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BIFLATNESS AND BIPROJECTIVITY OF LAU PRODUCT OF BANACH ALGEBRAS

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Abstract. We give sufficient and necessary conditions for the Lau product of Banach algebras to be biflat or biprojective.

1. **Introduction and Preliminaries**

Let A and B be Banach algebras with spectrum $\sigma(B) \neq \emptyset$. Let $\theta \in \sigma(B)$ then the direct product $A \times B$ equipped with the algebra multiplication

$$(a, b) \cdot (c, d) = (ac + \theta(d)a + \theta(b)c, bd), \quad (a, c \in A, b, d \in B),$$

and the ℓ^1-norm is a Banach algebra which is called the θ-Lau product of A and B and is denoted by $A \times_\theta B$. This type of product was introduced by Lau [4] for certain class of Banach algebras and was extended by Sangani Monfared [6] for the general case. For example, the unitization $A^\sharp = A \times_i \mathbb{C}$ of A can be regarded as the $i-$Lau product of A and $\mathbb{C}$ where $i \in \sigma(\mathbb{C})$ is the identity character. If one includes the possibility that $\theta = 0$ then the usual direct product of Banach algebras will be obtained.

This product provides not only new examples of Banach algebras by themselves, but also it has the potential to serve as (counter) examples in

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different branches of functional analysis. From the homological algebra point of view, $A \times_\theta B$ is a strongly splitting Banach algebra extension of B by A that exhibits many properties that are not shared, in general, by arbitrary strongly splitting extensions. For instance, commutativity is not preserved by a general strongly splitting extension. However, $A \times_\theta B$ is commutative if and only if both A and B are commutative [6, Proposition 2.3 (ii)].

Many basic properties of $A \times_\theta B$ such as characterizations of bounded approximate identity, spectrum, topological center, and the ideal structure are investigated in [6]. Character (inner) amenability of $A \times_\theta B$ was also studied in [7] ([2]). The main aim of this paper is to study some homological properties of $A \times_\theta B$, specifically, the concepts of biflatness and biprojectivity.

We denote the multiplication map for a Banach algebra A by $\Delta : A\hat{\otimes}A \to A$ or, for emphasis, Δ_A. It is clear that Δ is an $A-$bimodule map (i.e. a bounded linear map which preserves the module operations) with respect to the canonical module structure on the projective tensor product $A\hat{\otimes}A$. A Banach algebra A is called biprojective if $\Delta : A\hat{\otimes}A \to A$ has a bounded right inverse which is an $A-$bimodule map. A Banach algebra A is said to be biflat if the adjoint $\Delta^* : A^* \to (A\hat{\otimes}A)^*$ of Δ has a bounded left inverse which is an $A-$bimodule map. Taking adjoints implies that every biprojective Banach algebra is biflat. The basic properties of biprojectivity and biflatness are investigated in [3]; see also [1, 5].

A standard argument shows that the usual direct product $A \times B$ (which comes from the case $\theta = 0$) is biflat (resp. biprojective) if and only if both A and B are biflat (resp. biprojective). However, this is not the case for a general $\theta \in \sigma(B)$. For instance, the unitization $A^\sharp$ of A (as the $i-$Lau product of A and $\mathbb{C}$) is biprojective if and only if A is biprojective and unital (or equivalently, A is contractible); [1, Theorem 2.8.48]. For the biflatness of $A^\sharp$, one can also verify that: if A is biflat then $A^\sharp$ is biflat if and only if A has a bounded approximate identity (or equivalently, A is amenable); [5].

Inspired by this special example, as the main purpose of this paper we present the next result which clarifies the relation between biflatness (resp. biprojectivity) of $A \times_\theta B$ and those of A and B in the case where A is unital.

