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TOTAL DOMINATION IN K_r-COVERED GRAPHS

A. P. KAZEMI

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ABSTRACT. The inflation G_I of a graph G with $n(G)$ vertices and $m(G)$ edges is obtained from G by replacing every vertex of degree d of G by a clique, which is isomorphic to the complete graph K_d, and each edge (x_i, x_j) of G is replaced by an edge (u, v) in such a way that $u \in X_i$, $v \in X_j$, and two different edges of G are replaced by non-adjacent edges of G_I. The total domination number $\gamma_t(G)$ of a graph G is the minimum cardinality of a total dominating set, which is a set of vertices such that every vertex of G is adjacent to one vertex of it. A graph is K_r-covered if every vertex of it is contained in a clique K_r. Cockayne et al. in [Total domination in K_r-covered graphs, Ars Combin. 71 (2004) 289-303] conjectured that the total domination number of every K_r-covered graph with n vertices and no K_r-component is at most $\frac{2n}{r+1}$. This conjecture has been proved only for $3 \leq r \leq 6$. In this paper, we prove this conjecture for a big family of K_r-covered graphs.

1. **Preliminaries**

Let $G = (V, E)$ be a simple graph with *vertex set* V of *order* $n(G)$ and *edge set* E of *size* $m(G)$. The *open neighborhood* of a vertex v in G is the set $N_G(v) = \{u \in V \mid uv \in E\}$. The *degree* of a vertex v is $d(v) =\mid N_G(v) \mid$. The minimum and maximum degree among the vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. We write K_n for the *complete graph* of order n. A *clique* with n vertices in a graph G

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is the induced subgraph of G that is isomorphic to the complete graph K_n. A vertex of degree 1 in G is called a *leaf* of G. A graph H is a *spanning subgraph* of a graph G if $V(H) = V(G)$ and $E(H) \subseteq E(G)$.

An edge subset M in G is called a *matching* in G if no two edges of M has any vertex in common. If $e = vw \in M$, then we say either M *saturates* two vertices v and w or v and w are *M-saturated* (by e). A matching M is a *maximum matching* if there is no other matching M' with $| M' |>| M |$. In a graph G the number of edges in a maximum matching is denoted by $\alpha'(G)$.

We define $\phi_L(G)$ as the maximum possible number of leaves of G that are M-unmatched taken over all maximum matchings M in G. Recall that a subset S of V is *independent* if no two vertices of S are adjacent and a graph is *K_r-covered* if every vertex of it is contained in a clique K_r.

Definition 1.1. The *inflation* or *inflated graph* G_I of a graph G without isolated vertices is obtained as follows: each vertex x_i of degree $d(x_i)$ of G is replaced by a clique $X_i \cong K_{d(x_i)}$ and each edge (x_i, x_j) of G is replaced by an edge (u, v) in such a way that $u \in X_i$, $v \in X_j$, and two different edges of G are replaced by non-adjacent edges of G_I.

An obvious consequence of the definition is that

$$\delta(G_I) = \delta(G),\ \Delta(G_I) = \Delta(G)$$

and

$$n(G_I) = \sum_{x_i \in V(G)} d_G(x_i) = 2m(G).$$

There are two different kinds of edges in G_I. The edges of the clique X_i are colored red and the X_i's are called the *red cliques* (a red clique X_i is reduced to a point if x_i is a leaf of G). The other ones, which correspond to the edges of G, are colored *blue* and they form a perfect matching of G_I . Every vertex of G_I belongs to exactly one red clique and one blue edge. Two adjacent vertices of G_I are said to *red-adjacent* if they belong to the same red clique, *blue-adjacent* otherwise. In general, we adopt the following notation: if x_i and x_j are two adjacent vertices of G, the end vertices of the blue edge of G_I replacing the edge (x_i, x_j) of G are called $x_i x_j$ in X_i and $x_j x_i$ in X_j, and this blue edge is $(x_i x_j, x_j x_i)$. Clearly an inflation is claw-free. More precisely, G_I is the line-graph $L(S(G))$ where the subdivision $S(G)$ of G is obtained by replacing each edge of G by a path of length 2. Also a subgraph H of G that is an inflated graph is called an H-inflated subgraph of G. The study of various domination parameters in inflated graphs was originated by Dunbar and Haynes in [2]. Results related to the domination parameters in inflated graphs can be found in [3], [4] and [9].

