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SOLUTIONS OF VARIATIONAL INEQUALITIES ON FIXED POINTS OF NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper , we propose a generalized iterative method for finding a common element of the set of fixed points of a single nonexpannsive mapping and the set of solutions of two variational inequalities with inverse strongly monotone mappings and strictly pseudo-contractive of Browder-Petryshyn type mapping. Our results improve and extend the results announced by many others.

1. Introduction

Throughout this paper, we assume that H is a real Hilbert space with inner product and norm are denoted by $\langle .,. \rangle$ and $\| \, . \, \|$, respectively. Let C be a nonempty closed convex subset of H and $A\colon C \to H$ be a nonlinear mapping.

Recall the following definitions

Definition 1.1. *A is called strongly positive with constant $\overline{\gamma}$ if there is a constant $\overline{\gamma} > 0$ such that*

$$\langle Ax, x \rangle \geq \overline{\gamma} \parallel x \parallel^2, \qquad \forall x \in C.$$

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Definition 1.2. *A is called monotone*

$$\langle Ax - Ay, x - y\rangle \geq 0, \qquad \forall x, y \in C.$$

Definition 1.3. *A is called $\eta-$strongly monotone if there exists a positive constant η such that*

$$\langle Ax - Ay, x - y\rangle \geq \eta \parallel x - y \parallel^2, \qquad \forall x, y \in C. \tag{1.1}$$

Definition 1.4. *A is called $k-$Lipschitzian if there exist a positive constant k such that*

$$\parallel Ax - Ay \parallel \leq k \parallel x - y \parallel, \qquad \forall x, y \in C.$$

Definition 1.5. *A is called $\alpha-$inverse strongly monotone if there exists a positive real number $\alpha > 0$ such that*

$$\langle Ax - Ay, x - y\rangle \geq \alpha \parallel Ax - Ay \parallel^2, \qquad \forall x, y \in C.$$

It is obvious that any $\alpha-$inverse strongly monotone mapping A is $\frac{1}{\alpha}-$Lipschitzian.

The classical variational inequality problem is to fined $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solution of (1.2) is denoted by $VI(C, A)$, that is,

$$VI(C, A) = \{x \in C : \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C\}. \tag{1.3}$$

Let $T : C \to C$ be a mapping. In this paper, we use $Fix(T)$ to denote the set of fixed point of T. Recall the following definition

Definition 1.6. *T is called $\alpha-$contractive if there exists a constant $\alpha \in (0, 1)$ such that*

$$\parallel Tx - Ty \parallel \leq \alpha \parallel x - y \parallel^2, \qquad \forall x, y \in C.$$

Definition 1.7. *T is called nonexpansive if*

$$\parallel Tx - Ty \parallel \leq \parallel x - y \parallel^2, \qquad \forall x, y \in C.$$

Definition 1.8. *T is called λ-strictly pseudo-contractive of Browder and Petryshyn type* [2, 3, 4] *if there exists a constant $\alpha \in (0, 1)$ such that*

(1.4)

$$\parallel Tx - Ty \parallel^2 \leq \parallel x - y \parallel^2 + \lambda \parallel (I - T)x - (I - T)y \parallel^2, \qquad \forall x, y \in C.$$

It is well-known that the last inequality is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \| x - y \|^2 - \frac{1-\lambda}{2} \| (I - T)x - (I - T)y \|^2, \forall x, y \in C.$$

Moudafi [12] introduced the viscosity approximation method for fixed point of nonexpansive mappings (see [16] for further developments in both Hilbert and Banach spaces). Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n f(x_n), \qquad n \geq 0, \tag{1.5}$$

where, f is a contraction on H, $\{\alpha_n\}$ is a sequence in $(0, 1)$. It is proved in [12, 16] that, under appropriate conditions imposed on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.5), converges strongly to the unique solution x^* in $Fix(T)$ of the variational inequality:

$$\langle (I - f)x^*, x - x^*\rangle \geq 0, \qquad \forall x \in Fix(T).$$

Marino and Xu [13] introduced the following general iterative methods:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \qquad n \geq 0, \tag{1.6}$$

where $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. They proved that if $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

- (C_1) $\alpha_n \to 0$,
- (C_2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
- (C_3) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Then, the sequence $\{x_n\}$ generated by (1.6) converges strongly to the unique solution of the variational inequality:

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \qquad \forall x \in Fix(T),$$

which is the optimality condition for minimization problem

$$\min_{x\in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$, for all $x \in H$).

Further, Yao and Yao [17] introduced an iterative method for finding a common element of the set of fixed point of a single nonexpansive mapping and the set of solutions of variational inequality for an $\alpha-$inverse strongly monotone mapping. To be more precise, they introduce the following iterative

$$(1.7) \qquad \begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T P_C(I - \lambda_n A) x_n, \ n \geq 1. \end{cases}$$

where P_C is a metric projection of H onto C, $A\colon C \to H$ an $\alpha-$inverse strongly monotone mapping, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $[0,2\alpha]$. under suitable conditions of these parameters they proved strong convergence of the scheme (1.7) to $P_{\mathcal{F}}u$, where $\mathcal{F} = Fix(T) \cap VI(C,A)$.

In this paper, motivated and inspired by the iterative algorithms introduced by Marino and Xu [13], Katchang and Kumam [9, 10], Kumam [11] and Yao and Yao [17], we introduce the iterative below, with the initial guess $x_0 \in C$ chosen arbitrarily,

$$\begin{cases} y_n = \beta_n P_C(I - \beta_{1,n} A_1) x_n + (1 - \beta_n) P_C(I - \beta_{2,n} A_2) x_n, \\ x_{n+1} = \alpha_n \gamma f(P_C(I - \beta_{1,n} A_1) y_n) + (I - \alpha_n F) T P_C(I - \beta_{2,n} A_2) y_n, n \geq 1. \end{cases}$$

where, P_C is a metric projection of H onto C, for $i = 1, 2$, $A_i\colon C \to H$ a δ_i-inverse strongly monotone mapping, F a mapping on H which is both δ- strongly accretive and λ- strictly pseudo-contractive of Browder-Petryshyn type such that $\delta > \frac{1+\lambda}{2}$, f is a contraction on H with coefficient $0 < \alpha < 1$ and γ is a positive real number such that $\gamma < (1 - \sqrt{\frac{2-2\delta}{1-\lambda}})/\alpha$. Our purpose in this paper is to introduce this general iterative algorithm for approximating a fixed point of a single nonexpansive mapping, which solves two variational inequalities. Our results improve and extend the results of Halpern [6], Marino and Xu [13], Xu [16], and many others.

2. **Preliminaries**

This section collects some prerequisites which will be used later.

Lemma 2.1. [15] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - b_n) a_n + b_n c_n, \qquad n \geq 0,$$

where $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers satisfying the following conditions:

(i) $\{b_n\} \subset (0,1)$, $\sum_{n=0}^{\infty} b_n = \infty$,

(ii) *either* $\limsup_{n\to\infty} c_n \leq 0$ *or* $\sum_{n=0}^{\infty} |b_n c_n| < \infty$.
Then, $\lim_{n\to\infty} a_n = 0$.

Let C be a nonempty subset of a Hilbert space H and $T\colon C \to H$ a mapping. Then T is said to be demiclosed at $v \in H$ if, for any sequence $\{x_n\}$ in C, the following implication holds:

$$x_n \rightharpoonup u \in C \qquad \textit{and} \qquad Tx_n \to v \qquad \textit{imply} \qquad Tu = v,$$

where $\to$ (resp. $\rightharpoonup$) denotes strong (resp. weak) convergence.

