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LOCALIZATION OPERATORS ON HOMOGENEOUS SPACES

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Abstract. Let G be a locally compact group, H a compact subgroup of G and ϖ a representation of the homogeneous space G/H on a Hilbert space $\mathcal{H}$. For $\psi \in L^p(G/H)$, $1 \leq p \leq \infty$, and an admissible wavelet ζ for ϖ, we define the localization operator $L_{\psi,\zeta}$ on $\mathcal{H}$ and we show that it is a bounded operator. Moreover, we prove that the localization operator is in Schatten p-class and it is a compact operator for $1 \leq p \leq \infty$.

1. Introduction and Preliminaries

Recently, localization operators have been a subject of study in quantum mechanics, in PDE and signal analysis. In engineering, a natural language is given by time-frequency analysis. A linear operator $D_{\psi,\zeta} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ associated to ψ in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$ and ζ in $L^2(\mathbb{R}^n)$ with $\|\zeta\|_2 = 1$, which is called a Daubechies operator, is the same as the localization operator $L_{\psi,\zeta}$ associated to admissible wavelet ζ for the Schrodinger representation of the Weyl-Heisenberg group on

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$L^2(\mathbb{R}^n)$. In fact,

$$< D_{\psi,\zeta} f, g >_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi(q,p) < f, \zeta_{q,p} >< \zeta_{q,p}, g > dqdp,$$

for all f, g in $L^2(\mathbb{R}^n)$, where $\zeta_{q,p}(x) = e^{ip.x}\zeta(x-q)$, $x \in \mathbb{R}^n$, for all q, p in $\mathbb{R}^n$ [1,4,5,10,12]. Motivated by Daubechies operators, a localization operator is defined on locally compact groups [15]. Since, in many cases working with too large groups, the group representations are not square integrable, it is necessary to make the group smaller which can be performed by factoring out a suitable subgroup, that is, one has to work with homogenous spaces. Localization operators have been studied extensively on $\mathbb{R}^n$ and also on locally compact groups in [3–7, 14]. M. W. Wong in [15] has studied the localization operators in the setting of homogeneous spaces with a G-invariant measure. In this paper we do this task with a completely different approach considering a relatively invariant measure on a homogeneous space. The continuous wavelet transform on homogenous spaces has been studied in [8] whereas in this paper, we give a systematic study of localization operators on homogenous spaces which are related to the continuous wavelet transform.

Let G be a locally compact group and H let be a closed subgroup of G. Consider G/H as a homogeneous space on which G acts from the left and μ as a Radon measure on it. For $g \in G$ and a Borel subset E of G/H, we define the translation μ_g of μ by $\mu_g(E) = \mu(gE)$. A measure μ is said to be G-invariant if $\mu_g = \mu$, for all $g \in G$. A measure μ is said to be strongly quasi invariant provided that a continuous function $\lambda : G \times G/H \to (0, \infty)$ exists which satisfies

$$d\mu_g(kH) = \lambda(g, kH)d\mu(kH),$$

for all $g, k \in G$. If the functions $\lambda(g, .)$ reduce to constants, then μ is called relatively invariant under G (for a detailed account of homogeneous spaces, the reader is refered to [9]). A rho-function for the pair (G, H) is defined to be a continuous function $\rho : G \to (0, \infty)$ which satisfies

$$\rho(gh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(g) \quad (g \in G, h \in H),$$

where Δ_G, Δ_H are the modular functions on G and H, respectively. It is well known (see [9]), that any pair (G, H) admits a rho-function and for each rho-function ρ there is a strongly quasi invariant measure μ on

G/H such that

$$\frac{d\mu_g}{d\mu}(kH) = \frac{\rho(gk)}{\rho(k)} \quad (g, k \in G).$$

As has been shown in [9], every strongly quasi invariant measure on G/H, arises from a rho-function and all such measures are strongly equivalent. That is, there exists a positive continuous function τ on G/H such that $\frac{d\acute{\mu}}{d\mu} = \tau$, where μ and $\acute{\mu}$ are strongly quasi invariant measures on G/H.

