n(t))-\xi(t)+\sigma_n(f_n(t)-\eta_n(t))\|\leq c.\end{aligned} \tag{3.11}$$

Both inequalties (3.10) and (3.11) imply that

$$\begin{aligned}\lim_{n\to\infty}&\|(1-\alpha_n)(\eta_n(t)-\xi(t)+\sigma_n(f_n(t)-\eta_n(t)))\\ &+\alpha_n(T_1(PT_1)^{n-1}(t,\eta_n(t))-\xi(t)+\sigma_n(f_n(t)-\eta_n(t)))\|=c.\end{aligned} \tag{3.12}$$

It follows from (3.8),(3.9),(3.12) and Lemma (2.11) that

$$\lim_{n\to\infty}\|T_1(PT_1)^{n-1}(t,\eta_n(t))-\eta_n(t)\|=0. \tag{3.13}$$

In addition we have $\|T_3(PT_3)^{n-1}(t,\xi_n(t))-\xi(t)\|\leq r_{3_n}(t)\|\xi_n(t)-\xi(t)\|$. Taking lim sup on both sides in the above inequality, we have

$$\limsup_{n\to\infty}\|T_3(PT_3)^{n-1}(t,\xi_n(t))-\xi(t)\|\leq c. \tag{3.14}$$

Again using (2.1) we get

$$\begin{aligned}\|\xi_{n+1}(t)-\xi(t)\| &= \|P((1-\alpha_n-\sigma_n)\eta_n(t)+\alpha_n T_1(PT_1)^{n-1}(t,\eta_n(t))+\sigma_n f_n(t))-\xi(t)\|\\ &\leq (1-\alpha_n-\sigma_n)\|\eta_n(t)-\xi(t)\|+\alpha_n\|T_1(PT_1)^{n-1}(t,\eta_n(t))-\xi(t)\|\\ &+ \sigma_n\|f_n(t))-\xi(t)\|\\ &= (1-\alpha_n-\sigma_n)\|\eta_n(t)-\xi(t)\|\\ &+ \alpha_n\|T_1(PT_1)^{n-1}(t,\eta_n(t))-\eta_n(t)+\eta_n(t)-\xi(t)\|+\sigma_n\|f_n(t)-\xi(t)\|\\ &\leq (1-\alpha_n-\sigma_n)\|\eta_n(t)-\xi(t)\|+\alpha_n\|T_1(PT_1)^{n-1}(t,\eta_n(t))-\eta_n(t)\|\\ &+ \alpha_n\|\eta_n(t)-\xi(t)\|+\sigma_n\|f_n(t)-\xi(t)\|\\ &\leq (1-\sigma_n)\|\eta_n(t)-\xi(t)\|+\|T_1(PT_1)^{n-1}(t,\eta_n(t))-\eta_n(t)\|+\sigma_n\|f_n(t)-\xi(t)\|\\ &\leq \|\eta_n(t)-\xi(t)\|+\|T_1(PT_1)^{n-1}(t,\eta_n(t))-\eta_n(t)\|+\sigma_n\|f_n(t)-\xi(t)\|.\end{aligned}$$

Since $\lim_{n\to\infty}\|\xi_{n+1}(t)-\xi(t)\| = c$, by taking $\liminf$ on both sides of the above inequality we have

$$\liminf_{n\to\infty}\|\eta_n(t)-\xi(t)\| \geq c. \tag{3.15}$$

It follows from (3.6) and (3.15) that

$$\lim_{n\to\infty}\|\eta_n(t)-\xi(t)\| = c. \tag{3.16}$$

Thus

$$\begin{aligned}\|\eta_n(t)-\xi(t)\| &= \|P((1-\beta_n-\delta_n)\zeta_n(t)+\beta_n T_2(PT_2)^{n-1}(t,\zeta_n(t))+\delta_n g_n(t))-\xi(t)\|\\ &\leq \|(1-\beta_n-\delta_n)\zeta_n(t)+\beta_n T_2(PT_2)^{n-1}(t,\zeta_n(t))+\delta_n g_n(t)-\xi(t)\|\\ &= \|(1-\beta_n)(\zeta_n(t)-\xi(t)+\delta_n(g_n(t)-\zeta_n(t)))\\ &+ \beta_n(T_2(PT_2)^{n-1}(t,\zeta_n(t))-\xi(t)+\delta_n(g_n(t)-\zeta_n(t)))\|.\end{aligned}$$

Taking $\liminf$ on both sides in the above inequality and using (3.16) we have

$$\begin{aligned}c &\leq \liminf_{n\to\infty}\|(1-\beta_n)(\zeta_n(t)-\xi(t)+\delta_n(g_n(t)-\zeta_n(t)))\\ &+ \beta_n(T_2(PT_2)^{n-1}(t,\zeta_n(t))-\xi(t)+\delta_n(g_n(t)-\zeta_n(t))\|.\end{aligned} \tag{3.17}$$

