Bulletin of the Iranian Mathematical Society Vol. 39 *No. 4* (*2013*), *pp 779-788.*

MORE ABOUT MEASURES AND JACOBIANS OF SINGULAR RANDOM MATRICES

J. A. DÍAZ-GARCÍA

Communicated by Teo Mora

Abstract. In this work are studied the Jacobians of certain singular transformations and the corresponding measures which support the jacobian computations.

1. **Introduction**

First consider the following notation: Let $\mathcal{L}_{m,N}(q)$ be the space of all $N \times m$ real matrices of rank $q \leq \min(N, m)$ and $\mathcal{L}^{+}_{m,N}(q)$ be the space of all $N \times m$ real matrices of rank $q \leq \min(N, m)$, with q distinct singular values. The set of matrices $\mathbf{H}_1 \in \mathcal{L}_{m,N}(m)$ such that $\mathbf{H}'_1\mathbf{H}_1 = \mathbf{I}_m$ is a manifold denoted by $\mathcal{V}_{m,N}$, called Stiefel manifold. In particular, $\mathcal{V}_{m,m}$ is the group of orthogonal matrices $\mathcal{O}(m)$. Denote by $\mathcal{S}_m$, the homogeneous space of $m \times m$ positive definite symmetric matrices; and by $\mathcal{S}^{+}_m(q)$, the $(mq - q(q-1)/2)$-dimensional manifold of rank q positive semidefinite $m \times m$ symmetric matrices with q distinct positive eigenvalues.

Assuming that $\mathbf{X} \in \mathcal{L}^{+}_{m,N}(q)$, [4] proposed the Jacobian of nonsingular part of the singular value decomposition, $\mathbf{X} = \mathbf{H}_1\mathbf{D}\mathbf{W}'_1$, where $\mathbf{H}_1 \in \mathcal{V}_{q,N}$, $\mathbf{D}$ is a diagonal matrix with $D_1 > D_2 > \cdots D_q > 0$ and $\mathbf{W}_1 \in \mathcal{V}_{q,m}$. Also, note that the Jacobian itself defines the factorization of Hausdorff's measure $(d\mathbf{X})$ (or Lebesgue's measure defined on the

MSC(2010): Primary: 14H40; Secondary: 15A09, 15A23.
Keywords: Singular random matrices; Jacobian of transformation; Hausdorff measure; Lebesgue measure; multiplicity.
Received: 20 February 2012, Accepted: 8 June 2012.

manifold $\mathcal{L}^{+}_{m,N}(q)$, see [1, p. 249]). Analogous results for $\mathbf{V} \in \mathcal{S}^{+}_{m}(q)$ considering the nonsingular part of the spectral decomposition of $\mathbf{V}$ were proposed by [14] and [5]. Based on these two results, [8] and [9] computed the Jacobians of the transformations $\mathbf{Y} = \mathbf{X}^{+}$ and $\mathbf{W} = \mathbf{V}^{+}$ (see also [11]), where $\mathbf{A}^{+}$ denotes the Moore-Penrose inverse of $\mathbf{A}$, see [12, p.49].

In the present work, assuming that $\mathbf{X} \in \mathcal{L}^{+}_{m,N}(q)$, we propose the Jacobian of nonsingular part of the singular value decomposition assuming multiplicity in the singular values of $\mathbf{X}$ and the corresponding Jacobian of $\mathbf{Y} = \mathbf{X}^{+}$ under the same conditions. Analogous results for $\mathbf{V} \in \mathcal{S}^{+}_{m}(q)$ and $\mathbf{W} = \mathbf{V}^{+}$ considering the nonsingular part of the spectral decomposition of $\mathbf{V}$ are proposed assuming multiplicity in the eigenvalues of $\mathbf{V}$ and/or assuming that $\mathbf{V}$ is an indefinite singular matrix. Also we will determine the explicit measures with respect the Jacobians are we give computation.

2. Jacobian of symmetric matrices

Consider again $\mathbf{A} \in \mathcal{S}_m$, it remains to study: $\mathbf{A}$ as a (nonsingular) indefinite matrix, i.e., $\mathbf{A} \in \mathcal{S}^{\pm}_{m}(m_1, m_2)$, with $m_1 + m_2 = m$, where m_1 is the number of positive eigenvalues and m_2 is the number of negative eigenvalues; and $\mathbf{A}$ as a (singular) semi-indefinite matrix, i.e. $\mathbf{A} \in \mathcal{S}^{\pm}_{m}(q, q_1, q_2)$, with $q_1 + q_2 = q$, here q_1 is the number of positive eigenvalues and q_2 is the number of negative eigenvalues.