Theorem. *Let A be a unital Banach algebra, let B be a Banach algebra and $\theta \in \sigma(B)$. Then $A \times_\theta B$ is biflat (biprojective, respectively) if and only if both A and B are biflat (biprojective, respectively).*

To provide the proof we need some prerequisites. Recall that the dual A^* of a Banach algebra A is a Banach $A-$bimodule under the module operations defined by $\langle f \cdot a, b\rangle = \langle f, ab\rangle,\ \langle a \cdot f, b\rangle = \langle f, ba\rangle,\ \ (f \in A^*, a, b \in A)$. The basic properties of these module operations are discussed in [1]. It is easy to verify that the dual space $(A \times_\theta B)^*$ can be identified with $A^* \times B^*$ via $\langle (f, g), (a, b)\rangle = f(a)+g(b), (a \in A,\ b \in B,\ f \in A^*,\ g \in B^*)$. Moreover, a direct computation shows that the $(A \times_\theta B)-$bimodule operations of $(A \times_\theta B)^*$ are

$$(1.1) \qquad (a, b) \cdot (f, g) = (a \cdot f + \theta(b)f, f(a)\theta + b \cdot g) \ \text{ and}$$

$$(1.2) \qquad (f, g) \cdot (a, b) = (f \cdot a + \theta(b)f, f(a)\theta + g \cdot b).$$

Furthermore, $L =: A \times_\theta B$ is a Banach $A-$bimodule under the module actions $c \cdot (a, b) =: (c, 0) \cdot (a, b)$ and $(a, b) \cdot c =: (a, b) \cdot (c, 0),\ (a, c \in A, b \in B)$. Furthermore, L can also be made into a Banach $B-$bimodule in a similar fashion. We define some mappings that will be used frequently in the proof of the above theorem. Let $p_A : L \to A$ and $p_B : L \to B$ be the usual projections which are defined by $p_A((a, b)) = a$ and $p_B((a, b)) = b$, $(a \in A, b \in B)$, respectively. We define the usual injections $q_A : A \to L$ and $q_B : B \to L$ by $q_A(a) = (a, 0)$ and $q_B(b) = (0, b)$, respectively. It is easy to verify that p_B, q_B are $B-$bimodule maps and q_A is an $A-$bimodule map (however, p_A is not an $A-$bimodule map, in general). The mappings p_B and q_A induce the unique $B-$bimodule map $p_B \otimes p_B : L\hat{\otimes}L \to B\hat{\otimes}B$ and the unique $A-$bimodule map $q_A \otimes q_A : A\hat{\otimes}A \to L\hat{\otimes}L$ satisfying $(p_B \otimes p_B)((a, b) \otimes (c, d)) = b \otimes d$ and $(q_A \otimes q_A)(a \otimes c) = (a, 0) \otimes (c, 0)$, respectively.

In the case where A is unital we shall also deal with the mappings $r_A : L \to A$ and $s_B : B \to L$ which depend on the character θ and play a key role in the proof of the Theorem. We define r_A and s_B by $r_A((a, b)) = a + \theta(b)$ and $s_B(b) = (-\theta(b), b)$ $(a \in A, b \in B)$, respectively. Trivially r_A is an $A-$bimodule map and s_B is a $B-$bimodule map. Furthermore, they induce a unique $A-$bimodule map $r_A \otimes r_A : L\hat{\otimes}L \to A\hat{\otimes}A$ satisfying $(r_A \otimes r_A)((a, b) \otimes (c, d)) = (a + \theta(b)) \otimes (c + \theta(d))$ and $(s_B \otimes s_B)(b \otimes d) = (-\theta(b), b) \otimes (-\theta(d), d)$, respectively.

2. **Proof of the theorem**

For the convenience of citation and a better exposition, we provide the proof in four steps. Throughout the proof A and B are Banach algebras and $\theta \in \sigma(B)$. We write L for $A \times_\theta B$.

Step 1. *If A and B are biflat and A is unital then L is biflat.*

Proof. Since A and B are biflat there exist an $A-$bimodule map $\lambda_A : (A\hat{\otimes}A)^* \to A^*$ and a $B-$bimodule map $\lambda_B : (B\hat{\otimes}B)^* \to B^*$ such that $\lambda_A \circ (\Delta_A)^* = i_{A^*}$ and $\lambda_B \circ (\Delta_B)^* = i_{B^*}$. A direct verification shows that the left hand side squares in Diagram 1 commute; i.e.,

$$\begin{aligned}&(q_A \otimes q_A)^* \circ (\Delta_L)^* = (\Delta_A)^* \circ (q_A)^*, \quad \text{and}\\ &(s_B \otimes s_B)^* \circ (\Delta_L)^* = (\Delta_B)^* \circ (s_B)^*.\end{aligned}$$