Lemma 2.2. [8] *Let C be a nonempty closed convex subset of a Hilbert space H and suppose that $T\colon C \to H$ is nonexpansive. Then, the mapping $I - T$ is demiclosed at zero.*

Recall that the (nearest point) projection P_C from H into C assigns to $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\| x - P_C x \| = \inf_{y \in C} \| x - y \| .$$

The following characterizes the projection P_C.

In order to prove our main result, we need the following lemmas.

Lemma 2.3. [7] *For a given $x \in H$, $y \in C$,*

$$y = P_C x \Leftrightarrow \langle y - x, z - y \rangle \geq 0, \quad \forall z \in C.$$

It is well known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\| P_C x - P_C y \|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H \tag{2.1}$$

Moreover, P_C is characterized by the following properties: $P_C x \in C$ and for all $x \in H$, $y \in C$,

$$\langle x - P_C x, y - P_C x \rangle \leq 0. \tag{2.2}$$

It is easy to see that (2.2) is equivalent to the following inequality

$$\| x - y \|^2 \geq \| x - P_C x \|^2 + \| y - P_C x \|^2 . \tag{2.3}$$

Using Lemma 2.3, one can see that the variational inequality (1.2) is equivalent to a fixed point problem.

It is easy to see that the following is true:

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda A u), \quad \lambda > 0. \tag{2.4}$$

A set-valued mapping $U\colon H \to 2^H$ is called monotone if for all $x, y \in H, f \in Ux$ and $g \in Uy$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $U\colon H \to 2^H$ is maximal if the graph of $G(U)$ of U is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping U is maximal if and only if for $(x, f) \in H \times H, \langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(U)$ implies that $f \in Ux$. Let A be a monotone mapping of C into H and let $N_C x$ be the normal cone to C at $x \in C$, that is, $N_C x = \{y \in H : \langle y, x-z\rangle \leq 0, \forall z \in C\}$ and define

$$Ux = \begin{cases} Ax + N_C x, & x \in C, \\ \emptyset & x \notin C. \end{cases} \tag{2.5}$$

Then U is the maximal monotone and $0 \in Ux$ if and only if $x \in VI(C, A)$; see [14].

Lemma 2.4. [1] *Let H be a real Hilbert space. Then, for all $x, y \in H$*

(i) $\| x - y \|^2 \leq \| x \|^2 + 2\langle y, x + y\rangle$,
(ii) $\| x - y \|^2 \geq \| x \|^2 + 2\langle y, x\rangle$.

The following Lemma will be frequently used throughout the paper. For the sake of completeness, we include its proof.

Lemma 2.5. *Let C be a nonempty closed convex subset of a real Hilbert space H .*

(i) *If $F\colon C \to C$ is a mapping which is both δ-strongly monotone and λ-strictly pseudo-contractive of Browder-Petryshyn type such that $\delta > \frac{1+\lambda}{2}$. Then, $I - F$ is contractive with constant $\sqrt{\frac{2-2\delta}{1-\lambda}}$.*
(ii) *If $F\colon C \to C$ is a mapping which is both δ-strongly monotone and λ-strictly pseudo-contractive of Browder-Petryshyn type such that $\delta > \frac{1+\lambda}{2}$. Then, for any fixed number $\tau \in (0, 1)$, $I-\tau F$ is contractive with constant $1 - \tau\left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}}\right)$.*

Proof. We will employ the same method as used by Ceng et al. [5].

(i) From (1.2) and (1.4), we obtain

$$\begin{aligned} \frac{1-\lambda}{2} \| (I - F)x - (I - F)y \|^2 &\leq \| x - y \|^2 - \langle Fx - Fy, J(x - y)\rangle \\ &\leq (1 - \delta) \| x - y \|^2 . \end{aligned}$$

Because $\delta > \frac{1+\lambda}{2} \Leftrightarrow \delta + \frac{1-\lambda}{2} > 1 \Leftrightarrow \sqrt{\frac{2-2\delta}{1-\lambda}} \in (0,1)$, we have

$$\| (I-F)x-(I-F)y \| \le \sqrt{\frac{2-2\delta}{1-\lambda}} \| x-y \|$$

and, therefore, $I-F$ is contractive with constant $\sqrt{\frac{2-2\delta}{1-\lambda}}$.

(ii) Because $I-F$ is contractive with constant $\sqrt{\frac{2-2\delta}{1-\lambda}}$, for each fixed number $\tau \in (0,1)$, we have

$$\begin{aligned}\| x-y-\tau(F(x)-F(y)) \| &= \| (1-\tau)(x-y)+\tau[(I-F)x-(I-F)y] \| \\ &\le (1-\tau) \| x-y \| + \tau \| (I-F)x-(I-F)y \| \\ &\le (1-\tau) \| x-y \| + \tau\sqrt{\frac{2-2\delta}{1-\lambda}} \| x-y \| \\ &= \left(1-\tau\left(1-\sqrt{\frac{2-2\delta}{1-\lambda}}\right)\right) \| x-y \| .\end{aligned}$$

This shows that $I-\tau F$ is contractive with constant $1-\tau\left(1-\sqrt{\frac{2-2\delta}{1-\lambda}}\right)$.

3. **Strong convergence theorems**

The following is our main result.

Theorem 3.1. *Let $F\colon C \to H$ be a mapping which is both δ- strongly monotone and λ- strictly pseudo-contractive of Browder-Petryshyn type such that $\delta > (1+\lambda)/2$, f a contraction on H with coefficient $0 < \alpha < 1$ and γ be a positive real number such that $\gamma < (1-\sqrt{\frac{2-2\delta}{1-\lambda}})/\alpha$. Let $T\colon C \to C$ be a nonexpansive mapping and for each $i=1,2$, let $A_i\colon C \to H$ be δ_i-inverse strongly monotone mapping and $\mathcal{F} = VI(C,A_1) \cap VI(C,A_2) \cap Fix(T)) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_{i,n}\}_{i=1,n=1}^{2,\ \infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $(0,1)$ satisfy the following conditions:*

(B_1) *$\{\beta_{i,n}\} \subset (0,\delta_i)$ for $i=1,2$.*
(B_2) *$\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}\beta_n = \beta \in (0,1)$.*
(B_3) *$\sum_{n=1}^{\infty} | \beta_{n+1}-\beta_n | < \infty$, $\sum_{n=1}^{\infty} | \beta_{i,n+1}-\beta_{i,n} | < \infty$, for $i=1,2$.*

If $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences generated by $x_0 \in C$ and

$$\begin{cases} y_n = \beta_n P_C(I - \beta_{1,n} A_1)x_n + (1 - \beta_n) P_C(I - \beta_{2,n} A_2)x_n, \\ x_{n+1} = \alpha_n \gamma f(P_C(I - \beta_{1,n} A_1)y_n) + (I - \alpha_n F) T P_C(I - \beta_{2,n} A_2)y_n, \ n \geq 1. \end{cases}$$

Then $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *converge strongly to* $x^* \in \mathcal{F}$*, which is the unique solution of the system of variational inequalities:*

$$\begin{cases} \langle (F - \gamma f)x^*, x - x^* \rangle \geq 0, \\ \langle A_i x^*, x - x^* \rangle \geq 0 \qquad , \qquad \forall x \in \mathcal{F}, i = 1, 2. \end{cases}$$