First suppose $\mathbf{A} \in \mathcal{S}^{\pm}_{m}(m_1, m_2)$ such that $\mathbf{A} = \mathbf{HDH}'$, is the spectral decomposition (SD) of $\mathbf{A}$, where $\mathbf{H} \in \mathcal{O}(m)$, $\mathbf{D}$ is a diagonal matrix. Without loss of generality, let $\lambda_1 > \cdots > \lambda_{m_1} > 0$ and $0 > -\delta_1 > \cdots > -\delta_{m_2}$, explicitly

$$\mathbf{A} = \mathbf{H} \begin{bmatrix} \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{m_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\delta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & -\delta_{m_2} \end{bmatrix} \mathbf{H}'.$$

Now let $\mathbf{A} \in \mathcal{S}^{\pm}_{3}(1, 2)$ and let $\mathbf{A} = \mathbf{HDH}'$ be its SD, then

$$d\mathbf{A} = d\mathbf{H}\mathbf{D}\mathbf{H}' + \mathbf{H}d\mathbf{D}\mathbf{H}' + \mathbf{H}\mathbf{D}d\mathbf{H}',$$

thus by the skew symmetry of $\mathbf{H}'d\mathbf{H}$ we have, see [10, p. 105]

$$\mathbf{H}'d\mathbf{A}\mathbf{H} = \mathbf{H}'d\mathbf{H}\mathbf{D} + d\mathbf{D} + \mathbf{D}d\mathbf{H}'\mathbf{H} = \mathbf{H}'d\mathbf{H}\mathbf{D} + d\mathbf{D} - \mathbf{D}\mathbf{H}'d\mathbf{H}$$

Moreover,

$$\begin{aligned}
\mathbf{H}'d\mathbf{A}\mathbf{H} &= \begin{bmatrix} 0 & -h_2'dh_1 & -h_3'dh_1 \\ h_2'dh_1 & 0 & -h_3'dh_2 \\ h_3'dh_1 & h_3'dh_2 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & -\delta_1 & 0 \\ 0 & 0 & -\delta_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} d\lambda_1 & 0 & 0 \\ 0 & -d\delta_1 & 0 \\ 0 & 0 & -d\delta_2 \end{bmatrix} \\
&\quad - \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & -\delta_1 & 0 \\ 0 & 0 & -\delta_2 \end{bmatrix} \begin{bmatrix} 0 & -h_2'dh_1 & -h_3'dh_1 \\ h_2'dh_1 & 0 & -h_3'dh_2 \\ h_3'dh_1 & h_3'dh_2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \delta_1 h_2'dh_1 & \delta_2 h_3'dh_1 \\ \lambda_1 h_2'dh_1 & 0 & \delta_2 h_3'dh_2 \\ \lambda_1 h_3'dh_1 & -\delta_1 h_3'dh_2 & 0 \end{bmatrix} + \begin{bmatrix} d\lambda_1 & 0 & 0 \\ 0 & -d\delta_1 & 0 \\ 0 & 0 & -d\delta_2 \end{bmatrix} \\
&\qquad - \begin{bmatrix} 0 & -\lambda_1 h_2'dh_1 & -\lambda_1 h_3'dh_1 \\ -\delta_1 h_2'dh_1 & 0 & \delta_1 h_3'dh_2 \\ -\delta_2 h_3'dh_1 & -\delta_2 h_3'dh_2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} d\lambda_1 & (\lambda_1+\delta_1)h_2'dh_1 & (\lambda_1+\delta_2)h_3'dh_1 \\ (\lambda_1+\delta_1)h_2'dh_1 & -d\delta_1 & (\delta_2-\delta_1)h_3'dh_2 \\ (\lambda_1+\delta_2)h_3'dh_1 & (-\delta_1+\delta_2)h_3'dh_2 & -d\delta_2 \end{bmatrix}.
\end{aligned}$$