Define $\boldsymbol{\lambda_L} : (L\hat{\otimes}L)^* \to L^*$ by

$$\begin{aligned}\boldsymbol{\lambda_L}(h) \quad &=: \Big((\lambda_A \circ (q_A \otimes q_A)^*)(h),\ (\lambda_B \circ (s_B \otimes s_B)^*)(h)\\ &\quad +\langle(\lambda_A \circ (q_A \otimes q_A)^*)(h), 1\rangle\theta\Big), \qquad (h \in (L\hat{\otimes}L)^*).\end{aligned}$$

Then $\boldsymbol{\lambda_L} \circ (\Delta_L)^* = i_{L^*}$. Indeed, for every $(f,g) \in (A^* \times B^*) \cong L^*$,

$$\begin{aligned}\big(\boldsymbol{\lambda_L} \circ (\Delta_L)^*\big)((f,g)) \quad &= \quad \Big(\big(\lambda_A \circ (q_A \otimes q_A)^* \circ (\Delta_L)^*\big)((f,g)),\\ &\qquad \big(\lambda_B \circ (s_B \otimes s_B)^* \circ (\Delta_L)^*\big)((f,g))\\ &\qquad +\langle\big(\lambda_A \circ (q_A \otimes q_A)^* \circ (\Delta_L)^*\big)((f,g)), 1\rangle\theta\Big)\\ &= \quad \Big(\big(\lambda_A \circ (\Delta_A)^* \circ (q_A)^*\big)((f,g)),\\ &\qquad \big(\lambda_B \circ (\Delta_B)^* \circ (s_B)^*\big)((f,g))\\ &\qquad +\langle\big(\lambda_A \circ (\Delta_A)^* \circ (q_A)^*\big)((f,g)), 1\rangle\theta\Big)\\ &= \quad \Big((i_{A^*} \circ (q_A)^*)((f,g)), (i_{B^*} \circ (s_B)^*)((f,g))\\ &\qquad +\langle(i_{A^*} \circ (q_A)^*)((f,g)), 1\rangle\theta\Big)\\ &= \quad \big(f, (f,g) \circ s_B + \langle f, 1\rangle\theta\big)\\ &= \quad (f, -\langle f, 1\rangle\theta + g + \langle f, 1\rangle\theta)\\ &= \quad (f,g).\end{aligned}$$

A straightforward computation shows that the following identities hold.

$$\begin{aligned}
&(q_A\otimes q_A)^*(h\cdot(a,b)) = (q_A\otimes q_A)^*(h)\cdot(a+\theta(b)),\\
&(q_A\otimes q_A)^*((a,b)\cdot h) = (a+\theta(b))\cdot(q_A\otimes q_A)^*(h),\\
&(s_B\otimes s_B)^*(h\cdot(a,b)) = (s_B\otimes s_B)^*(h)\cdot b \qquad \text{and}\\
&(s_B\otimes s_B)^*((a,b)\cdot h) = b\cdot(s_B\otimes s_B)^*(h); \quad ((a,b)\in L \text{ and } h\in(L\hat{\otimes}L)^*).
\end{aligned}$$

Now we use these identities to show that $\boldsymbol{\lambda_L}$ is a $L-$bimodule map. To this end, let $(a,b)\in L$ and $h\in(L\hat{\otimes}L)^*$, then