Proof. Since $\{\beta_{i,n}\}_{i=1,n=1}^{2,\ \infty}$ satisfies the in condition (B_1) and A_i is δ_i-inverse strongly monotone mapping, for any $x, y \in C$, we have

$$\begin{aligned} &\| (I - \beta_{i,n} A_i)x - (I - \beta_{i,n} A_i)y \|^2 \\ &\quad = \| (x - y) - \beta_{i,n}(A_i x - A_i y) \|^2 \\ &\quad = \| x - y \|^2 - 2\beta_{i,n} \langle x - y, A_i x - A_i y \rangle + \beta_{i,n}^2 \| A_i x - A_i y \|^2 \\ &\quad \leq \| x - y \|^2 - 2\beta_{i,n} \delta_i \| A_i x - A_i y \|^2 + \beta_{i,n}^2 \| A_i x - A_i y \|^2 \\ &\quad = \| x - y \|^2 + \beta_{i,n}(\beta_{i,n} - 2\delta_i) \| A_i x - A_i y \|^2 \\ &\quad \leq \| x - y \|^2 \end{aligned}$$

It follows that

$$(3.1) \qquad \| (I - \beta_{i,n} A_i)x - (I - \beta_{i,n} A_i)y \| \leq \| x - y \|, \qquad i = 1, 2.$$

Let $p \in \mathcal{F}$, in the context of the variational inequality problem the characterization of projection (2.4) implies that $p = P_C(I - \beta_{i,n} A_i)p, \ \ i = 1, 2$. Using (2.4) and (3.1), we get

$$\begin{aligned} \| y_n - p \| &= \| \beta_n P_C(I - \beta_{1,n} A_1)x_n + (1 - \beta_n) P_C(I - \beta_{2,n} A_2)x_n - p \| \\ &= \| \beta_n [P_C(I - \beta_{1,n} A_1)x_n - P_C(I - \beta_{1,n} A_1)p] \\ &\quad + (1 - \beta_n)[P_C(I - \beta_{2,n} A_2)x_n - P_C(I - \beta_{2,n} A_2)p] \| \\ &\leq \beta_n \| P_C(I - \beta_{1,n} A_1)x_n - P_C(I - \beta_{2,n} A_2)p \| \\ &\quad + (1 - \beta_n) \| P_C(I - \beta_{2,n} A_2)x_n - P_C(I - \beta_{2,n} A_2)p] \| \\ &\leq \beta_n \| x_n - p \| + (1 - \beta_n) \| x_n - p \| = \| x_n - p \| \qquad (3.2) \end{aligned}$$

First we show that $\{x_n\}$ is bounded. Indeed, we take $p \in \mathcal{F}$. Then using (3.2) and Lemma 2.4, we have

$$
\begin{aligned}
&\| x_{n+1} - p \| \\
&\quad =\| \alpha_n \gamma f(P_C(I-\beta_{1,n}A_1)y_n) + (I-\alpha_n F)TP_C(I-\beta_{2,n}A_2)y_n - p \| \\
&\quad =\| (I-\alpha_n F)TP_C(I-\beta_{2,n}A_2)y_n - (I-\alpha_n F)p \\
&\qquad + \alpha_n[\gamma f(P_C(I-\beta_{1,n}A_1)y_n) - F(p)] \| \\
&\quad \leq\| (I-\alpha_n F)TP_C(I-\beta_{2,n}A_2)y_n - (I-\alpha_n F)p \| \\
&\qquad + \alpha_n \| \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - F(p) \| \\
&\quad \leq \left(1-\alpha_n\left(1-\sqrt{\frac{2-2\delta}{1-\lambda}}\right)\right) \| TP_C(I-\beta_{2,n}A_2)y_n - p \| \\
&\qquad + \alpha_n \| \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - \gamma f(p) \| + \alpha_n \| \gamma f(p) - F(p) \| \\
&\quad \leq \left(1-\alpha_n\left(1-\sqrt{\frac{2-2\delta}{1-\lambda}}-\gamma\alpha\right)\right) \| y_n - p \| \\
&\qquad + \alpha_n \| \gamma f(p) - F(p) \| \\
&\quad \leq \left(1-\alpha_n\left(1-\sqrt{\frac{2-2\delta}{1-\lambda}}-\gamma\alpha\right)\right) \| x_n - p \| \\
&\qquad + \frac{\alpha_n\left(1-\sqrt{\frac{2-2\delta}{1-\lambda}}-\gamma\alpha\right)}{\left(1-\sqrt{\frac{2-2\delta}{1-\lambda}}-\gamma\alpha\right)} \| \gamma f(p) - F(p) \| \\
&\quad \leq \max\left\{\left(1-\sqrt{\frac{2-2\delta}{1-\lambda}}-\gamma\alpha\right)^{-1} \| \gamma f(p) - F(p) \|, \| x_n - p \| \right\}.
\end{aligned}
$$

By induction,

$$
\| x_n - p \| \leq \max\left\{\left(1-\sqrt{\frac{2-2\delta}{1-\lambda}}-\gamma\alpha\right)^{-1} \| \gamma f(p) - F(p) \|, \| x_0 - p \| \right\}.
$$

Therefore, $\{x_n\}$ is bounded and so are the sequences $\{y_n\}$, $\{FT(y_n)\}$ and $\{f(y_n)\}$. Now we claim that

$$
\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0. \tag{3.3}
$$

Indeed, we have (for some approximation constant $M_1 > 0$)

$$
\begin{aligned}
&\| x_{n+1} - x_n \| \\
&= \| \alpha_n \gamma f(P_C(I - \beta_{1,n}A_1)y_n) + (I - \alpha_n F)TP_C(I - \beta_{2,n}A_2)y_n \\
&\quad - \alpha_{n-1}\gamma f(P_C(I - \beta_{1,n-1}A_1)y_{n-1}) \\
&\quad - (I - \alpha_{n-1}F)TP_C(I - \beta_{2,n-1}A_2)y_{n-1} \| \\
&\le \| (I - \alpha_n F)TP_C(I - \beta_{2,n}A_2)y_n \\
&\quad - (I - \alpha_n F)TP_C(I - \beta_{2,n}A_2)y_{n-1} \| \\
&\quad + \| (I - \alpha_n F)TP_C(I - \beta_{2,n}A_2)y_{n-1} \\
&\quad - (I - \alpha_n F)TP_C(I - \beta_{2,n-1}A_2)y_{n-1} \| \\
&\quad + \| (I - \alpha_n F)TP_C(I - \beta_{2,n-1}A_2)y_{n-1} \\
&\quad - (I - \alpha_{n-1} F)TP_C(I - \beta_{2,n-1}A_2)y_{n-1} \| \\
&\quad + \| \alpha_n \gamma f(P_C(I - \beta_{1,n}A_1)y_n) - \alpha_n \gamma f(P_C(I - \beta_{1,n}A_1)y_{n-1}) \| \\
&\quad + \| \alpha_n \gamma f(P_C(I - \beta_{1,n}A_1)y_{n-1}) - \alpha_n \gamma f(P_C(I - \beta_{1,n-1}A_1)y_{n-1}) \| \\
&\quad + \| \alpha_n \gamma f(P_C(I - \beta_{1,n-1}A_1)y_{n-1}) - \alpha_{n-1} \gamma f(P_C(I - \beta_{1,n-1}A_1)y_{n-1}) \| \\
&\le \left(1 - \alpha_n \left(1 - \sqrt{\frac{2 - 2\delta}{1 - \lambda}}\right)\right) \| y_n - y_{n-1} \| \\
&\quad + \left(1 - \alpha_n \left(1 - \sqrt{\frac{2 - 2\delta}{1 - \lambda}}\right)\right) | \beta_{2,n} - \beta_{2,n-1} | \| A_2 y_{n-1} \| \\
&\quad + | \alpha_{n-1} - \alpha_n | \| FTP_C(I - \beta_{2,n-1}A_2)y_{n-1} \| + \alpha_n \gamma \alpha \| y_n - y_{n-1} \| \\
&\quad + \alpha_n \gamma \alpha | \beta_{1,n} - \beta_{1,n-1} | \| A y_{n-1} \| \\
&\quad + | \alpha_{n-1} - \alpha_n | \| f(P_C(I - \beta_{1,n-1}A_1)y_{n-1}) \| \\
&\le \left(1 - \alpha_n \left(1 - \sqrt{\frac{2 - 2\delta}{1 - \lambda}} - \gamma\alpha\right)\right) \| y_n - y_{n-1} \| \\
&\quad + [| \alpha_{n-1} - \alpha_n | + | \beta_{1,n} - \beta_{1,n-1} | + | \beta_{2,n} - \beta_{2,n-1} |] M_1.
\end{aligned}
$$