For the reader's convenience, we recall from [8] the basic concepts in the theory of unitary representations of homogeneous spaces. A continuous unitary representation of a homogeneous space G/H is a map ϖ from G/H into $U(\mathcal{H})$, the group of all unitary operators on some nonzero Hilbert space $\mathcal{H}$, for which the function $gH \mapsto < \varpi(gH)x, y >$ is continuous, for each $x, y \in \mathcal{H}$ and

$$\varpi(gkH) = \varpi(gH)\varpi(kH), \quad \varpi(g^{-1}H) = \varpi(gH)^*,$$

for each $g, k \in G$ (see Example 3.4). Moreover, a closed subspace M of $\mathcal{H}$ is said to be invariant with respect to ϖ if $\varpi(gH)M \subseteq M$, for all $g \in G$. A continuous unitary representation ϖ is said to be irreducible if the only invariant subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$. In the sequel we always mean by a representation, a continuous unitary representation. An irreducible representation ϖ of G/H on $\mathcal{H}$ is said to be *square integrable* if there exists a nonzero element $\zeta \in \mathcal{H}$ such that

$$\int_{G/H} \frac{\rho(e)}{\rho(g)} \ | < \zeta, \varpi(gH)\zeta > |^2 d\mu(gH) < \infty, \tag{1.1}$$

where μ is a relatively invariant measure on G/H which arises from a rho function $\rho : G \to (0, \infty)$. If ζ satisfies (1.1), it is called an *admissible vector*. An admissible vector $\zeta \in \mathcal{H}$ is called *admissible wavelet* if $\|\zeta\| = 1$. In this case, we define the wavelet constant c_ζ as

$$c_\zeta := \int_{G/H} \frac{\rho(e)}{\rho(g)} \ | < \zeta, \varpi(gH)\zeta > |^2 d\mu(gH). \tag{1.2}$$

We call c_ζ the *wavelet constant* associated to the admissible wavelet ζ. It is worthwhile to note that there is a close relation between the representation on homogeneous spaces G/H, where H is a compact subgroup of G, and the representation of G. More precisely if ϖ is a representation on G/H, then it defines a representation π of G in which the subgroup

H is considered to be contained in the kernel of π. Conversely, any representation π of G which is trivial on H induces a representation ϖ of G/H by letting $\varpi(gH) = \pi(g)$.

Recall that if T is a compact operator on a separable Hilbert space $\mathcal{H}$, then there exist orthonormal sets $\{\alpha_n\}$ and $\{\beta_n\}$ in $\mathcal{H}$ such that

$$T\xi = \sum_n \lambda_n < \xi, \alpha_n > \beta_n, \quad \xi \in \mathcal{H},$$

where λ_n is the nth singular value of T [16].
Given $0 < p < +\infty$, the Schatten p-class S_p of $\mathcal{H}$ is defined to be the space of all compact operators T on $\mathcal{H}$ with its singular value sequence $\{\lambda_n\}$ belonging to l^p, the p-summable sequence space. When $1 \leq p \leq \infty$, S_p is a Banach space with the norm

$$\|T\|_p = [\sum_n |\lambda_n|^p]^{1/p}.$$

Two special cases S_1 and S_2, which are called the trace class and the Hilbert-Schmidt class, respectively, are worth mentioning (for more detailed on Schatten p-class see [16]).

In this paper we study the localization operator $L_{\psi,\zeta}$ where $\psi \in L^p(G/H, \mu), \quad 1 \leq p \leq \infty$, and ζ is admissible wavelet in a separable Hilbert space $\mathcal{H}$ (see Definition 2.1). We investigate some significant properties of a localization operator, such as boundedness and compactness. This paper is organized as follows: In section 2, we show that for $\psi \in L^p(G/H), \quad 1 \leq p \leq \infty$, and an admissible wavelet $\zeta \in \mathcal{H}$, the localization operator $L_{\psi,\zeta}$ on $\mathcal{H}$ is bounded. Section 3 is devoted to proving that $L_{\psi,\zeta}$ is a compact operator which is in Schatten p-class.

2. **Boundedness of localization operators on** G/H

Throughout this paper, we assume that the notation will be as the previous section. Let G a locally compact group and let H a compact subgroup of G. Consider G/H as a homogeneous space associated with a relatively invariant measure μ which arises from a rho-function ρ. Let $\mathcal{H}$ be a separable Hilbert space, ϖ a square integrable representation of G/H on $\mathcal{H}$ and ζ an admissible wavelet for ϖ. In this section, we introduce the localization operator $L_{\psi,\zeta}$ which is related to the continuous wavelet transform $W_\zeta : \mathcal{H} \to L^2(G/H), \quad W_\zeta(x)(gH) = (\frac{\rho(e)}{\rho(g)})^{1/2} < x, \varpi(gH)\zeta >$, for each $\psi \in L^p(G/H), \quad 1 \leq p \leq \infty$. In this setting, we investigate the boundednees properties of localization operators. Now, we define the linear operator $L_{\psi,\zeta} : \mathcal{H} \to \mathcal{H}$ as follows.