On the other hand, using (3.4) and (3.5) we obtain

$$\begin{array}{ll}\limsup_{n\to\infty} & \|(1-\beta_n)(\zeta_n(t)-\xi(t)+\delta_n(g_n(t)-\zeta_n(t))) \\ + & \beta_n(T_2(PT_2)^{n-1}(t,\zeta_n(t))-\xi(t)+\delta_n(g_n(t)-\zeta_n(t)))\| \\ \leq & \limsup_{n\to\infty}(1-\beta_n)\|\zeta_n(t)-\xi(t)+\delta_n(g_n(t)-\zeta_n(t))\| \\ + & \beta_n \limsup_{n\to\infty}\|T_2(PT_2)^{n-1}(t,\zeta_n(t))-\xi(t)+\delta_n(g_n(t)-\zeta_n(t))\|\leq c.\end{array}$$

(3.18)

Both inequalities (3.17) and (3.18) imply that

$$\lim_{n\to\infty}\|(1-\beta_n)(\zeta_n(t)-\xi(t)+\delta_n(g_n(t)-\zeta_n(t))) +\beta_n(T_2(PT_2)^{n-1}(t,\zeta_n(t))-\xi(t)+\delta_n(g_n(t)-\zeta_n(t)))\|=c$$

Using lemma 2.11, we have

$$\lim_{n\to\infty}\|T_2(PT_2)^{n-1}(t,\zeta_n(t))-\zeta_n(t)\|=0. \tag{3.19}$$

It follows that

$$\begin{array}{lll}\|\eta_n(t)-\xi(t)\| & = & \|P((1-\beta_n-\delta_n)\zeta_n(t)+\beta_n T_2(PT_2)^{n-1}(t,\zeta_n(t))+\delta_n g_n(t))-\xi(t)\| \\ & \leq & (1-\beta_n-\delta_n)\|\zeta_n(t)-\xi(t)\|+\beta_n\|T_2(PT_2)^{n-1}(t,\zeta_n(t))-\zeta_n(t)+\zeta_n(t)-\xi(t))\| \\ & + & \delta_n\|g_n(t)-\xi(t)\| \\ & \leq & (1-\beta_n-\delta_n)\|\zeta_n(t)-\xi(t)\|+\beta_n\|T_2(PT_2)^{n-1}(t,\zeta_n(t))-\zeta_n(t)\| \\ & + & \beta_n\|\zeta_n(t)-\xi(t))\|+\delta_n\|g_n(t)-\xi(t)\| \\ & \leq & \|\zeta_n(t)-\xi(t)\|+\|T_2(PT_2)^{n-1}(t,\zeta_n(t))-\zeta_n(t)\|+\delta_n\|g_n(t)-\xi(t)\|.\end{array}$$

Taking lim inf on both sides in the above inequality and using (3.19) we get

$$c \leq \liminf_{n\to\infty}\|\zeta_n(t)-\xi(t)\|. \tag{3.20}$$

From (3.4) and (3.20), we get that

$$lim_{n\to\infty}\|\zeta_n(t)-\xi(t)\|=c. \tag{3.21}$$

Similarly, by using (3.14), (3.21) and the same arguments as above we get

$$\begin{array}{ll}\lim_{n\to\infty} & \|(1-\gamma_n)(\xi_n(t)-\xi(t)+\lambda_n(h_n(t)-\xi_n(t))) \\ + & \gamma_n(T_3(PT_3)^{n-1}(t,\xi_n(t))-\xi(t)+\lambda_n(h_n(t)-\xi_n(t)))\|=c.\end{array}$$

By Lemma (2.11) this implies that

$$lim_{n\to\infty}\|(T_3(PT_3)^{n-1}(t,\xi_n(t))-\xi_n(t)\|=0. \tag{3.22}$$

Now, using (3.22) we obtain

$$\begin{aligned}\|\zeta_n(t)-\xi_n(t)\| &= \|P((1-\gamma_n-\lambda_n)\xi_n(t)+\gamma_n T_3(PT_3)^{n-1}(t,\xi_n(t))+\lambda_n h_n(t))-\xi_n(t)\| \\ &\leq (1-\gamma_n-\lambda_n)\|\xi_n(t)-\xi_n(t)\|+\gamma_n\|T_3(PT_3)^{n-1}(t,\xi_n(t))-\xi_n(t)\| \\ &+ \lambda_n\|h_n(t)-\xi_n(t)\|. \\ &\to 0(as\ n\to\infty). \end{aligned} \tag{3.23}$$