On the other hand, by taking $v_{i,n} = P_C(I - \beta_{i,n}A_i)x_n$ for $i = 1, 2$ and definition of $\{y_n\}$, we have (for some approximation constant $M_2 > 0$)

$$
\begin{aligned}
&\| y_n - y_{n-1} \| \\
&\quad = \| \beta_n v_{1,n} + (1 - \beta_n)v_{2,n} - \beta_{n-1}v_{1,n-1} - (1 - \beta_{n-1})v_{2,n-1} \| \\
&\quad = \| \beta_n (v_{1,n} - v_{1,n-1}) + (\beta_n - \beta_{n-1})v_{1,n-1} + (1 - \beta_n)v_{2,n}
\end{aligned}
$$

$$
\begin{aligned}
&\quad - (1-\beta_n)v_{2,n-1} + (\beta_{n-1}-\beta_n)v_{2,n-1} \parallel \\
&\le \beta_n \parallel v_{1,n} - v_{1,n-1} \parallel + \mid \beta_n - \beta_{n-1} \mid (\parallel v_{1,n-1} \parallel + \parallel v_{2,n-1} \parallel) \\
&\quad + (1-\beta_n) \parallel v_{2,n} - v_{2,n-1} \parallel \\
&= \beta_n \parallel P_C(I - \beta_{1,n}A_1)x_n - P_C(I - \beta_{1,n-1}A_1)x_{n-1} \parallel \\
&\quad + \mid \beta_n - \beta_{n-1} \mid (\parallel v_{1,n-1} \parallel + \parallel v_{2,n-1} \parallel) \\
&\quad + (1-\beta_n) \parallel P_C(I - \beta_{2,n}A_2)x_n - P_C(I - \beta_{2,n-1}A_2)x_{n-1} \parallel \\
&= \beta_n \parallel P_C(I - \beta_{1,n}A_1)x_n - P_C(I - \beta_{1,n}A_1)x_{n-1} \\
&\quad + P_C(I - \beta_{1,n}A_1)x_{n-1} - P_C(I - \beta_{1,n-1}A_1)x_{n-1} \parallel \\
&\quad + \mid \beta_n - \beta_{n-1} \mid (\parallel v_{1,n-1} \parallel + \parallel v_{2,n-1} \parallel) \\
&\quad + (1-\beta_n) \parallel P_C(I - \beta_{2,n}A_2)x_n - P_C(I - \beta_{2,n}A_2)x_{n-1} \\
&\quad + P_C(I - \beta_{2,n}A_2)x_{n-1} - P_C(I - \beta_{2,n-1}A_2)x_{n-1} \parallel \\
&\le \beta_n \parallel x_n - x_{n+1} \parallel + \beta_n \mid \beta_{1,n} - \beta_{1,n-1} \mid \parallel A_1 x_{n-1} \parallel \\
&\quad + \mid \beta_n - \beta_{n-1} \mid (\parallel v_{1,n-1} \parallel + \parallel v_{2,n-1} \parallel) + (1-\beta_n) \parallel x_n - x_{n-1} \parallel \\
&\quad + (1-\beta_n) \mid \beta_{2,n} - \beta_{2,n-1} \mid \parallel A_2 x_{n-1} \parallel \\
&\le \parallel x_n - x_{n-1} \parallel + (\mid \beta_{1,n} - \beta_{1,n-1} \mid + \mid \beta_n - \beta_{n-1} \mid + \mid \beta_{2,n} - \beta_{2,n-1} \mid) M_2.
\end{aligned}
$$

Therefore, we have(for some approximation constant $M > 0$)

$$
\begin{aligned}
\parallel x_{n+1} - x_n \parallel \le &\left(1 - \alpha_n \left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}} - \gamma\alpha\right)\right) \parallel x_n - x_{n-1} \parallel \\
&+ (\mid \beta_{1,n} - \beta_{1,n-1} \mid + \mid \beta_{2,n} - \beta_{2,n-1} \mid \\
&+ \mid \beta_n - \beta_{n-1} \mid + \mid \alpha_n - \alpha_{n-1} \mid) M. \qquad (3.4)
\end{aligned}
$$

Thus, using conditions (B_2) and (B_3) and Lemma 2.1 to (3.4), we conclude that $\parallel x_{n+1} - x_n \parallel \to 0$ as $n \to \infty$. In this stage we well show that

$$
\lim_{n\to\infty} \parallel v_{i,n} - x_n \parallel = 0 \qquad i = 1, 2. \qquad (3.5)
$$

Let $p \in \mathcal{F}$, from definition of $\{x_n\}$, we have

$$
\begin{aligned}
&\parallel x_{n+1} - p \parallel^2 \\
&= \parallel \alpha_n \gamma f(P_C(I - \beta_{1,n}A_1)y_n) + (I - \alpha_n F)TP_C(I - \beta_{2,n}A_2)y_n - p \parallel^2 \\
&= \parallel \alpha_n[\gamma f(P_C(I - \beta_{1,n}A_1)y_n) - FTP_C(I - \beta_{2,n}A_2)y_n] \\
&\quad + [TP_C(I - \beta_{2,n}A_2)y_n - p] \parallel^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \alpha_n^2 \parallel \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - FTP_C(I-\beta_{2,n}A_2)y_n \parallel^2 \\
&\quad + \parallel TP_C(I-\beta_{2,n}A_2)y_n - p \parallel^2 + 2\alpha_n\langle \gamma f(P_C(I-\beta_{1,n}A_1)y_n) \\
&\quad - FTP_C(I-\beta_{2,n}A_2)y_n, TP_C(I-\beta_{2,n}A_2)y_n - p\rangle \\
&\leq \alpha_n^2 \parallel \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - FTP_C(I-\beta_{2,n}A_2)y_n \parallel^2 \\
&\quad + \parallel y_n - p \parallel^2 + 2\alpha_n\langle \gamma f(P_C(I-\beta_{1,n}A_1)y_n) \\
&\quad - FTP_C(I-\beta_{2,n}A_2)y_n, TP_C(I-\beta_{2,n}A_2)y_n - p\rangle \\
&= \alpha_n^2 \parallel \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - FTP_C(I-\beta_{2,n}A_2)y_n \parallel^2 \\
&\quad + \parallel \beta_n P_C(I-\beta_{1,n}A_1)x_n + (1-\beta_n)P_C(I-\beta_{2,n}A_2)x_n - p \parallel^2 \\
&\quad + 2\alpha_n\langle \gamma f(P_C(I-\beta_{1,n}A_1)y_n) \\
&\quad - FTP_C(I-\beta_{2,n}A_2)y_n, TP_C(I-\beta_{2,n}A_2)y_n - p\rangle \\
&\leq \alpha_n^2 \parallel \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - FTP_C(I-\beta_{2,n}A_2)y_n \parallel^2 \\
&\quad + \beta_n \parallel P_C(I-\beta_{1,n}A_1)x_n - p \parallel^2 + (1-\beta_n) \parallel P_C(I-\beta_{2,n}A_2)x_n - p \parallel^2 \\
&\quad + 2\alpha_n\langle \gamma f(P_C(I-\beta_{1,n}A_1)y_n) \\
&\quad - FTP_C(I-\beta_{2,n}A_2)y_n, TP_C(I-\beta_{2,n}A_2)y_n - p\rangle
\end{aligned}
$$