Domination in graphs is now well studied in graph theory. The literature on this subject has been surveyed and detailed in two books by Haynes, Hedetniemi, and Slater [6] and [7]. A famous type of domination is total domination, and a recent survey of it can be found in [8].

Definition 1.2. A *total dominating set,* abbreviated TDS, of a graph G is a set S of vertices of G such that every vertex of G is adjacent to a vertex in S. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a TDS of G. A TDS of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-*set.*

Cockayne and et al. have conjectured the following *r-CC conjecture* in [1].

Conjecture 1.3. [1] *Every K_r-covered graph G of order n with no K_r-component satisfies* $\gamma_t(G) \leq \frac{2n}{r+1}$.

This conjecture has been proved only for $3 \leq r \leq 6$ (see [1] and [5]). In this paper, we will prove it for a big family of graphs in the next Theorem 1.4 which will be proved in the next section.

Theorem 1.4. *Let G be a K_r-covered graph with no K_r-component and no isolated vertex that contains H_I as greatest spanning inflated subgraph. If H is regular or satisfies*

$$\phi_L(H) \geq 2\alpha'(H)\left(\frac{\Delta(H) - \delta(H)}{\delta(H) + 1}\right), \tag{1.1}$$

then G satisfies the r-CC conjecture.

2. Main Results

We first state the following observation without its proof . It gives a sufficient condition for verifying when a graph G satisfies the r-CC conjecture.

Observation 2.1. *If inflated graphs satisfy the r-CC conjecture, then every K_r-covered graph G that contains a spanning inflated subgraph satisfies the r-CC conjecture.*

Lemma 2.2. *If G is a graph with no isolated vertex, then*

$$\gamma_t(G_I) \leq 2n(G) - 2\alpha'(G) - \phi_L(G).$$

Proof. Among all maximum matchings in G, let M be one that maximizes the number of leaves that are M-unmatched. If w is an M-unmatched leaf and v is its unique neighbor, then v is M-saturated, by the maximality of M. Form a set D of G_I as follows.

For each $x_ix_j \in M$, let x_ix_j,$x_jx_i \in D$. Since $x_ix_j \in X_i$ and $x_jx_i \in X_j$, these $2\alpha'(G)$ vertices dominate $\cup_{x_i \in V(M)} X_i$. If x_i is an M-unsaturated leaf and x_j is its unique neighbor in G, then x_j is M-saturated by an edge $x_jx_k \in M$, for some $k \neq i, j$. Let $x_jx_k \in D$. Hence x_jx_i is adjacent to vertex x_jx_k in D. Let $x_jx_i \in D$ also. If x_i is an M-unsaturated vertex of degree at least two, place two arbitrary vertices x_ix_j and x_ix_k of X_i in D such that $x_j, x_k \in N_G(x_i)$.

Then D is a total dominating set of G_I and $\mid D \mid = 2n(G) - 2\alpha'(G) - \phi_L(G)$. To justify this counting, observe that D contains two vertices from each X_i except when x_i is one of the $2\alpha'(G)$ M-saturated vertices or one of the $\phi_L(G)$ M-unsaturated leaves. $\square$

We use Lemma 2.2 to prove the following theorem.