Also, (3.19) implies

$$\begin{aligned}\|\eta_n(t)-\zeta_n(t)\| &= \|P((1-\beta_n-\delta_n)\zeta_n(t)+\beta_n T_2(PT_2)^{n-1}(t,\zeta_n(t))+\delta_n g_n(t))-\zeta_n(t)\| \\ &\leq (1-\beta_n-\delta_n)\|\zeta_n(t)-\zeta_n(t)\|+\beta_n\|T_2(PT_2)^{n-1}(t,\zeta_n(t))-\zeta_n(t)\| \\ &+ \delta_n\|g_n(t)-\zeta_n(t)\|. \\ &\to 0(as\ n\to\infty). \end{aligned} \tag{3.24}$$

From (3.23) and (3.24), we obtain

$$\begin{aligned}\|\eta_n(t)-\xi_n(t)\| &\leq \|\eta_n(t)-\zeta_n(t)\|+\|\zeta_n(t)-\xi_n(t)\|. \\ &\to 0\ (as\ n\to\infty). \end{aligned} \tag{3.25}$$

In addition we have

$$\begin{aligned}\|T_1(PT_1)^{n-1}(t,\xi_n(t))-\xi_n(t)\| &\leq \|T_1(PT_1)^{n-1}(t,\xi_n(t))-\eta_n(t)\|+\|\eta_n(t)-\xi_n(t)\| \\ &= \|T_1(PT_1)^{n-1}(t,\xi_n(t))-T_1(PT_1)^{n-1}(t,\eta_n(t)) \\ &+ T_1(PT_1)^{n-1}(t,\eta_n(t))-\eta_n(t)\|+\|\eta_n(t)-\xi_n(t)\| \\ &\leq \|T_1(PT_1)^{n-1}(t,\xi_n(t))-T_1(PT_1)^{n-1}(t,\eta_n(t))\| \\ &+ \|T_1(PT_1)^{n-1}(t,\eta_n(t))-\eta_n(t)\|+\|\eta_n(t)-\xi_n(t)\| \\ &\leq r_{1_n}\|\xi_n(t)-\eta_n(t))\|+\|T_1(PT_1)^{n-1}(t,\eta_n(t))-\eta_n(t)\| \\ &+ \|\eta_n(t)-\xi_n(t)\|. \\ &\to 0\ (as\ n\to\infty). \end{aligned} \tag{3.26}$$

Using (3.13) and (3.25) we have

$$\begin{aligned}
\|\xi_{n+1}(t)-\xi_n(t)\| &\leq (1-\alpha_n-\sigma_n)\|\eta_n(t)-\xi_n(t)\|+\alpha_n\|T_1(PT_1)^{n-1}(t,\eta_n(t))-\xi_n(t)\| \\
&+ \sigma_n\|f_n(t))-\xi_n(t)\| \\
&\leq (1-\alpha_n-\sigma_n)\|\eta_n(t)-\xi_n(t)\|+\alpha_n\|T_1(PT_1)^{n-1}(t,\eta_n(t))-\eta_n(t)\| \\
&+ \alpha_n\|\eta_n(t)-\xi_n(t)\|+\sigma_n\|f_n(t))-\xi_n(t)\| \\
&\leq \|\eta_n(t)-\xi_n(t)\|+\|T_1(PT_1)^{n-1}(t,\eta_n(t))-\eta_n(t)\| \\
&+ \sigma_n\|f_n(t))-\xi_n(t)\| \\
&\to 0 \ (as\ n\to\infty). \qquad (3.27)
\end{aligned}$$

Also using (3.27) and (3.26) we get,

$$\begin{aligned}
\|\xi_{n+1}(t)-T_1(PT_1)^{n-1}(t,\xi_{n+1}(t))\| &= \|\xi_{n+1}(t)-\xi_n(t)+\xi_n(t)-T_1(PT_1)^{n-1}(t,\xi_n(t)) \\
&+ T_1(PT_1)^{n-1}(t,\xi_n(t))-T_1(PT_1)^{n-1}(t,\xi_{n+1}(t))\| \\
&\leq \|\xi_{n+1}(t)-\xi_n(t)\| \\
&+ \|T_1(PT_1)^{n-1}(t,\xi_{n+1}(t))-T_1(PT_1)^{n-1}(t,\xi_n(t))\| \\
&+ \|T_1(PT_1)^{n-1}(t,\xi_n(t))-\xi_n(t)\| \\
&\leq \|\xi_{n+1}(t)-\xi_n(t)\|+r_{1_n}(t)\|\xi_{n+1}(t)-\xi_n(t)\| \\
&+ \|T_1(PT_1)^{n-1}(t,\xi_n(t))-\xi_n(t)\| \\
&\to 0 \ (as\ n\to\infty). \qquad (3.28)
\end{aligned}$$