(3.6)

Using (2.4) and (3.6), we have

$$
\begin{aligned}
&\parallel x_{n+1} - p \parallel^2 \\
&\quad \leq \alpha_n^2 \parallel \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - FTP_C(I-\beta_{2,n}A_2)y_n \parallel^2 \\
&\qquad + \beta_n \parallel (x_n - p) - \beta_{1,n}(A_1x_n - A_1p) \parallel^2 + (1-\beta_n) \parallel x_n - p \parallel^2 \\
&\qquad + 2\alpha_n\langle \gamma f(P_C(I-\beta_{1,n}A_1)y_n) \\
&\qquad - FTP_C(I-\beta_{2,n}A_2)y_n, TP_C(I-\beta_{2,n}A_2)y_n - p\rangle \\
&\quad = \alpha_n^2 \parallel \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - FTP_C(I-\beta_{2,n}A_2)y_n \parallel^2 \\
&\qquad + \beta_n \parallel x_n - p \parallel^2 + \beta_n\beta_{1,n}^2 \parallel A_1x_n - A_1p \parallel^2 \\
&\qquad - 2\beta_n\beta_{1,n}\langle A_1x_n - A_1p, x_n - p\rangle + (1-\beta_n) \parallel x_n - p \parallel^2 \\
&\qquad + 2\alpha_n\langle \gamma f(P_C(I-\beta_{1,n}A_1)y_n) \\
&\qquad - FTP_C(I-\beta_{2,n}A_2)y_n, TP_C(I-\beta_{2,n}A_2)y_n - p\rangle \\
&\quad \leq \alpha_n^2 \parallel \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - FTP_C(I-\beta_{2,n}A_2)y_n \parallel^2 \\
&\qquad + \beta_n \parallel x_n - p \parallel^2 + \beta_n\beta_{1,n}^2 \parallel A_1x_n - A_2p \parallel^2 - 2\beta_n\beta_{1,n}\delta_1 \parallel A_1x_n - A_1p \parallel^2
\end{aligned}
$$

$$+ (1-\beta_n) \parallel x_n - p \parallel^2 + 2\alpha_n\langle \gamma f(P_C(I-\beta_{1,n}A_1)y_n) \\ - FTP_C(I-\beta_{2,n}A_2)y_n, TP_C(I-\beta_{2,n}A_2)y_n - p\rangle$$

and

$$\begin{aligned} &\parallel x_{n+1} - p \parallel^2 \\ &\quad \le \alpha_n^2 \parallel \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - FTP_C(I-\beta_{2,n}A_2)y_n \parallel^2 \\ &\qquad + \beta_n \parallel x_n - p \parallel^2 + (1-\beta_n) \parallel (x_n - p) - \beta_{2,n}(A_2x - n - A_2p \parallel^2 \\ &\qquad + 2\alpha_n\langle \gamma f(P_C(I-\beta_{1,n}A_1)y_n) \\ &\qquad - FTP_C(I-\beta_{2,n}A_2)y_n, TP_C(I-\beta_{2,n}A_2)y_n - p\rangle \\ &\quad = \alpha_n^2 \parallel \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - FTP_C(I-\beta_{2,n}A_2)y_n \parallel^2 \\ &\qquad + \beta_n \parallel x_n - p \parallel^2 \\ &\qquad + (1-\beta_n) \parallel x_n - p \parallel^2 + (1-\beta_n)\beta_{2,n}^2 \parallel A_2x_n - A_2p \parallel^2 \\ &\qquad - 2(1-\beta_n)\beta_{2,n}\langle A_2x_n - A_2p, x_n - p\rangle + 2\alpha_n\langle \gamma f(P_C(I-\beta_{1,n}A_1)y_n) \\ &\qquad - FTP_C(I-\beta_{2,n}A_2)y_n, TP_C(I-\beta_{2,n}A_2)y_n - p\rangle \\ &\quad \le \alpha_n^2 \parallel \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - FTP_C(I-\beta_{2,n}A_2)y_n \parallel^2 \\ &\qquad + \beta_n \parallel x_n - p \parallel^2 + (1-\beta_n) \parallel x_n - p \parallel^2 \\ &\qquad + (1-\beta_n)\beta_{2,n}^2 \parallel A_2x_n - A_2p \parallel^2 \\ &\qquad - 2(1-\beta_n)\beta_{2,n}\delta_2 \parallel A_2x_n - A_2 \parallel^2 \\ &\qquad + 2\alpha_n\langle \gamma f(P_C(I-\beta_{1,n}A_1)y_n) \\ &\qquad - FTP_C(I-\beta_{2,n}A_2)y_n, TP_C(I-\beta_{2,n}A_2)y_n - p\rangle \end{aligned}$$

It follows that

$$\begin{aligned} &- \beta_n\beta_{1,n}(\beta_{1,n} - 2\delta_1) \parallel A_1x_n - A_1p \parallel^2 \\ &\quad \le \alpha_n^2 \parallel \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - FTP_C(I-\beta_{2,n}A_2)y_n \parallel^2 \\ &\qquad + [\parallel x_n - p \parallel + \parallel x_{n+1} - p \parallel] \parallel x_{n+1} - x_n \parallel \\ &\qquad + 2\alpha_n\langle \gamma f(P_C(I-\beta_{1,n}A_1)y_n) \\ &\qquad - FTP_C(I-\beta_{2,n}A_2)y_n, TP_C(I-\beta_{2,n}A_2)y_n - p\rangle \end{aligned}$$

and

$$
\begin{aligned}
&-(1-\beta_n)\beta_{2,n}(\beta_{2,n}-2\delta_2) \parallel A_2x_n - A_2p, x_n - p \parallel^2 \\
&\quad \leq \alpha_n^2 \parallel \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - FTP_C(I-\beta_{2,n}A_2)y_n \parallel^2 \\
&\qquad + [\parallel x_n - p \parallel + \parallel x_{n+1} - p \parallel] \parallel x_{n+1} - x_n \parallel \\
&\qquad + 2\alpha_n \langle \gamma f(P_C(I-\beta_{1,n}A_1)y_n) \\
&\qquad - FTP_C(I-\beta_{2,n}A_2)y_n, TP_C(I-\beta_{2,n}A_2)y_n - p\rangle
\end{aligned}
$$

Therefore, from condition B_1 and B_2, we get

$$
\lim_{n\to\infty} \parallel A_ix_n - A_ip \parallel = 0 \qquad i = 1, 2. \tag{3.7}
$$