$$\begin{aligned}
&\boldsymbol{\lambda_L}(h\cdot(a,b))\\
=\ &\Big((\lambda_A\circ(q_A\otimes q_A)^*)(h\cdot(a,b)),\ (\lambda_B\circ(s_B\otimes s_B)^*)(h\cdot(a,b))\\
&+\langle(\lambda_A\circ(q_A\otimes q_A)^*)(h\cdot(a,b)),1\rangle\theta\Big)\\
=\ &\Big(\lambda_A((q_A\otimes q_A)^*(h)\cdot(a+\theta(b))),\ \lambda_B((s_B\otimes s_B)^*(h)\cdot b)\\
&+\langle\lambda_A((q_A\otimes q_A)^*(h)\cdot(a+\theta(b))),1\rangle\theta\Big)\\
=\ &\Big([(\lambda_A\circ(q_A\otimes q_A)^*)(h)]\cdot a+\theta(b)[(\lambda_A\circ(q_A\otimes q_A)^*)(h)],\\
&\langle(\lambda_A\circ(q_A\otimes q_A)^*)(h),a\rangle\theta+[(\lambda_B\circ(s_B\otimes s_B)^*)(h)\\
&+\langle(\lambda_A\circ(q_A\otimes q_A)^*)(h),1\rangle\theta]\cdot b\Big)\\
=\ &\Big((\lambda_A\circ(q_A\otimes q_A)^*)(h),\ (\lambda_B\circ(s_B\otimes s_B)^*)(h)\\
&+\langle(\lambda_A\circ(q_A\otimes q_A)^*)(h),1\rangle\theta\Big)\boldsymbol{\cdot}(a,b)\\
=\ &\boldsymbol{\lambda_L}(h)\boldsymbol{\cdot}(a,b).
\end{aligned}$$

$$\begin{tikzcd}
B^* \arrow[r, "(\Delta_B)^*"] & (B\hat{\otimes}B)^* \arrow[r, "\lambda_B"] & B^* \arrow[d, "(p_B)^*"'] \\
L^* \arrow[u, "(s_B)^*"] \arrow[d, "(q_A)^*"'] \arrow[r, "(\Delta_L)^*"] & (L\hat{\otimes}L)^* \arrow[u, "(s_B\otimes s_B)^*"] \arrow[d, "(q_A\otimes q_A)^*"'] \arrow[r, "\boldsymbol{\lambda_L}"] & L^* \\
A^* \arrow[r, "(\Delta_A)^*"] & (A\hat{\otimes}A)^* \arrow[r, "\lambda_A"] & A^* \arrow[u, "(p_A)^*"']
\end{tikzcd}$$

(Diagram 1)

This completes the proof of step 1. $\square$

Step 2. *Let L be biflat then B is biflat. If in addition A is unital then A is also biflat.*

Proof. Since L is biflat there exists an $L-$bimodule map $\lambda_L : (L\hat{\otimes}L)^* \to L^*$ such that $\lambda_L \circ (\Delta_L)^* = i_{L^*}$. One can directly check that the left hand side squares in Diagram 2 commute; i.e.,

$$(p_B \otimes p_B)^* \circ (\Delta_B)^* = (\Delta_L)^* \circ (p_B)^*,$$

and

$$(\Delta_L)^* \circ (r_A)^* = (r_A \otimes r_A)^* \circ (\Delta_A)^*.$$

Define $\boldsymbol{\lambda_B} : (B\hat{\otimes}B)^* \to B^*$ and $\boldsymbol{\lambda_A} : (A\hat{\otimes}A)^* \to A^*$ by $\boldsymbol{\lambda_B} =: (q_B)^* \circ \lambda_L \circ (p_B \otimes p_B)^*$ and $\boldsymbol{\lambda_A} =: (q_A)^* \circ \lambda_L \circ (r_A \otimes r_A)^*$, respectively.

$$\begin{array}{ccccc}
B^* & \xrightarrow{(\Delta_B)^*} & (B\hat{\otimes}B)^* & \xrightarrow{\boldsymbol{\lambda_B}} & B^* \\
{\scriptstyle (p_B)^*}\downarrow & & {\scriptstyle (p_B\otimes p_B)^*}\downarrow & & {\scriptstyle (q_B)^*}\uparrow \\
L^* & \xrightarrow{(\Delta_L)^*} & (L\hat{\otimes}L)^* & \xrightarrow{\lambda_L} & L^* \\
{\scriptstyle (r_A)^*}\uparrow & & {\scriptstyle (r_A\otimes r_A)^*}\uparrow & & {\scriptstyle (q_A)^*}\downarrow \\
A^* & \xrightarrow{(\Delta_A)^*} & (A\hat{\otimes}A)^* & \xrightarrow{\boldsymbol{\lambda_A}} & A^*
\end{array}$$

(Diagram 2)