In addition we have

$$\begin{aligned}
\|\xi_{n+1}(t)-T_1(PT_1)^{n-2}(t,\xi_{n+1}(t))\| &= \|\xi_{n+1}(t)-\xi_n(t)+\xi_n(t)-T_1(PT_1)^{n-2}(t,\xi_n(t)) \\
&+ T_1(PT_1)^{n-2}(t,\xi_n(t))-T_1(PT_1)^{n-2}(t,\xi_{n+1}(t))\| \\
&\leq \|\xi_{n+1}(t)-\xi_n(t)\| \\
&+ \|T_1(PT_1)^{n-2}(t,\xi_{n+1}(t))-T_1(PT_1)^{n-2}(t,\xi_n(t))\| \\
&+ \|T_1(PT_1)^{n-2}(t,\xi_n(t))-\xi_n(t)\| \\
&\leq \|\xi_{n+1}(t)-\xi_n(t)\| \\
&+ L\|\xi_{n+1}(t)-\xi_n(t)\|+\|T_1(PT_1)^{n-2}(t,\xi_n(t))-\xi_n(t)\|,
\end{aligned}$$

where $L = sup\ \{r_{1_n}(t) : n \geq 1, t \in \Omega\}$. It follows from (3.27) and (3.28) that

$$\lim_{n\to\infty} \|\xi_{n+1}(t) - T_1(PT_1)^{n-2}(t, \xi_{n+1}(t))\| = 0. \qquad (3.29)$$

We denote the random identity map by $I = (PT_1)^{1-1} : \Omega \times C \to C$. It follows by the inequalites (3.28) and (3.29) that

$$\begin{aligned}
\|\xi_{n+1}(t) - T_1(t,\xi_{n+1}(t))\| &\leq \|\xi_{n+1}(t) - T_1(PT_1)^{n-1}(t,\xi_{n+1}(t))\| \\
&+ \|T_1(PT_1)^{n-1}(t,\xi_{n+1}(t)) - T_1(t,\xi_{n+1}(t))\| \\
&\leq \|\xi_{n+1}(t) - T_1(PT_1)^{n-1}(t,\xi_{n+1}(t))\| \\
&+ \|T_1(PT_1)^{1-1}(PT_1)^{n-1}(t,\xi_{n+1}(t)) - T_1(PT_1)^{1-1}(t,\xi_{n+1}(t))\| \\
&\leq \|\xi_{n+1}(t) - T_1(PT_1)^{n-1}(t,\xi_{n+1}(t))\| \\
&+ L\|(PT_1)^{n-1}(t,\xi_{n+1}(t)) - \xi_{n+1}(t)\| \\
&= \|\xi_{n+1}(t) - T_1(PT_1)^{n-1}(t,\xi_{n+1}(t))\| \\
&+ L\|(PT_1)(PT_1)^{n-2}(t,\xi_{n+1}(t)) - P(\xi_{n+1}(t))\| \\
&\leq \|\xi_{n+1}(t) - T_1(PT_1)^{n-1}(t,\xi_{n+1}(t))\| \\
&+ L\|T_1(PT_1)^{n-2}(t,\xi_{n+1}(t)) - \xi_{n+1}(t)\| \\
&\to 0 \ (as\ n \to \infty).
\end{aligned}$$

It follows that $\lim_{n\to\infty} \|\xi_n(t) - T_1(t,\xi_n(t))\| = 0$. Similarly, we can prove that $\lim_{n\to\infty} \|\xi_n(t) - T_2(t,\xi_n(t))\| = 0$ and $\lim_{n\to\infty} \|\xi_n(t) - T_3(t,\xi_n(t))\| = 0$. □

Theorem 3.3. *Let E be a real uniformly convex separable Banach space and let C be a nonempty closed convex subset of E with P as a nonexpansive retraction. Let $T_i : \Omega \times C \to E$, $i = 1,2,3$ be three asymptotically nonexpansive nonself random mappings with sequences of measurable mappings $\{r_{i_n}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(r_{i_n}(t) - 1) < \infty$, $r_{i_n}(t) \to 1$ as $n \to \infty$, for all $t \in \Omega$ and $i = 1,2,3$. Suppose that $\bigcap_{i=1}^{3} RF(T_i) \neq \emptyset$ and $\{\xi_n(t)\}$, $\{\eta_n(t)\}$ and $\{\zeta_n(t)\}$ are the sequences defined in (2.1) where α_n, β_n, γ_n are sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon > 0$, $\sum_{n=1}^{\infty}\sigma_n < \infty$, $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty}\lambda_n < \infty$. If one of T_is, $i = 1,2,3$ is completely continuous then $\{\xi_n(t)\}$,$\{\eta_n(t)\}$ and $\{\zeta_n(t)\}$ converge to a common random fixed point of T_1, T_2 and T_3.*