Theorem 2.3. *Let G be a graph with no isolated vertex and size m. If G is regular or satisfies formula* (1.1)*, then $\gamma_t(G_I) \leq 4m/(\delta(G)+1)$.*

Proof. Let $n = n(G)$, $\alpha' = \alpha'(G)$, $\delta = \delta(G)$, $\Delta = \Delta(G)$ and let $\phi_L = \phi_L(G)$. Among all maximum matchings in G, let M be one that maximizes the number of leaves that are M-unmatched. Let $V(G) = V(M) \cup V_0$ be a partition. Then the induced subgraph $G[V_0]$ is an independent set, and since $\delta(G) \geq 1$, every vertex of V_0 is adjacent to at least one vertex of $V(M)$. Let $xy \in M$. We claim that if $N_G(x) - V(M) \neq \emptyset$ and $N_G(y) - V(M) \neq \emptyset$, then $N_G(y) - V(M) \subseteq N_G(x) - V(M)$ or $N_G(x) - V(M) \subseteq N_G(y) - V(M)$. Otherwise, if $v \in N_G(x) - V(M)$, $w \in N_G(y) - V(M)$ and $v \neq w$, then $M' = (M - \{xy\}) \cup \{xv, yw\}$ is a matching of G with $\mid M' \mid > \mid M \mid$. Therefore we have a partition $V(M) = V_1 \cup V_2$ such that $\mid V_1 \mid = \mid V_2 \mid = \alpha'$ and every edge of M has a vertex in V_1 and other vertex in V_2. We may also assume that every vertex of V_0 is adjacent to at least one vertex of V_1. Let $xy \in M$ such that $x \in V_1$ and $y \in V_2$. Since x is adjacent to at most $\Delta - 1$ vertices of V_0, $\mid V_0 \mid \leq \alpha'(\Delta - 1)$.

If G is regular, then $| V_0 |\leq \alpha'(\delta-1)$ implies that $\alpha'\delta \geq n-\alpha'$, and so

$$\begin{aligned} 2m &= \textstyle\sum_{v\in V(G)} \deg_G(v) \\ &= \textstyle\sum_{v\in V_1} \deg_G(v) + \sum_{v\in V_0\cup V2} \deg_G(v) \\ &= \alpha'\delta + (n-\alpha')\delta \\ &\geq (n-\alpha')(\delta+1). \end{aligned}$$

By Lemma 2.2,

$$\begin{aligned} \gamma_t(G_I) &\leq 2n-2\alpha'-\phi_L \\ &\leq 2n-2\alpha' \\ &\leq 4m/(\delta+1). \end{aligned}$$

If G is not regular, then $\phi_L \geq \frac{2\alpha'(\Delta-\delta)}{\delta+1}$ and $| V_0 |\leq \alpha'(\Delta-1)$, which implies that

$$\begin{aligned} 2m &= \textstyle\sum_{v\in V(G)} \deg_G(v) \\ &= \textstyle\sum_{v\in V_1} \deg_G(v) + \sum_{v\in V_0\cup V2} \deg_G(v) \\ &\geq \alpha'\delta + (n-\alpha')\delta \\ &\geq (n-\alpha')(\delta+1) - \alpha'(\Delta-\delta). \end{aligned}$$

Again by Lemma 2.2,

$$\begin{aligned} \gamma_t(G_I) &\leq 2n-2\alpha'-\phi_L \\ &\leq 2n-2\alpha'-\tfrac{2\alpha'(\Delta-\delta)}{\delta+1} \\ &\leq 4m/(\delta+1). \end{aligned}$$

□

Corollary 2.4. *Let G be a graph with no isolated vertex and size m such that G is regular or satisfies formula* (1.1)*. Then for every $1\leq r\leq \delta(G)$, $\gamma_t(G_I)\leq 4m/(r+1)$.*

Now Observation 2.1 and Corollary 2.4 prove Theorem 1.4.

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Adel P. Kazemi

Department of Mathematics, University of Mohaghegh Ardabili, P.O. Box 5619911367, Ardabil, Iran

Email: `a.p.kazemi@uma.ac.ir, adelpkazemi@yahoo.com`