That $\boldsymbol{\lambda_B}$ is a $B-$bimodule map follows from the fact that it is a composition of three $B-$bimodule maps. Similarly $\boldsymbol{\lambda_A}$ is an $A-$bimodule map. Moreover

$$\begin{aligned}
\boldsymbol{\lambda_A} \circ (\Delta_A)^* &= [(q_A)^* \circ \lambda_L \circ (r_A \otimes r_A)^*] \circ (\Delta_A)^* \\
&= (q_A)^* \circ \lambda_L \circ [(r_A \otimes r_A)^* \circ (\Delta_A)^*] \\
&= (q_A)^* \circ \lambda_L \circ [(\Delta_L)^* \circ (r_A)^*] \\
&= (q_A)^* \circ [\lambda_L \circ (\Delta_L)^*] \circ (r_A)^* \\
&= (q_A)^* \circ i_{L^*} \circ (r_A)^* \\
&= (r_A \circ q_A)^* \\
&= i_{A^*}.
\end{aligned}$$

A similar argument can be applied to show that $\boldsymbol{\lambda_B} \circ (\Delta_B)^* = i_{B^*}$. Therefore, A and B are biflat, as claimed. $\square$

Step 3. *If A and B are biprojective and A is unital then L is biprojective.*

Proof. Since A and B are biprojective, there exist an $A-$bimodule map $\rho_A : A \to A\hat{\otimes}A$ and a $B-$bimodule map $\rho_B : B \to B\hat{\otimes}B$ such that $\Delta_A \circ \rho_A = i_A$ and $\Delta_B \circ \rho_B = i_B$. Recall that the mapping $s_B : B \to L$ defined by $s_B(b) = (-\theta(b), b)$, $(b \in B)$ is a $B-$bimodule map, and so it induces the unique $B-$bimodule map $s_B \otimes s_B : B\hat{\otimes}B \to L\hat{\otimes}L$ such that $s_B \circ \Delta_B = \Delta_L \circ (s_B \otimes s_B)$. A similar argument shows that, $q_A \circ \Delta_A = \Delta_L \circ (q_A \otimes q_A)$. In other words, the right hand side squares in Diagram 3 commute. We define $\boldsymbol{\rho_L} : L \to L\hat{\otimes}L$ by

$$\boldsymbol{\rho_L}((a,b)) =: (a,b)\cdot\big((q_A\otimes q_A)(\rho_A(1))\big)+(s_B\otimes s_B)(\rho_B(b)),\ (a \in A, b \in B);$$

where 1 is the identity of A.

$$\begin{array}{ccccc}
B & \xrightarrow{\rho_B} & B\hat{\otimes}B & \xrightarrow{\Delta_B} & B \\
{\scriptstyle p_B}\Big\uparrow & & {\scriptstyle s_B\otimes s_B}\Big\downarrow & & {\scriptstyle s_B}\Big\downarrow \\
L & \xrightarrow{\boldsymbol{\rho_L}} & L\hat{\otimes}L & \xrightarrow{\Delta_L} & L \\
{\scriptstyle p_A}\Big\downarrow & & {\scriptstyle q_A\otimes q_A}\Big\uparrow & & {\scriptstyle q_A}\Big\uparrow \\
A & \xrightarrow{\rho_A} & A\hat{\otimes}A & \xrightarrow{\Delta_A} & A
\end{array}$$

(DIAGRAM 3)

Then $\Delta_L \circ \boldsymbol{\rho_L} = i_L$; indeed,

$$\begin{aligned}
(\Delta_L \circ \boldsymbol{\rho_L})((a,b)) &= \Delta_L\Big((a,b)\cdot\big((q_A \otimes q_A)(\rho_A(1))\big)+(s_B \otimes s_B)(\rho_B(b))\Big) \\
&= (a,b)\cdot\Big(\big(\Delta_L \circ (q_A \otimes q_A)\big)(\rho_A(1))\Big) \\
&\quad +\big(\Delta_L \circ (s_B \otimes s_B)\big)(\rho_B(b)) \\
&= (a,b)\cdot q_A \circ (\Delta_A \circ \rho_A)(1) + s_B \circ (\Delta_B \circ \rho_B)(b) \\
&= (a,b)\cdot q_A(1) + s_B(b) \\
&= (a,b)\cdot(1,0) + (-\theta(b), b) \\
&= (a+\theta(b), 0) + (-\theta(b), b) \\
&= (a,b).
\end{aligned}$$