Definition 2.1. *Let $\mathcal{H}$ be a Hilbert space and let ϖ be a square integrable representation of G/H on $\mathcal{H}$ with an admissible wavelet ζ. Define the linear operator $L_{\psi,\zeta}$ on $\mathcal{H}$ as:*

(2.1)

$$< L_{\psi,\zeta}x, y >= \frac{1}{c_\zeta} \int_{G/H} \frac{\rho(e)}{\rho(g)} \psi(gH) < x, \varpi(gH)\zeta >< \varpi(gH)\zeta, y > d\mu(gH),$$

for all $\psi \in L^p(G/H)$ and $x, y \in \mathcal{H}$, where c_ζ is the wavelet constant defined as (1.2). We call $L_{\psi,\zeta}$ the localization operator.

First, we show that for $\psi \in L^\infty(G/H)$ the localization operator $L_{\psi,\zeta}$ is bounded. For this, we need to recall the reconstruction formula for square integrable representation ϖ of G/H, which is (Theorem 2.1, [8]).

Theorem 2.2. *Let ϖ be a square integrable representation of G/H on $\mathcal{H}$. If ζ is an admissible wavelet for ϖ, then*

(2.2)

$$< x, y >= \frac{1}{c_\zeta} \int_{G/H} \frac{\rho(e)}{\rho(g)} < x, \varpi(gH)\zeta >< \varpi(gH)\zeta, y > d\mu(gH),$$

where c_ζ is as in (1.2).

Now, we are ready to prove boundedness of $L_{\psi,\zeta}$ for $\psi \in L^\infty(G/H)$.

Proposition 2.3. *Let $\psi \in L^\infty(G/H)$ and let $\zeta \in \mathcal{H}$ be an admissible wavelet. Then $L_{\psi,\zeta} : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator and $\|L_{\psi,\zeta}\| \leq \|\psi\|_\infty$.*

Proof. Using Theorem 2.2 and the Schwarz inequality we get,

$$\begin{aligned}
&| < L_{\psi,\zeta}x, y > | \leq \\
&\tfrac{1}{c_\zeta} \textstyle\int_{G/H} \frac{\rho(e)}{\rho(g)} \ |\psi(gH)| \ | < x, \varpi(gH)\zeta > | \ | < \varpi(gH)\zeta, y > | d\mu(gH) \leq \\
&\|\psi\|_\infty (\tfrac{1}{c_\zeta} \textstyle\int_{G/H} \frac{\rho(e)}{\rho(g)} \ | < x, \varpi(gH)\zeta > |^2 d\mu(gH))^{1/2} \\
&(\tfrac{1}{c_\zeta} \textstyle\int_{G/H} \frac{\rho(e)}{\rho(g)} \ | < \varpi(gH)\zeta, y > |^2 d\mu(gH))^{1/2} \leq \\
&\|\psi\|_\infty \|x\| \|y\|,
\end{aligned}$$

for all $x, y \in \mathcal{H}$. So $\|L_{\psi,\zeta}\| \leq \|\psi\|_\infty$. □

Secondly, let $\psi \in L^1(G/H)$ where G/H is considered to be given a G-invariant measure $\acute{\mu}$. Note that since H is compact, G/H admits such a G-invariant measure.

Proposition 2.4. *Let $\psi \in L^1(G/H)$. Then $L_{\psi,\zeta}$ is a bounded linear operator and*

$$\|L_{\psi,\zeta}\| \leq \frac{\rho(e)}{c_\zeta}\|\psi\|_1.$$

Proof. Consider G/H with a G-invariant measure $\acute{\mu}$ which arises from rho-function $\acute{\rho} \equiv 1$ [9]. Then,