Proof. By Lemma 3.1, $\{\xi_n(t)\}$ is bounded. In addition, by lemma 3.2, $\lim_{n\to\infty} \|\xi_n(t) - T_i(t,\xi_n(t))\| = 0$, $i = 1,2,3$, and $\{T_i(t,\xi_n(t))\}, i = 1,2,3$

are also bounded. If T_1 is completely continuous, there exists a subsequence $\{T_1(t,\xi_{n_j}(t))\}$ of $\{T_1(t,\xi_n(t))\}$ such that $T_1(t,\xi_{n_j}(t)) \to \xi(t)$ as $j \to \infty$. From Lemma 3.2, we have that $\lim_{n\to\infty} \|\xi_{n_j}(t) - T_i(t,\xi_{n_j}(t))\| = 0$, $i = 1,2,3$. It follows from Lemma 2.12, that $\xi(t) = T_i(t,\xi(t))$, $i = 1,2,3$. So by the continuity of T_1, for $i = 1,2,3$ we have

$$\begin{aligned} \|\xi_{n_j}(t) - \xi(t)\| &\leq \|\xi_{n_j}(t) - T_i(t,\xi_{n_j}(t))\| + \|T_i(t,\xi_{n_j}(t)) - \xi(t)\| \\ &\to 0 \ (as\ n \to \infty). \end{aligned}$$

Furthermore, by Lemma 3.1, $\lim_{n\to\infty} \|\xi_n(t)) - \xi(t)\|$ exists. Thus $\lim_{n\to\infty} \|\xi_n(t)) - \xi(t)\| = 0$, for all $t \in \Omega$ and since $\xi(t)$ is a pointwise limit of the measurable mapping sequence $\{\xi_n(t)\}$, $\xi(t)$ is measurable and therefore $\xi(t) \in \bigcap_{i=1}^{3} RF(T_i)$. It follows from (3.25) that

$$\begin{aligned} \|\eta_n(t) - \xi(t)\| &\leq \|\eta_n(t) - \xi_n(t)\| + \|\xi_n(t) - \xi(t)\|, \\ &\to 0 \ (as\ n \to \infty), \end{aligned}$$

and using (3.23) we get

$$\begin{aligned} \|\zeta_n(t) - \xi(t)\| &\leq \|\zeta_n(t) - \xi_n(t)\| + \|\xi_n(t) - \xi(t)\| \\ &\to 0 \ (as\ n \to \infty). \end{aligned}$$

□

Theorem 3.4. *Let E be a real uniformly convex separable Banach space and let C be a nonempty closed convex subset of E with P as a nonexpansive retraction. Let $T_i : \Omega \times C \to E$, $i = 1,2,3$ be three asymptotically nonexpansive nonself random mappings with sequences of measurable mappings $\{r_{i_n}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty} (r_{i_n}(t) - 1) < \infty$, $r_{i_n}(t) \to 1$ as $n \to \infty$, for all $t \in \Omega$ and $i = 1,2,3$. Suppose that $\bigcap_{i=1}^{3} RF(T_i) \neq \emptyset$ and, $\{\xi_n(t)\}$, $\{\eta_n(t)\}$ and $\{\zeta_n(t)\}$ are the sequences defined in (2.1), where α_n, β_n, γ_n are sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon > 0$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$, $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$. If one of T_is, $i = 1,2,3$, is semicompact, then $\{\xi_n(t)\}$, $\{\eta_n(t)\}$ and $\{\zeta_n(t)\}$ converge to a common random fixed point of T_1, T_2 and T_3.*

Proof. By Lemmas 3.1 and 3.2, $\{\xi_n(t)\}$ is bounded and $\lim_{n\to\infty} \|\xi_n(t) - T_i(t,\xi_n(t))\| = 0, i = 1,2,3$. Since T_1 or T_2 or T_2 is semicompact, there exists subsequence $\{\xi_{n_j}(t)\}$ of $\{\xi_n(t)\}$ such that $\{\xi_{n_j}(t)\}$ converges strongly to $\{\xi(t)\}$ for all $t \in \Omega$. Then by Lemma 2.12 we have $\xi(t) = T_i(t, \xi(t)), i = 1, 2, 3,$ and since $\xi(t)$ is a pointwise limit of the measurable mapping sequence $\{\xi_{n_j}(t)\}$, $\xi(t)$ is measurable and therefore $\xi(t) \in \bigcap_{i=1}^{3} RF(T_i)$. Thus by Lemma 3.1, $\lim_{n\to\infty} \|\xi_n(t)) - \xi(t)\|$ exists, and $\lim_{n\to\infty} \|\xi_n(t)) - \xi(t)\| = 0$, for all $t \in \Omega$. From (3.25) and (3.23), we have $\lim_{n\to\infty} \|\eta_n(t) - \xi(t)\| = 0$ and $\lim_{n\to\infty} \|\zeta_n(t) - \xi(t)\| = 0$. □

In the next result, we prove the strong convergence of the scheme (2.1) under condition B which is weaker than the compactness of the domain of the mappings.