It remains to show that $\boldsymbol{\rho_L}$ is a $L-$bimodule map. Clearly $\boldsymbol{\rho_L}$ is bounded. So it is enough to show that $\boldsymbol{\rho_L}((a,b)\cdot(c,d)) = (a,b)\cdot\boldsymbol{\rho_L}((c,d))$

and $\boldsymbol{\rho_L}((a,b)\cdot(c,d)) = \boldsymbol{\rho_L}((a,b))\cdot(c,d)$, for all $(a,b),(c,d)\in L$. We have,

$$\begin{aligned}\boldsymbol{\rho_L}((a,b)\cdot(c,d)) &= (a,b)\cdot(c,d)\cdot(q_A\otimes q_A)(\rho_A(1)) + (s_B\otimes s_B)(\rho_B(bd))\\ &= (a,b)\cdot(c,d)\cdot(q_A\otimes q_A)(\rho_A(1))\\ &\quad +(s_B\otimes s_B)(b\cdot\rho_B(d))\\ &= (a,b)\cdot(c,d)\cdot(q_A\otimes q_A)(\rho_A(1))\\ &\quad +(0,b)\cdot(s_B\otimes s_B)(\rho_B(d))\\ &= (a,b)\cdot[(c,d)\cdot(q_A\otimes q_A)(\rho_A(1))\\ &\quad +(s_B\otimes s_B)(\rho_B(d))] - (a,0)\cdot(s_B\otimes s_B)(\rho_B(d))\\ &= (a,b)\cdot\boldsymbol{\rho_L}((c,d)) - (a,0)\cdot(s_B\otimes s_B)(\rho_B(d)).\end{aligned}$$

But $(a,0)\cdot(s_B\otimes s_B)(\rho_B(d)) = 0$. Indeed, let $\rho_B(d) = \sum_{j=1}^{\infty} b_j\otimes d_j$, for some sequences $\{b_j\},\{d_j\}$ in B with $\sum_{j=1}^{\infty}\|b_j\|\|d_j\| < \infty$ then

$$\begin{aligned}(a,0)\cdot(s_B\otimes s_B)(\rho_B(d)) &= (a,0)\cdot(s_B\otimes s_B)(\sum_{j=1}^{\infty} b_j\otimes d_j)\\ &= (a,0)\cdot(\sum_{j=1}^{\infty} s_B(b_j)\otimes s_B(d_j))\\ &= \sum_{j=1}^{\infty}(a,0)\cdot[(-\theta(b_j),b_j)\otimes(-\theta(d_j),d_j)] = 0.\end{aligned}$$

Therefore, $\boldsymbol{\rho_L}((a,b)\cdot(c,d)) = (a,b)\cdot\boldsymbol{\rho_L}((c,d))$. Similarly since $(q_A\otimes q_A)(\rho_A(1))$ commutes with each $(c,d)\in L$ we have $\boldsymbol{\rho_L}((a,b)\cdot(c,d)) = \boldsymbol{\rho_L}((a,b))\cdot(c,d)$. Thus L is biprojective, as required. □

Step 4. *If L is biprojective then B is biprojective. If in addition A is unital then A is also biprojective.*

Proof. Let L be biprojective. So there exists a $L-$bimodule map $\rho_L : L\to L\hat{\otimes}L$ such that $\Delta_L\circ\rho_L = i_L$. A direct verification reveals that the upper right hand side square in Diagram 4 commutes, i.e., $p_B\circ\Delta_L = \Delta_B\circ(p_B\otimes p_B)$. Define $\boldsymbol{\rho_B} : B\to B\hat{\otimes}B$ by $\boldsymbol{\rho_B} =: (p_B\otimes p_B)\circ\rho_L\circ q_B$. That $\boldsymbol{\rho_B}$ is a $B-$bimodule map follows from the fact that $\boldsymbol{\rho_B}$ is a composition of various bimodule maps. Moreover, $\Delta_B \circ \boldsymbol{\rho_B} = i_B$. Indeed,

$$\Delta_B \circ \boldsymbol{\rho_B} = \Delta_B \circ \big((p_B \otimes p_B) \circ \rho_L \circ q_B\big) = p_B \circ (\Delta_L \circ \rho_L) \circ q_B = p_B \circ i_L \circ q_B = i_B.$$