From (2.1), we have

$$
\begin{aligned}
&\parallel v_{i,n} - p \parallel^2 \\
&\quad = \parallel P_C(I-\beta_{i,n}A_i)x_n - P_C(I-\beta_{i,n}A_ip) \parallel^2 \\
&\quad \leq \langle (I-\beta_{i,n}A_i)x_n - (I-\beta_{i,n}A_i)p, v_{i,n} - p\rangle \\
&\quad = \frac{1}{2}[\parallel (I-\beta_{i,n}A_i)x_n - (I-\beta_{i,n}A_i)p \parallel^2 + \parallel v_{i,n} - p \parallel^2 \\
&\qquad - \parallel (I-\beta_{i,n}A_i)x_n - (I-\beta_{i,n}A_i)p - (v_{i,n} - p) \parallel^2] \\
&\quad \leq \frac{1}{2}[\parallel x_n - p \parallel^2 + \parallel v_{i,n} - p \parallel^2 \\
&\qquad - \parallel (I-\beta_{i,n}A_i)x_n - (I-\beta_{i,n}A_i)p - (v_{i,n} - p) \parallel^2] \\
&\quad = \frac{1}{2}[\parallel x_n - p \parallel^2 + \parallel v_{i,n} - p \parallel^2 - \parallel x_n - v_{i,n} \parallel^2 \\
&\qquad + 2\beta_{i,n}\langle x_n - v_{i,n}, A_ix_n - A_ip\rangle - \beta_{i,n}^2 \parallel A_ix_n - A_ip \parallel^2].
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
&\parallel v_{i,n} - p \parallel^2 \\
&\quad \leq \parallel x_n - p \parallel^2 - \parallel x_n - v_{i,n} \parallel^2 \\
&\qquad + 2\beta_{i,n}\langle x_n - v_{i,n}, A_ix_n - A_ip\rangle \\
&\qquad - \beta_{i,n}^2 \parallel A_ix_n - A_ip \parallel^2, \qquad i = 1, 2.
\end{aligned} \tag{3.8}
$$

From (2.4), (3.8) and definition of $\{y_n\}$, we have

$$
\parallel y_n - p \parallel^2
$$

$$
\begin{aligned}
&\le \beta_n \parallel v_{2,n}x_n - p \parallel^2 + (1-\beta_n) \parallel v_{2,n}x_n - p \parallel^2 \\
&\le \beta_n[\parallel x_n - p \parallel^2 - \parallel x_n - v_{1,n} \parallel^2 + 2\beta_{1,n}\langle x_n - v_{1,n}, A_1x_n - A_1p\rangle \\
&\quad - \beta_{1,n}^2 \parallel A_1x_n - A_1p \parallel^2] + (1-\beta_n)[\parallel x_n - p \parallel^2 - \parallel x_n - v_{2,n} \parallel^2 \\
&\quad + 2\beta_{2,n}\langle x_n - v_{2,n}, A_2x_n - A_2p\rangle - \beta_{2,n}^2 \parallel A_2x_n - A_2p \parallel^2] \\
&= \parallel x_n - p \parallel^2 + \beta_n[- \parallel x_n - v_{1,n} \parallel^2 + 2\beta_{1,n}\langle x_n - v_{1,n}, A_1x_n - A_1p\rangle \\
&\quad - \beta_{1,n}^2 \parallel A_1x_n - A_1p \parallel^2] + (1-\beta_n)[- \parallel x_n - v_{2,n} \parallel^2 \\
&\quad + 2\beta_{2,n}\langle x_n - v_{2,n}, A_2x_n - A_2p\rangle - \beta_{2,n}^2 \parallel A_2x_n - A_2p \parallel^2]
\end{aligned}
\tag{3.9}
$$

From (2.4), (3.9) and definition of $\{x_n\}$, we have

$$
\begin{aligned}
&\parallel x_{n+1} - p \parallel^2 \\
&\le \alpha_n^2 \parallel \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - FTP_C(I-\beta_{2,n}A_2)y_n \parallel^2 + \parallel y_n - p \parallel^2 \\
&\quad + 2\alpha_n\langle \gamma f(P_C(I-\beta_{1,n}A_1)y_n) \\
&\quad - FTP_C(I-\beta_{2,n}A_2)y_n, TP_C(I-\beta_{2,n}A_2)y_n - p\rangle \\
&\le \alpha_n^2 \parallel \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - FTP_C(I-\beta_{2,n}A_2)y_n \parallel^2 \\
&\quad + \parallel x_n - p \parallel^2 + \beta_n[- \parallel x_n - v_{1,n} \parallel^2 + 2\beta_{1,n}\langle x_n - v_{1,n}, A_1x_n - A_1p\rangle \\
&\quad - \beta_{1,n}^2 \parallel A_1x_n - A_1p \parallel^2] + (1-\beta_n)[- \parallel x_n - v_{2,n} \parallel^2 \\
&\quad + 2\beta_{2,n}\langle x_n - v_{2,n}, A_2x_n - A_2p\rangle - \beta_{2,n}^2 \parallel A_2x_n - A_2p \parallel^2] \\
&\quad + 2\alpha_n\langle \gamma f(P_C(I-\beta_{1,n}A_1)y_n) \\
&\quad - FTP_C(I-\beta_{2,n}A_2)y_n, TP_C(I-\beta_{2,n}A_2)y_n - p\rangle
\end{aligned}
$$

Which implies that

$$
\begin{aligned}
&\beta_n \parallel x_n - v_{1,n} \parallel^2 \\
&\le \alpha_n^2 \parallel \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - FTP_C(I-\beta_{2,n}A_2)y_n \parallel^2 \\
&\quad + [\parallel x_n - p \parallel + \parallel x_{n+1} - p \parallel] \parallel x_{n+1} - x_n \parallel \\
&\quad + \beta_n[2\beta_{1,n}\langle x_n - v_{1,n}, A_2x_n - A_2p\rangle \\
&\quad - \beta_{1,n}^2 \parallel A_1x_n - A_1p \parallel^2] + (1-\beta_n)[- \parallel x_n - v_{2,n} \parallel^2 \\
&\quad + 2\beta_{2,n}\langle x_n - v_{2,n}, A_2x_n - A_2p\rangle - \beta_{2,n}^2 \parallel A_2x_n - A_2p \parallel^2] \\
&\quad + 2\alpha_n\langle \gamma f(P_C(I-\beta_{1,n}A_1)y_n)
\end{aligned}
$$

$$- FTP_C(I - \beta_{2,n}A_2)y_n, TP_C(I - \beta_{2,n}A_2)y_n - p\rangle$$

and

$$\begin{aligned}
&(1-\beta_n) \parallel x_n - v_{2,n} \parallel^2 \\
&\quad \leq \alpha_n^2 \parallel \gamma f(P_C(I - \beta_{1,n}A_1)y_n) - FTP_C(I - \beta_{2,n}A_2)y_n \parallel^2 \\
&\qquad + [\parallel x_n - p \parallel + \parallel x_{n+1} - p \parallel] \parallel x_{n+1} - x_n \parallel + \beta_n[- \parallel x_n - v_{1,n} \parallel^2 \\
&\qquad + 2\beta_{1,n}\langle x_n - v_{1,n}, A_1x_n - A_1p\rangle - \beta_{1,n}^2 \parallel A_1x_n - A_1p \parallel^2] \\
&\qquad + (1-\beta_n)[2\beta_{2,n}\langle x_n - v_{2,n}, A_2x_n - A_2p\rangle - \beta_{2,n}^2 \parallel A_2x_n - A_2p \parallel^2] \\
&\qquad + 2\alpha_n\langle \gamma f(P_C(I - \beta_{1,n}A_1)y_n) \\
&\qquad - FTP_C(I - \beta_{2,n}A_2)y_n, TP_C(I - \beta_{2,n}A_2)y_n - p\rangle
\end{aligned}$$