Theorem 3.5. *Let E be a real uniformly convex separable Banach space and let C be a nonempty closed convex subset of E with P as a nonexpansive retraction. Let $T_i : \Omega \times C \to E$, $i = 1, 2, 3$, be three asymptotically nonexpansive nonself random mappings with sequences of measurable mappings $\{r_{i_n}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (r_{i_n}(t) - 1) < \infty$, $r_{i_n}(t) \to 1$ as $n \to \infty$, for all $t \in \Omega$ and $i = 1, 2, 3$. Suppose that $F = \bigcap_{i=1}^{3} RF(T_i) \neq \emptyset$ and $\{\xi_n(t)\}$,$\{\eta_n(t)\}$ and $\{\zeta_n(t)\}$ are the sequences defined in (2.1) where α_n,β_n,γ_n are sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon > 0$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$, $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$. If T_i satisfies the condition B for all $t \in \Omega$ and $i = 1, 2, 3$, then $\{\xi_n(t)\}$,$\{\eta_n(t)\}$ and $\{\zeta_n(t)\}$ converge strongly to a common random fixed point of T_1, T_2 and T_3.*

Proof. By Lemma 3.2, we have $\lim_{n\to\infty} \|\xi_n(t) - T_i(t, \xi_n(t))\| = 0, i = 1, 2, 3.$ Since $\{T_i : i = 1, 2, 3\}$ satisfy the condition B, we have that

$$\lim_{n\to\infty} f(d(\xi_n(t), F)) = 0.$$

Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$, we obtain that $\lim_{n\to\infty} f(d(\xi_n(t), F)) = 0$. Next we claim that $\{\xi_n(t)\}$ is a Cauchy sequence. Indeed, from Lemma 3.1 we have that $\|\xi_{n+1}(t) - \xi(t)\| \leq (1 + B_n(t))\|\xi_n(t) - \xi(t)\| + D_n(t)$, so, for each $t \in \Omega$ and for all natural numbers m, n, we have

$$\begin{aligned}
\|\xi_{n+m}(t) - \xi(t)\| &\leq (1 + B_{n+m-1}(t))\|\xi_{n+m-1}(t) - \xi(t)\| + D_{n+m-1}(t) \\
&\leq e^{B_{n+m-1}(t)}\|\xi_{n+m-1}(t) - \xi(t)\| + D_{n+m-1}(t) \\
&\leq e^{B_{n+m-1}(t)+B_{n+m-2}(t)}\|\xi_{n+m-2}(t)-\xi(t)\| \\
&+ e^{B_{n+m-1}(t)}D_{n+m-2}(t)+D_{n+m-1}(t) \\
&. \\
&. \\
&. \\
&\leq e^{\sum_{i=n}^{n+m-1}(B_i(t))}\|\xi_n(t)-\xi(t)\|+\sum_{k=n}^{n+m-1} D_k(t)e^{\sum_{i=k}^{n+m}(B_i(t))} \\
&+ D_{n+m-1}(t) \\
&\leq R(t)\|\xi_n(t)-\xi(t)\|+R(t)\sum_{k=n}^{\infty} D_k(t),
\end{aligned}$$

where $R(t) = e^{\sum_{n=1}^{\infty} B_n(t)} < \infty$. Therefore, for any $\xi(t) \in \bigcap_{i=1}^{3} RF(T_i)$, we have

$$\begin{aligned}
\|\xi_{n+m}(t) - \xi_n(t)\| &\leq \|\xi_{n+m}(t)-\xi(t)\|+\|\xi_n(t)-\xi(t)\| \\
&\leq R(t)\|\xi_n(t)-\xi(t)\|+R(t)\sum_{k=n}^{\infty} D_k(t)+\|\xi_n(t)-\xi(t)\| \\
&= (R(t)+1)\|\xi_n(t)-\xi(t)\|+R(t)\sum_{k=n}^{\infty} D_k(t). \qquad (3.30)
\end{aligned}$$

Since $\lim_{n\to\infty} f(d(\xi_n(t), F)) = 0$, and $\sum_{n=1}^{\infty} D_n(t) < \infty$, given $\epsilon > 0$ there exists a natural number n_0 such that $d(\xi_n(t), F) < \frac{\epsilon}{2(R(t)+1)}$ and $\sum_{n=1}^{\infty} D_n(t) < \frac{\epsilon}{2R(t)}$ for all $n \geq n_0$. So there exists $\xi^*(t) \in F$ such that $\|\xi_n(t) - \xi^*(t)\| < \frac{\epsilon}{2(R(t)+1)}$ for all $n \geq n_0$. Therefore from (3.30), for all $n \geq n_0$ we have that

$$\begin{aligned}
\|\xi_{n+m}(t) - \xi_n(t)\| &\leq (R(t) + 1)\|\xi_n(t) - \xi^*(t)\| + R(t)\sum_{k=n}^{\infty}(D_k(t)) \\
&< (R(t) + 1)\frac{\epsilon}{2(R(t) + 1)} + R(t)\frac{\epsilon}{2R(t)} = \epsilon,
\end{aligned}$$

which implies that $\{\xi_n(t)\}$ is a Cauchy sequence for each $t \in \Omega$ and so is convergent since E is complete. Let $\lim_{n\to\infty} \xi_n(t) = p(t)$. Now we show that $p(t) \in F$. Since $\lim_{n\to\infty} d(\xi_n(t), F) = 0$ gives that $d(p(t), F) = 0$. Since F is closed, we have $p(t) \in F$. From (3.25) and (3.23) we have $\lim_{n\to\infty} \|\eta_n(t) - p(t)\| = 0$ and $\lim_{n\to\infty} \|\zeta_n(t) - p(t)\| = 0$. □