Therefore, B is biprojective.

$$\begin{tikzcd}
B \arrow[r, "\boldsymbol{\rho_B}"] \arrow[d, "q_B"'] & B\hat{\otimes}B \arrow[r, "\Delta_B"] & B \\
L \arrow[r, "\rho_L"] & L\hat{\otimes}L \arrow[r, "\Delta_L"] \arrow[u, "p_B\otimes p_B"'] \arrow[d, "r_A\otimes r_A"'] & L \arrow[u, "p_B"'] \arrow[d, "r_A"'] \\
A \arrow[r, "\boldsymbol{\rho_A}"] \arrow[u, "q_A"] & A\hat{\otimes}A \arrow[r, "\Delta_A"] & A
\end{tikzcd}$$

(Diagram 4)

If A is unital, we use the $A-$bimodule map $r_A : L \to A$ defined by $r_A((a,b)) = a + \theta(b),\ (a \in A, b \in B)$. Then trivially r_A induces a unique $A-$bimodule map $r_A \otimes r_A : L\hat{\otimes}L \to A\hat{\otimes}A$ such that the lower right hand side square in Diagram 4 commutes; i.e., $r_A \circ \Delta_L = \Delta_A \circ (r_A \otimes r_A)$. Define $\boldsymbol{\rho_A} : A \to A\hat{\otimes}A$ by $\boldsymbol{\rho_A} =: (r_A \otimes r_A) \circ \rho_L \circ q_A$. That $\boldsymbol{\rho_A}$ is an $A-$bimodule map follows from the fact that $\boldsymbol{\rho_A}$ is a composition of various bimodule maps. Moreover, $\Delta_A \circ \boldsymbol{\rho_A} = i_A$. Indeed,

$$\Delta_A \circ \boldsymbol{\rho_A} = \Delta_A \circ \big((r_A \otimes r_A) \circ \rho_L \circ q_A\big) = r_A \circ (\Delta_L \circ \rho_L) \circ q_A = r_A \circ i_L \circ q_A = i_A.$$

Therefore, A is also biprojective, as claimed. □

Remark 2.1. *(i) In step 4, we have presented a direct proof for biprojectivity of B. It is worthwhile mentioning that biprojectivity of B can be obtained from [1, Proposition 2.8.41(iii)] and the fact that L is a strongly splitting Banach algebra extension of B by the essential closed ideal A.*

(ii) The current proof of the Theorem heavily is based on the hypothesis that A is unital. The fact that, the unitization $A^\sharp$ of A is biprojective if and only if A is biprojective and unital, shows that the condition "A is unital" is essential for biprojectivity of L and can not be removed in general. However, for the biflatness we know nothing about the essentiality of this hypothesis.

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References

[1] H. G. Dales, Banach Algebras and Automatic Continuity, London Math. Soc. Monogr. Ser., 24, The Clarendon Press, Oxford University Press, New York, 2000.

[2] H. R. Ebrahimi Vishki and A. R. Khoddami, Character inner amenability of certain Banach algebras, *Colloq. Math.* **122** (2011) 225–232.

[3] A. Ya. Helemskii, The Homology of Banach and Topological Algebras, Kluwer Academic Publishers Group, Dordrecht, 1986.

[4] A. T.-M. Lau, Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups, *Fund. Math.* **118** (1983), no. 3, 161–175.

[5] V. Runde, Lectures on Amenability, Lecture Notes in Mathematics, 1774, Springer-Verlag, Berlin, 2002.

[6] M. Sangani Monfared, On certain products of Banach algebras with applications to harmonic analysis, *Studia Math.* **178** (2007), no. 3, 277–294.

[7] M. Sangani Monfared, Character amenability of Banach algebras, *Math. Proc. Cambridge Philos. Soc.* **144** (2008), no. 3, 697–706.

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