$$\frac{d\mu}{d\acute{\mu}} = \tau, \quad \rho(g) = \tau(gH),$$

where μ is a relatively invariant measure which arises from ρ. We have

$$|< L_{\psi,\zeta}x, y >| \leq \tfrac{1}{c_\zeta}\int_{G/H} \tfrac{\rho(e)}{\rho(g)} \ \ |\psi(gH)| \, |< x, \varpi(gH)\zeta >| \, |< \varpi(gH)\zeta,$$
$$[1ex] y > |d\mu(gH)$$
$$\leq \tfrac{1}{c_\zeta}\int_{G/H} \tfrac{\tau(eH)}{\tau(gH)} \ \ |\psi(gH)| \, |< x, \varpi(gH)\zeta >| \, |< \varpi(gH)\zeta,$$
$$y > |\tau(gH)d\acute{\mu}(gH)$$
$$\leq \tfrac{1}{c_\zeta}\rho(e)\|\psi\|_1\|x\|\|y\|,$$

where $\|\psi\|_1 = \int_{G/H}\psi(gH)d\acute{\mu}(gH)$. □

Now we intend to show that if $\psi \in L^p(G/H)$, $1 < p < \infty$, then $L_{\psi,\zeta}$ is a bounded linear operator.

Theorem 2.5. *Let $\psi \in L^p(G/H)$, for $1 < p < \infty$. Then there exists a unique bounded linear operator $L_{\psi,\zeta} : \mathcal{H} \to \mathcal{H}$ such that*

$$\|L_{\psi,\zeta}\| \leq (\frac{\rho(e)}{c_\zeta})^{1/p}\|\psi\|_p, \tag{2.3}$$

where $\|\psi\|_p$ is defined with respect to a G-invariant measure and $L_{\psi,\zeta}$ is given by (2.1) for all $x \in \mathcal{H}$ and all simple functions ψ on G/H for which $\mu(\{gH \in G/H; \quad \psi(gH) \neq 0\}) < \infty$.

Proof. Let $\Gamma : \mathcal{H} \to L^2(\mathbb{R}^n)$ be a unitary operator and $\psi \in L^1(G/H)$. Then the linear operator $\tilde{L}_{\psi,\zeta} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by

$$\tilde{L}_{\psi,\zeta} = \Gamma L_{\psi,\zeta}\Gamma^{-1}, \tag{2.4}$$

is bounded and $\|\tilde{L}_{\psi,\zeta}\| \leq \frac{\rho(e)}{c_\zeta}\|\psi\|_1$. Also, if $\psi \in L^\infty(G/H)$, then $\tilde{L}_{\psi,\zeta}$ on $L^2(\mathbb{R}^n)$ defined as (2.4) is bounded and $\|\tilde{L}_{\psi,\zeta}\| \leq \|\psi\|_\infty$.
Denote by $\mathfrak{A}$, the set of all simple functions ψ on G/H such that

$$\mu(\{gH \in G/H; \psi(gH) \neq 0\}) < \infty.$$

Let $g \in L^2(\mathbb{R}^n)$ and let Φ be a linear transformation from $\mathfrak{A}$ into the set of all Lebesgue measurable functions on $\mathbb{R}^n$ defined as $\Phi_g(\psi) = \tilde{L}_{\psi,\zeta}(g)$. Then for all $\psi \in L^1(G/H)$

$$\|\Phi_g(\psi)\|_2 = \|\tilde{L}_{\psi,\zeta}(g)\|_2 \leq \|\tilde{L}_{\psi,\zeta}\|\|g\|_2 \leq \frac{\rho(e)}{c_\zeta}\|\psi\|_1\|g\|_2.$$

Similarly for all $\psi \in L^\infty(G/H)$,

$$\|\Phi_g(\psi)\|_2 \leq \|\psi\|_\infty\|g\|_2.$$

By the Riesz Thorin Interpolation Theorem [16], we get,

$$\|\Phi_g(\psi)\|_2 \leq (\frac{\rho(e)}{c_\zeta})^{1/p}\|\psi\|_p\|g\|_2.$$

Therefore,

$$\|\tilde{L}_{\psi,\zeta}(g)\|_2 \leq (\frac{\rho(e)}{c_\zeta})^{1/p}\|\psi\|_p\|g\|_2.$$