Therefore using condition B_2, (3.3) and (3.7), we get

$$\lim_{n\to\infty} \parallel x_n - v_{i,n} \parallel = 0 \qquad i = 1, 2.$$

We now show that

$$\lim_{n\to\infty} \parallel x_n - Tx_n \parallel = 0. \tag{3.10}$$

Since T is nonexpansive, we get

$$\begin{aligned}
&\parallel x_n - Tx_n \parallel \\
&\quad \leq \parallel x_n - x_{n+1} \parallel + \parallel x_{n+1} - TP_C(I - \beta_{2,n-1}A_2)y_{n-1} \parallel \\
&\qquad + \parallel TP_C(I - \beta_{2,n-1}A_2)y_{n-1} - Tx_n \parallel \\
&\quad \leq \parallel x_n - x_{n+1} \parallel + \alpha_n \parallel \gamma f(P_C(I - \beta_{1,n}A_1)y_n) - FTP_C(I - \beta_{2,n}A_2)y_n \parallel \\
&\qquad + \parallel TP_C(I - \beta_{2,n}A_2)y_n - Tx_n \parallel \\
&\quad = \parallel x_n - x_{n+1} \parallel + \alpha_n \parallel \gamma f(P_C(I - \beta_{1,n}A_1)y_n) - FTP_C(I - \beta_{2,n}A_2)y_n \parallel \\
&\quad = \parallel v_{2,n} - x_n \parallel
\end{aligned} \tag{3.11}$$

Since $\{\alpha_n\}$ satisfies in B_2. From (3.3), (3.5) and (3.11), we get (3.10).

Next, let us show that, there exists a unique $x^* \in \mathcal{F}$ such that

$$\limsup_{n\to\infty} \langle (F - \gamma f)x^*, x^* - x_n\rangle \leq 0, \tag{3.12}$$

Let $Q = P_{\mathcal{F}}$. Then $Q(I - F + \gamma f)$ is a contraction of H into itself. In fact, we see that

$$\begin{aligned}
&\| Q(I - F + \gamma f)x - Q(I - F + \gamma f)y \| \\
&\quad \leq \| (I - F + \gamma f)x - (I - F + \gamma f)y \| \\
&\quad \leq \| (I - F)x - (I - F)y \| + \gamma \| f(x) - f(y) \| \\
&\quad = \lim_{n\to\infty} \| (I - (1 - \frac{1}{n})F)x - (I - (1 - \frac{1}{n})F)y \| + \gamma \| f(x) - f(y) \| \\
&\quad \leq \lim_{n\to\infty} (1 - (1 - \frac{1}{n})\tau) \| x - y \| + \gamma\alpha \| x - y \| \\
&\quad = (1 - \tau) \| x - y \| + \gamma\alpha \| x - y \|,
\end{aligned}$$

and hence $Q(I - F + \gamma f)$ is a contraction due to $(1 - (\tau - \gamma\alpha)) \in (0, 1)$. Therefore, by Banach's contraction principal, $P_{\mathcal{F}}(I - F + \gamma f)$ has a unique fixed point x^*. Then using Lemma (2.3), x^* is the unique solution of the variational inequality:

$$(3.13) \qquad \langle (\gamma f - F)x^*, x - x^* \rangle \leq 0, \qquad \forall x \in \mathcal{F}.$$

We can choose a a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

(3.14)
$$\limsup_{n\to\infty} \langle \gamma f(x^*) - \mu F x^*, x_n - x^* \rangle = \lim_{j\to\infty} \langle \gamma f(x^*) - \mu F x^*, x_{n_j} - x^* \rangle.$$

Because $\{x_{n_j}\}$ is bounded, therefore $\{x_{n_j}\}$ has subsequence $\{x_{n_{j_k}}\}$ such that $x_{n_{j_k}} \rightharpoonup z$. With no loss of generality, we may assume that $x_{n_j} \rightharpoonup z$. it follows from (3.10) and Lemma 2.2 that $z \in Fix(T)$.

Now, let us show that $z \in VI(C, A_1) \cap VI(C, A_2)$. Let for $i = 1, 2$, $U_i \colon H \to 2^H$ be a set-valued mapping is defined by

$$U_i x = \begin{cases} A_i x + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

where $N_C x$ is the normal cone to C at $x \in C$. Since A_i is monotone. Thus U is maximal monotone see [14]. Let $(x, y) \in G(U_i)$, hence $y - A_i x \in N_C x$ and since $v_{i,n} = P_C(I - \beta_{i,n} A_i)x_n$ therefore, $\langle x - v_{i,n}, y - A_i x \rangle \geq 0$. On the other hand from $v_{i,n} = P_C(x_n - \beta_{i,n} A_i x_n)$, we have

$$\langle x - v_{i,n}, v_{i,n} - (x_n - \beta_{i,n} A_i x_n) \rangle \geq 0,$$

that is

$$\langle x - v_{i,n}, \frac{v_{i,n} - x_n}{\beta_{i,n}} + A_i x_n) \rangle \geq 0$$

Therefore, we have

$$\begin{aligned}
&\langle x - v_{i,n_j}, y\rangle \\
&\quad \geq \langle x - v_{i,n_j}, A_i x\rangle \\
&\quad \geq \langle x - v_{i,n_j}, A_i x\rangle - \langle x - v_{i,n_j}, \frac{v_{i,n_j} - x_{n_j}}{\beta_{i,n_j}} + A_i x_{n_j}\rangle \\
&\quad = \langle x - v_{i,n_i}, A_i x - \frac{v_{i,n_j} - xz_{n_j}}{\beta_{i,n_j}} - A_i x_{n_j}\rangle \\
&\quad = \langle x - v_{i,n_j}, A_i x - A_i v_{i,n_j}\rangle + \langle x - v_{i,n_j}, A_i v_{i,n_j} - A_i x_{n_j}\rangle \\
&\qquad - \langle x - v_{i,n_j}, \frac{v_{i,n_j} - x_{n_j}}{\beta_{i,n_j}}\rangle \\
&\quad \geq \langle x - v_{i,n_j}, A_i v_{i,n_j} - A_i x_{n_j}\rangle - \langle x - v_{i,n_j}, \frac{v_{i,n_j} - x_{n_j}}{\beta_{i,n_j}}\rangle \\
&\quad \geq \langle x - v_{i,n_j}, A_i v_{i,n_j} - A_i x_{n_j}\rangle - \| x - v_{i,n_j} \| \| \frac{v_{i,n_j} - x_{n_j}}{\beta_{i,n_j}} \| .
\end{aligned}$$