Finally, we prove the weak convergence of the iterative scheme (2.1) for three asymptotically nonexpansive nonself random mappings in a uniformly convex separable Banach space satisfying Opial's condition.

Theorem 3.6. *Let E be a real uniformly convex separable Banach space which satisfies Opial's condition and let C be a nonempty closed convex subset of E with P as a nonexpansive retraction. Let $T_i : \Omega \times C \to E$, $i = 1, 2, 3$, be three asymptotically nonexpansive nonself random mappings with sequences of measurable mappings $\{r_{i_n}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (r_{i_n}(t) - 1) < \infty$, $r_{i_n}(t) \to 1$ as $n \to \infty$, for all $t \in \Omega$ and $i = 1, 2, 3$. Suppose that $F = \bigcap_{i=1}^{3} RF(T_i) \neq \emptyset$ and, $\{\xi_n(t)\}$, $\{\eta_n(t)\}$ and $\{\zeta_n(t)\}$ are the sequences defined in (2.1) where α_n, β_n, γ_n are sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon > 0$ and $\sum_{n=1}^{\infty} \sigma_n < \infty, \sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$. Then $\{\xi_n(t)\}$,$\{\eta_n(t)\}$ and $\{\zeta_n(t)\}$ converge weakly to a common random fixed point of T_1, T_2 and T_3.*

Proof. From Lemma 3.2, we have that $\lim_{n\to\infty} \|\xi_n(t) - T_i(t, \xi_n(t))\| = 0$ for $i = 1, 2, 3$. Since E is uniformly convex and $\{\xi_n(t)\}$ is bounded, without loss of generality we may assume that $\xi_n(t) \to u(t)$ weakly as $n \to \infty$. Hence by Lemma 2.12, we have $u(t) \in \bigcap_{i=1}^{3} RF(T_i)$. Suppose that subsequences $\xi_{n_k}(t), \xi_{m_k}(t)$ and $\xi_{l_k}(t)$ of $\xi_n(t)$ converge weakly to $u(t), v(t)$ and $w(t)$, respectively. Lemma 2.12 implies that $u(t), v(t), w(t) \in \bigcap_{i=1}^{3} RF(T_i)$ and by lemma 3.1, $\lim_{n\to\infty} \|\xi_n(t)-u(t)\|$, $\lim_{n\to\infty} \|\xi_n(t)-v(t)\|$ and $\lim_{n\to\infty} \|\xi_n(t)-w(t)\|$ exist. It follows from Lemma 2.13, that $u(t) = v(t) = w(t)$. Therefore $\{\xi_n(t)\}$ converges weakly to a common fixed point of T_1, T_2 and T_3. In addition by (3.25) and (3.23) we have $\eta_n(t) \to u(t)$ weakly as $n \to \infty$ and $\zeta_n(t) \to u(t)$ weakly as $n \to \infty$. □

References

[1] S. Banerjee and B. S. Choudhury, Composite implicit random iterations for approximating common random fixed point for a finite family asymptotically nonexpansive random operators, *Commun. Korean Math. Soc.* **26** (2011), no. 1, 23–35.

[2] I. Beg, Random fixed points of random operators satisfying semicontractivity conditions, *Math. Japon.* **46** (1997), no. 1, 151–155

[3] I. Beg, Approximation of random fixed points in normed spaces, *Nonlinear Anal.* **51** (2002), no. 8, 1363–1372.

[4] I. Beg, Minimal displacement of random variables under Lipschitz random maps, *Topol. Methods Nonlinear Anal.* **19** (2002), no. 2, 391–397.

[5] I. Beg and M. Abbas, Iterative procedure for solutions of random operator equations in Banach spaces, *J. Math. Anal. Appl.* **315** (2006) 181–201.

[6] I. Beg and N. Shahzad, Random fixed point theorems for nonexpansive and contractive-type random operators on Banach spaces, *J. Appl. Math. Stochastic Anal.* **7** (1994), no. 4, 569–580.

[7] A. T. Bharucha-Reid, Fixed point theorems in probabilistic analysis, *Bull. Amer. Math. Soc.* **82** (1976), no. 5, 641–657.