So,

$$\|\tilde{L}_{\psi,\zeta}\| \leq (\frac{\rho(e)}{c_\zeta})^{1/p}\|\psi\|_p,$$

for each $\psi \in \mathfrak{A}$.
Now, let $\psi \in L^p(G/H)$, for all $1 < p < \infty$. Then there exists a sequence $\{\psi_k\}_{k=1}^\infty$ of functions in $\mathfrak{A}$ such that ψ_k is convergent to ψ in $L^p(G/H)$ as $k \to \infty$. Also, $\{\tilde{L}_{\psi_k,\zeta}\}_{k=1}^\infty$ is a Cauchy sequence in $B(L^2(\mathbb{R}^n))$. Indeed,

$$\|\tilde{L}_{\psi_n,\zeta} - \tilde{L}_{\psi_m,\zeta}\| \leq (\frac{\rho(e)}{c_\zeta})^{1/p}\|\psi_n - \psi_m\|_p \to 0.$$

By completeness of $B(L^2(\mathbb{R}^n))$, there exists a bounded linear operator $\tilde{L}_{\psi,\zeta} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that $\tilde{L}_{\psi_k,\zeta}$ converges to $\tilde{L}_{\psi,\zeta}$ in $B(L^2(\mathbb{R}^n))$, in which

$$\|\tilde{L}_{\psi,\zeta}\| \leq (\frac{\rho(e)}{c_\zeta})^{1/p}\|\psi\|_p.$$

Thus, the linear operator $L_{\psi,\zeta}:\mathcal{H}\to\mathcal{H}$ where $L_{\psi,\zeta}=\Gamma^{-1}\tilde{L}_{\psi,\zeta}\Gamma$, is a bounded linear operator and

$$\|L_{\psi,\zeta}\|\le(\frac{\rho(e)}{c_\zeta})^{1/p}\|\psi\|_p.$$

To prove uniqueness, let $\psi\in L^p(G/H)$, $1<p<\infty$, and suppose that $P_{\psi,\zeta}$ is another bounded linear operator satisfying the conclusions of the theorem. Consider $\Theta:L^p(G/H)\to B(\mathcal{H})$ to be the linear operator defined by

$$\Theta(\psi)=L_{\psi,\zeta}-P_{\psi,\zeta},\quad \psi\in L^p(G/H).$$

Then by (2.3),

$$\|\Theta(\psi)\|\le 2(\frac{\rho(e)}{c_\zeta})^{1/p}\|\psi\|_p.$$

Moreover, $\Theta(\psi)$ is equal to the zero operator on $\mathcal{H}$ for all $\psi\in\mathfrak{A}$. Thus, $\Theta:L^p(G/H)\to B(\mathcal{H})$ is a bounded linear operator which is equal to zero on the dense subspace $\mathfrak{A}$ of $L^p(G/H)$. Therefore, $L_{\psi,\zeta}=P_{\psi,\zeta}$ for all $\psi\in L^p(G/H)$. □

3. Localization operator as an element of Schatten p-class operators

Our goal in this section is to give a complete account of the Schatten p-class property of localization operators. To this end, we note that if $T:\mathcal{H}\to\mathcal{H}$ is a positive operator such that

$$\sum_n <T\xi_n,\xi_n> <\infty,$$

for all orthonormal bases $\{\xi_n, n=1,2,...\}$ for $\mathcal{H}$, then $T:\mathcal{H}\to\mathcal{H}$ is in the trace class S_1. Moreover, $\|T\|_{S_1}=tr(T)=\sum_n <T\xi_n,\xi_n>$. The following proposition shows that $L_{\psi,\zeta}$ is in the trace class when $\psi\in(L^1(G/H),\acute{\mu})$ where $\acute{\mu}$ is a G-invariant measure on G/H.

Proposition 3.1. *Let $\psi\in L^1(G/H)$. Then the localization operator $L_{\psi,\zeta}:\mathcal{H}\to\mathcal{H}$ is in S_1 and*

$$\|L_{\psi,\zeta}\|_{S_1}\le 4\frac{\rho(e)}{c_\zeta}\|\psi\|_1,$$

where $\|\psi\|_1=\int_{G/H}\psi(gH)d\acute{\mu}(gH)$ and $\acute{\mu}$ is a G-invariant measure on G/H.