Noting that $\lim_{i\to\infty} \| v_{i,n_j} - x_{n_j} \| = 0$, $x_{n_j} \rightharpoonup z$ and A_i is $\frac{1}{\delta_i}-$ Lipschitzian, we obtain

$$\langle x - z, y\rangle \geq 0.$$

Since U is maximal monotone, we have $z \in U^{-1}0$, and hence for $= 1, 2$, $z \in VI(C, A_i)$. Therefore $z \in \mathcal{F}$ and applying (3.13) and (3.14), we have

$$\limsup_{n\to\infty} \langle (\gamma f - F)x^*, x_n - x^*\rangle \leq 0.$$

Finally, we prove that $x_n \to x^*$ as $n \to \infty$. Takeing $\tau = 1 - \sqrt{\frac{2-2\delta}{1-\lambda}}$ and using (2.4), (3.2) and Lemma 2.5, we have

$$\begin{aligned}
&\| x_{n+1} - x^* \|^2 \\
&\quad = \| \alpha_n \gamma f(P_C(I - \beta_{1,n}A_1)y_n) + (I - \alpha_n F)TP_C(I - \beta_{2,n}A_2)y_n - x^* \|^2 \\
&\quad = \| \alpha_n[\gamma f((P_C(I - \beta_{1,n}A_1)y_n)) - Fx^*] + [(I - \alpha_n F)TP_C(I - \beta_{2,n}A_2)y_n \\
&\qquad - (I - \alpha_n F)x^*] \|^2 \\
&\quad \leq \| (I - \alpha_n F)TP_C(I - \beta_{2,n}A_2)y_n - (I - \alpha_n F)x^* \|^2 \\
&\qquad + 2\alpha_n \langle \gamma f(P_C(I - \beta_{1,n}A_1)y_n) - Fx^*, x_{n+1} - x^*\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\alpha_n\tau)^2 \parallel y_n - x^* \parallel^2 \\
&\quad + 2\alpha_n\langle \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - Fx^*, x_{n+1}-x^*\rangle \\
&\leq (1-\alpha_n\tau)^2 \parallel y_n - x^* \parallel^2 \\
&\quad + 2\alpha_n\langle \gamma f(P_C(I-\beta_{1,n}A_1)y_n) - \gamma f(x^*), x_{n+1}-x^*\rangle \\
&\quad + 2\alpha_n\langle \gamma f(x^*) - Fx^*, x_{n+1}-x^*\rangle. \\
&\leq (1-\alpha_n\tau)^2 \parallel y_n - x^* \parallel^2 + \alpha_n\gamma\alpha[\parallel y_n - x^* \parallel^2 + \parallel x_{n+1} - x^* \parallel^2] \\
&\quad + 2\alpha_n\langle \gamma f(x^*) - \mu Fx^*, x_{n+1}-x^*\rangle. \\
&\leq (1-\alpha_n\tau)^2 \parallel x_n - x^* \parallel^2 + \alpha_n\gamma\alpha[\parallel x_n - x^* \parallel^2 + \parallel x_{n+1} - x^* \parallel^2] \\
&\quad + 2\alpha_n\langle \gamma f(x^*) - \mu Fx^*, x_{n+1}-x^*\rangle.
\end{aligned}$$

So we reach the following

$$\begin{aligned}
&\|x_{n+1} - x^* \parallel^2 \\
&\quad \leq \frac{1+\alpha^2\tau^2 - 2\alpha_n\tau + \alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha} \parallel x_n - x^* \parallel^2 \\
&\qquad + \frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle \gamma f(x^*) - Fx^*, x_{n+1}-x^*\rangle \\
&\quad \leq (1-\alpha_n\frac{2(\tau-\gamma\alpha)-\alpha_n\tau^2}{1-\alpha_n\gamma\alpha}) \parallel x_n - x^* \parallel^2 \\
&\qquad + \alpha_n\frac{2(\tau-\gamma\alpha)-\alpha_n\tau^2}{1-\alpha_n\gamma\alpha}\frac{2}{2(\tau-\gamma\alpha)-\alpha_n\tau^2}\langle \gamma f(x^*) - Fx^*, x_{n+1}-x^*\rangle
\end{aligned}$$

It follows that

$$\parallel x_{n+1} - x^* \parallel^2 \leq (1-b_n) \parallel x_n - x^* \parallel^2 + b_n c_n, \tag{3.15}$$

where

$$b_n = \alpha_n\frac{2(\tau-\gamma\alpha)-\alpha_n\tau^2}{1-\alpha_n\gamma\alpha}$$

and

$$c_n = \frac{2}{2(\tau-\gamma\alpha)-\alpha_n\tau^2}\langle \gamma f(x^*) - Fx^*, x_{n+1}-x^*\rangle$$

Since $\{\alpha_n\}$ satisfies in condition B_3, we have $\sum_{n=0}^{\infty} b_n = \infty$ and by condition B_1 and (3.12), we get $\limsup_{n\to\infty} c_n \leq 0$. Consequently, applying Lemma 2.1, to (3.15), we conclude that $x_n \to x^*$. Since $\parallel y_n - x^* \parallel \leq \parallel x_n - x^* \parallel$, we have $y_n \to x^*$.

Corollary 3.2. *(See H. K. Xu [16]) Let $T: C \to H$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$ and $f: C \to C$ be a contraction with coefficient $\alpha \in (0,1)$. Let $\{x_n\}$ be generated by the following algorithm:*

$$x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n f(x_n), \qquad n \geq 0.$$

Assume the sequence $\{\alpha_n\}$ satisfies conditions (B_2). Then, $\{x_n\}$ converges strongly, as $n \to \infty$, to $x^* \in Fix(T)$ which is the solution of the variational inequality:

$$\langle (I-f)x^*, x - x^* \rangle \geq 0, \qquad \forall x \in Fix(T).$$

Proof. It suffices to take $A_i = 0$, for $i = 1, 2$, $F = I$ and $\gamma = 1$ in Theorem 3.1.

Corollary 3.3. *(See B. Halpern [6]) Let $T: C \to H$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$ and $f: C \to C$ be a contraction with coefficient $\alpha \in (0,1)$. Let $\{x_n\}$ be generated by the following algorithm:*

$$x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n u, \qquad n \geq 0,$$

where $u \in H$ is arbitrary (but fixed) and the sequence $\{\alpha_n\}$ satisfies conditions (B_2). Then, $\{x_n\}$ converges strongly, as $n \to \infty$, to $x^* \in Fix(T)$ which is the solution of the variational inequality:

$$\langle (I-f)x^*, x - x^* \rangle \leq 0, \qquad \forall x \in Fix(T).$$

Proof. It suffices to take $A_i = 0$ for $i = 1, 2$, $F = I$ and $f = \frac{1}{\gamma}u$ in Theorem 3.1.

Corollary 3.4. *Let λ be a positive real number such that $\lambda < 1$. Suppose A be a strongly positive linear operator on H with coefficient $\overline{\gamma}$ such that $\overline{\gamma} > \frac{1+\lambda}{2}$ and $\| A \| \leq 1$. Let $0 < \zeta < \frac{1-\sqrt{\frac{2-2\overline{\gamma}}{1-\lambda}}}{\alpha}$ and $\{x_n\}$ be generated by the following algorithm:*

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \zeta f(x_n), \qquad n \geq 0.$$

Assume the sequence $\{\alpha_n\}$ satisfies conditions (B_2). Then, $\{x_n\}$ converges strongly, as $n \to \infty$, to x^* which is the solution of the variational inequality:

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \qquad \forall x \in Fix(T).$$

Proof. Because A is $\overline{\gamma}-$ strongly monotone and $\lambda-$strictly pseudo-contractive of Browder-Petryshyn type such that $\overline{\gamma} > (1+\lambda)/2$, by taking $A_i = 0$, for $i = 1, 2$, in Theorem 3.1 the proof is complete.

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