[8] S. S. Chang, Y.J. Cho and H. Zhou, Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings, *J. Korean Math. Soc.* **38** (2001), no. 6, 1245–1260.

[9] C. E. Chidume, E. U. Ofoedu and H. Zegeye, Strong and weak convergence theorems for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.* **280** (2003), no. 2, 364–374.

[10] B. S. Choudhury, Convergence of a random iteration scheme to a random fixed point, *J. Appl. Math. Stochastic Anal.* **8** (1995), no. 2, 139–142.

[11] K. Goebel and W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.* **35** (1972) 171–174.

[12] O. Hans, Reduzierende zulliallige transformaten, *Czechoslovak Math. J.* **7(82)** (1957) 154–158.

[13] O. Hans, Random operator equations, 185–202, Proc. 4th Berkeley Sympos. Math. Statist. and Prob. II Univ. California Press, Berkeley, 1961.

[14] S. Ishikawa, Fixed points and iteration of a nonexpansive mapping in a Banach space, *Proc. Amer. Math. Soc.* **59** (1976), no. 1, 61–71.

[15] S. Itoh, Random fixed point theorems with an application to random differential equations in Banach spaces, *J. Math. Anal. Appl.* **67** (1979), no. 2, 261–273.

[16] J. S. Jung and S. S. Kim, Strong convergence theorems for nonexpansive nonself-mappings in Banach spaces, *Nonlinear Anal.* **33** (1998), no. 3, 321–329.

[17] S. H. Khan and H. Fukhar-ud-din, Weak and strong convergence of a scheme with errors for two nonexpansive mappings, *Nonlinear Anal.* **61** (2005), no. 8, 1295–1301.

[18] S. Y. Matsushita and D. Kuroiwa, Strong convergence of averaging iteration of nonexpansive nonself-mappings, *J. Math. Anal. Appl.* **294** (2004), no. 1, 206–214.

[19] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.* **73** (1967) 591–597.

[20] M. O. Osilike and S. C. Aniagbosor, Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings, *Math. Comput. Modelling* **32** (2000), no. 10, 1181–1191.

[21] M. O. Osilike and A. Udomene, Demiclosedness principle and convergence theorems for strictly pseudocontractive mappings of Browder-Petryshyn type, *J. Math. Anal. Appl.* **256** (2001), no. 2, 431–445.

[22] N. S. Papageorgiou, Random fixed point theorems for measurable multifunction in Banach spaces, *Proc. Amer. Math. Soc.* **97** (1986), no. 3, 507–514.

[23] S. Plubtieng, P. Kumam and R. Wangkeeree, Random three-step iteration scheme and common random fixed point of three operators, *J. Appl. Math. Stoch. Anal.* **2007** (2006) 10 pages.

[24] S. Plubtieng, R. Wangkeeree and R. Punpaeng, On the convergence of modified Noor iterations with errors for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.* **322** (2006), no. 2, 1018–1029.

[25] R. A. Rashwan and S. M. Altwqi, Convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings, *Int. J. Pure Appl. Math* **70** (2011), no. 4, 503–520.

[26] J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mappings, *J. Math. Anal. Appl.* **158** (1991), no. 2, 407–413.

[27] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, *Bull. Austral. Math. Soc.* **43** (1991), no. 1, 153–159.

[28] N. Shahzad, Approximating fixed points of non-self nonexpansive mappings in Banach spaces, *Nonlinear Anal.* **61** (2005), no. 6, 1031–1039.

[29] A. Spacek, Zufallige gleichungen, *Czechoslovak Math. J.* **5** (1955) 462–466.

[30] S. Suantai, Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.* **311** (2005), no. 2, 506–517.

[31] W. Takahashi, G. E. Kim, Strong convergence of approximants to fixed points of nonexpansive nonself-mappings, *Nonlinear Anal.* **32** (1998), no. 3, 447–454.

[32] K. K. Tan and K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, *J. Math. Anal. Appl.* **178** (1993), no. 2, 301–308.

[33] S. Thianwan, Common fixed points of new iterations for two asymptotically nonexpansive nonself-mappings in a Banach space, *J. Comput. Appl. Math.* **224** (2009), no. 2, 688–695.

[34] L. Wang, Strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings, *J. Math. Anal. Appl.* **323** (2006), no. 1, 550–557.

[35] H. K. Xu, Some random fixed point theorems for condensing and nonexpansive operators, *Proc. Amer. Math. Soc.* **110** (1990), no. 2, 395–400.

[36] H. K. Xu and X. M. Yin, Strong convergence theorems for nonexpansive nonself-mappings, *Nonlinear Anal.* **24** (1995), no. 2, 223–228.

[37] X. Zhou and L. Wang, Approximation of random fixed points of non-self asymptotically nonexpansive random mappings, *Int. Math. Forum* **2** (2007), no. 37-40, 1859–1868.

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