Proof. We assume that $\psi \in L^1(G/H)$ is non-negative. Then

$$< L_{\psi,\zeta}x, x >= \frac{1}{c_\zeta}\int_{G/H}\frac{\rho(e)}{\rho(g)}\psi(gH)|< x, \varpi(gH)\zeta >|^2 d\mu(gH) \geq 0$$

for all $x \in \mathcal{H}$. That is, $L_{\psi,\zeta}$ is positive. Let $\{\zeta_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Then using the fact that $\acute{\mu}$ is a G-invariant measure on G/H which arises from $\acute{\rho}(g) = 1$ we get,

$$\begin{aligned} \|L_{\psi,\zeta}\|_{S_1} &= \textstyle\sum_{k=1}^{\infty} < L_{\psi,\zeta}\zeta_k, \zeta_k > \\ &= \textstyle\sum_{k=1}^{\infty} \frac{1}{c_\zeta}\int_{G/H}\frac{\rho(e)}{\rho(g)}\psi(gH)|< \zeta_k, \varpi(gH)\zeta >|^2 d\mu(gH) \\ &= \textstyle\frac{1}{c_\zeta}\int_{G/H}\rho(e)\psi(gH)\sum_{k=1}^{\infty}|< \zeta_k, \varpi(gH)\zeta >|^2 d\acute{\mu}(gH) \\ &= \textstyle\frac{\rho(e)}{c_\zeta}\int_{G/H}\psi(gH)d\acute{\mu}(gH) \\ &= \textstyle\frac{\rho(e)}{c_\zeta}\|\psi\|_1. \end{aligned}$$

Now if $\psi \in L^1(G/H)$ is real valued, then we write $\psi = \psi_+ - \psi_-$. So we have,

$$\|L_{\psi,\zeta}\|_{S_1} \leq 2\frac{\rho(e)}{c_\zeta}\|\psi\|_1.$$

Let $\psi \in L^1(G/H)$. Then $\psi = \psi_1 + i\psi_2$, where ψ_1, ψ_2 is real valued. So,

$$\|L_{\psi,\zeta}\|_{S_1} \leq 4\frac{\rho(e)}{c_\zeta}\|\psi\|_1.$$

□

Let S_∞ be the set of bounded linear operators on $\mathcal{H}$. Thus $\|.\|_{S_\infty}$ is the operator norm. By Proposition 2.3, if $\psi \in L^\infty(G/H)$, then $\|L_{\psi,\zeta}\|_{S_\infty} \leq \|\psi\|_\infty$. Now, the following proposition shows that $L_{\psi,\zeta}$ is in S_p for $\psi \in L^p(G/H)$, $1 < p < \infty$.

Theorem 3.2. *Let $\psi \in L^p(G/H)$, for $1 < p < \infty$. Then $L_{\psi,\zeta} : \mathcal{H} \to \mathcal{H}$ is in S_p and*

$$\|L_{\psi,\zeta}\|_{S_p} \leq (4\frac{\rho(e)}{c_\zeta})^{1/p}\|\psi\|_p,$$

where $\|\psi\|_p$ is given with respect to a G-invariant measure.

Proof. By Proposition 3.1 and Riesz Thorin Interpolation Theorem [16], the proof is obvious. □

In the following proposition we prove that the localization operator is compact.

Theorem 3.3. *Let $\psi \in L^p(G/H)$, for $1 \le p \le \infty$. Then $L_{\psi,\zeta} : \mathcal{H} \to \mathcal{H}$ is a compact operator.*

Proof. Let $\mathfrak{A}$ be the set of all simple functions ψ on G/H such that $\mu(\{gH \in G/H, \quad \psi(gH) \neq 0\}) < \infty$. Let $\{\psi_n\}_{n=1}^{\infty}$ be a sequence of functions in $\mathfrak{A}$ such that $\psi_n \to \psi$ in $L^p(G/H)$ as $n \to \infty$. Then

$$\|L_{\psi_n,\zeta} - L_{\psi,\zeta}\| \le (\frac{\rho(e)}{c_\zeta})^{1/p} \|\psi_n - \psi\|_p \to 0,$$

as $n \to \infty$. But $\{L_{\psi_n,\zeta}\}$ is in S_1. Since $S_p \subseteq S_q$, for $1 \le p \le q \le \infty$, we have $L_{\psi_n,\zeta} \in S_2$. So $\{L_{\psi_n,\zeta}\}$ is a sequence of compact operators and hence $L_{\psi,\zeta}$ is a compact operator. □

Here we intend to support our technical considerations developed in the previous discussion by giving some examples.

Example 3.4. *Consider the Euclidean group $G = SO(n) \times_\tau \mathbb{R}^n$ with group operations*

$$(R_1, p_1).(R_2, p_2) = (R_1R_2, R_1p_2 + p_1), \quad (R, p)^{-1} = (R^{-1}, -R^{-1}p).$$

Put $n = 2$ in G, i.e., $G = SO(2) \times_\tau \mathbb{R}^2$ and $\mathcal{H} = L^2(S^1) \simeq L^2[-\pi, \pi]$. In this setting any $R \in SO(2)$ and $s \in S^1$ are given explicitly by

$$R = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}$$

$$s = \begin{pmatrix} sin\gamma \\ cos\gamma \end{pmatrix}.$$

The representation ϖ of G/H, in which $H = \{(0, 0, p_2) \in G\}$, is defined as

$$\varpi(\theta, p_1)\psi(\gamma) = e^{ip_1 sin\gamma}\psi(\gamma - \theta),$$

for all $(\theta, p_1) \in G/H, \psi \in L^2(S^1)$. For an admissible wavelet $\psi \in L^2(S^1)$ and $F \in L^p(G/H), 1 \le p \le \infty$, the localization operator $L_{F,\psi} : L^2(S^1) \to L^2(S^1)$ is given by

$$< L_{F,\psi} f, g >= \frac{1}{c_\psi} \int_0^{2\pi} \int_{-\infty}^{\infty} F(\theta, p_1) < f, \psi_{\theta,p_1} >< \psi_{\theta,p_1}, g > d\theta dp_1,$$

where $\psi_{\theta,p_1}(\gamma) = e^{ip_1 sin\gamma}\psi(\gamma - \theta)$. Also, by Theorem 3.2

$$\|L_{F,\psi}\|_{S_p} \le (4\frac{\rho(e)}{c_\psi})^{1/p} \|F\|_p.$$

Example 3.5. *Denote by* $SO_o(3,1)$ *the connected component of Lorentz group. It is a non-abelian group which may be realized as the set of all real* 4×4 *pseudo orthogonal matrices* A*, i.e., matrices with the following property*

$$A^T\eta A=\eta, detA=1, A_{00}\geq 1, \eta=diag(-1,1,1,1).$$

Let $So(n)$ *be the group of rotations around the origin of* $\mathbb{R}^n$ *and* $\frac{SO_o(3,1)}{So(2)}$ *be identified whit 2-sphere* S^2*, which is not a group. It is well known that,* $SO_o(3,1)=KAH$ *in which*

$K\simeq SO(3), A\simeq SO_o(1,1)\simeq\mathbb{R}\simeq\mathbb{R}_*^+, H\simeq\mathbb{C}$*(Iwasawa decomposition).*

Since $\frac{SO_o(3,1)}{H}\simeq SO(3)A$*, then every element* $gH\in\frac{SO_o(3,1)}{H}$ *can be represented by* $gH\equiv(\gamma,a)$ *where* $\gamma\in SO(3), a\in A$*. Now define the representation* ϖ *of* $\frac{SO_o(3,1)}{H}$ *on* $L^2(S^2)$ *as follows*

$$\varpi:\frac{SO_o(3,1)}{H}\to U(L^2(S^2)), (\varpi(gH)f)(\xi)=\lambda(\gamma.a,\xi)^{1/2}f((\gamma.a)^{-1}\xi),$$

for $f\in L^2(S^2), \xi\in S^2$*, where* $\lambda(\gamma.a,\xi)$ *is the Radon Nikodym derivative. This representation is square integrable (see [2] for more details). For an admissible wavelet* $\psi\in L^2(S^2)$ *and* $F\in L^p(G/H), 1\leq p\leq\infty$*, the localization operator* $L_{F,\psi}:L^2(S^2)\to L^2(S^2)$ *is defined as follows:*

$$< L_{F,\psi}f,g>=\frac{1}{c_\psi}\int_{So_3}\int_A F(\gamma,a)< f,\varpi(\gamma,a)\psi>< \varpi(\gamma,a)\psi, g>\frac{d\mu(\gamma)da}{a^3},$$

for $f,g\in L^2(S^2)$*. Moreover, by Theorem 3.2 and Theorem 2.5 we have*

$$\|L_{F,\psi}\|_{S_p}\leq(4\frac{\rho(e)}{c_\psi})^{1/p}\|F\|_p,$$

and

$$\|L_{F,\psi}\|\leq(\frac{\rho(e)}{c_\psi})^{1/p}\|F\|_p.$$

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