We know that $(\mathbf{H}'d\mathbf{A}\mathbf{H}) = (d\mathbf{A})$, then a column by column computation of the exterior product of the subdiagonal elements of $\mathbf{H}'d\mathbf{H}\mathbf{D} + d\mathbf{D} - \mathbf{D}\mathbf{H}'d\mathbf{H}$ gives, ignoring the sign,

$$(d\mathbf{A}) = (\lambda_1+\delta_1)(\lambda_1+\delta_2)(-\delta_1+\delta_2)\left(\bigwedge_{i=1}^{3}\bigwedge_{j=i+1}^{3} h_j'dh_i\right)\wedge d\lambda_1\wedge -d\delta_1\wedge -d\delta_2.$$

Recall that, if for example, the first element in each column of $\mathbf{H}$ is nonnegative, so, $\mathbf{A} = \mathbf{H}\mathbf{D}\mathbf{H}'$ is one-to-one transformation. Then the corresponding Jacobian must be divided by 2^m, see [10, pp. 104-105]. Thus we have

$$(d\mathbf{A}) = 2^{-3}(\lambda_1+\delta_1)(\lambda_1+\delta_2)(\delta_1-\delta_2)(\mathbf{H}'d\mathbf{H})\wedge(d\mathbf{D}),$$

where $(\mathbf{H}'d\mathbf{H})$ is the Haar measure on $\mathcal{O}(m)$ and

$$(\mathbf{H}'d\mathbf{H}) = \bigwedge_{i<j}^{m} h_j'dh_i, \quad (d\mathbf{D}) = d\lambda_1\wedge d\delta_1\wedge d\delta_2,$$

$(d\mathbf{D})$ is an exterior product of all differentials $d\lambda_i$ and $d\delta_j$ ignoring the sign.

Analogously, if $\mathbf{A} \in \mathcal{S}_3^{\pm}(2,1)$,

$$(d\mathbf{A}) = 2^{-3}(\lambda_1 - \lambda_2)(\lambda_1 + \delta_1)(\lambda_2 + \delta_1)(\mathbf{H}'d\mathbf{H}) \wedge (d\mathbf{D}).$$

Similarly, let $\mathbf{A} \in \mathcal{S}_4^{\pm}(2,2)$, then

$$(d\mathbf{A}) = 2^{-4}(\lambda_1-\lambda_2)(\delta_1-\delta_2)(\lambda_1+\delta_1)(\lambda_1+\delta_2)(\lambda_2+\delta_1)(\lambda_2+\delta_2)(\mathbf{H}'d\mathbf{H})\wedge(d\mathbf{D}).$$

By generalization we get

Theorem 2.1. *Let $\mathbf{A} \in \mathcal{S}_m^{\pm}(m_1, m_2)$ such that $\mathbf{A} = \mathbf{HDH}'$, where $\mathbf{H} \in \mathcal{O}(m)$, $\mathbf{D}$ is a diagonal matrix with $\lambda_1 > \cdots > \lambda_{m_1} > 0$ and $0 > -\delta_1 > \cdots > -\delta_{m_2}$, $m_1 + m_2 = m$. Then*

$$(d\mathbf{A}) = 2^{-m} \prod_{i<j}^{m_1}(\lambda_i - \lambda_j) \prod_{k<l}^{m_2}(\delta_k - \delta_l) \prod_{i,k}^{m_1,m_2} (\lambda_i + \delta_k)(\mathbf{H}'d\mathbf{H}) \wedge (d\mathbf{D}).$$

where

$$\prod_{i,k}^{m_1,m_2} (\lambda_i + \delta_k) = \prod_{i=1}^{m_1}\prod_{k=1}^{m_2}(\lambda_i + \delta_k), \quad (\mathbf{H}'d\mathbf{H}) = \bigwedge_{i<j}^{m} h_j' dh_i,$$

$$(d\mathbf{D}) = \bigwedge_{i=1}^{m_1} d\lambda_i \bigwedge_{k=1}^{m_2} d\delta_k.$$

A similar procedure for $\mathbf{A} \in \mathcal{S}_m^{\pm}(q, q_1, q_2)$ gives:

Theorem 2.2. *Let $\mathbf{A} \in \mathcal{S}_m^{\pm}(q, q_1, q_2)$ such that $\mathbf{A} = \mathbf{H}_1\mathbf{D}\mathbf{H}_1'$, where $\mathbf{H}_1 \in \mathcal{V}_{q,m}$, $\mathbf{D}$ is a diagonal matrix with $\lambda_1 > \cdots > \lambda_{q_1} > 0$ and $0 > -\delta_1 > \cdots > -\delta_{q_2}$, $q_1 + q_2 = q$. Then*

$$(d\mathbf{A}) = 2^{-q} \prod_{i=1}^{q_1} \lambda_i^{m-q} \prod_{k=1}^{q_2} \delta_k^{m-q} \prod_{i<j}^{q_1}(\lambda_i - \lambda_j) \prod_{k<l}^{q_2}(\delta_k - \delta_l) \prod_{i,k}^{q_1,q_2} (\lambda_i + \delta_k)$$
$$\wedge(\mathbf{H}_1'd\mathbf{H}_1)\wedge(d\mathbf{D}),$$

where

$$\prod_{i,k}^{q_1,q_2} (\lambda_i + \delta_k) = \prod_{i=1}^{q_1}\prod_{k=1}^{q_2}(\lambda_i + \delta_k), \quad (\mathbf{H}_1'd\mathbf{H}_1) = \bigwedge_{i=1}^{m} \bigwedge_{j=i+1}^{q} h_j' dh_i,$$

$$(d\mathbf{D}) = \bigwedge_{i=1}^{q_1} d\lambda_i \bigwedge_{k=1}^{q_2} d\delta_k.$$

3. Jacobians of symmetric matrices repeated eigenvalues

As a motivation for this section, consider a general random matrix $\mathbf{A} \in \Re^{m\times m}$, explicitly

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix}.$$

Any density function of this matrix can be expressed as

$$dF_{\mathbf{A}}(\mathbf{A}) = f_{\mathbf{A}}(\mathbf{A})(d\mathbf{A}),$$

where $(d\mathbf{A})$ denotes the Lebesgue measure in $\Re^{m^2}$, which can be written by using the exterior product, as

$$(d\mathbf{A}) = \bigwedge_{i=1}^{m}\bigwedge_{j=1}^{m} da_{ij},$$

see [10].

However, if $\mathbf{A} \in \mathcal{S}_m$ and it is nonsingular, then the Lebesgue measure defined on $\mathcal{S}_m$ is given by

$$(d\mathbf{A}) = \bigwedge_{i\leq j}^{m} da_{ij}.$$

Remark 3.1. *Note that the above product is the Hausdorff measure on $\Re^{m^2}$ defined on the homogeneous space of positive definite symmetric matrices, see* [1].

In general, we can consider any factorization of the Lebesgue measure $(d\mathbf{A})$ on $\mathcal{S}_m^+$ as an alternative definition of $(d\mathbf{A})$ with respect to the corresponding coordinate system. For example, if we consider the spectral decomposition (SD), $\mathbf{A} = \mathbf{HDH}'$, where $\mathbf{H} \in \mathcal{O}(m)$, and $\mathbf{D}$ is a diagonal matrix with $D_1 > \cdots > D_m > 0$ or we consider the Cholesky decomposition $\mathbf{A} = \mathbf{T}'\mathbf{T}$, where $\mathbf{T}$ is upper-triangular with positive diagonal entries, then we have respectively

$$(d\mathbf{A}) = \begin{cases} 2^{-m}\displaystyle\prod_{i<j}^{m}(D_i - D_j)(\mathbf{H}'d\mathbf{H}) \wedge (d\mathbf{D}), & \text{Spectral decomposition;} \\ 2^{m}\displaystyle\prod_{i=1}^{m} t_{ii}^{m+1-i}(d\mathbf{T}), & \text{Cholesky decomposition,} \end{cases}$$

see [6], where

$$(\mathbf{H}'d\mathbf{H}) = \bigwedge_{i<j}^{m} h_j' dh_i, \quad (d\mathbf{D}) = \bigwedge_{i=1}^{m} dD_i \text{ and } (d\mathbf{T}) = \bigwedge_{i\leq j}^{m} dt_{ij}.$$

Given a matrix $\mathbf{A} \in \mathcal{S}_m^+(q)$, sometimes it is difficult to establish an explicit form of the Hausdorff measure in the original coordinate system, that is, to define $(d\mathbf{A})$ in terms of da_{ij}. In particular, if $\mathbf{A} \in \mathcal{S}_m^+(q)$ some unsuccessful efforts have been trailed, see [13] and [2]. A definition of such measure in terms of the SD is given by [14]:

$$(d\mathbf{A}) = 2^{-q} \prod_{i=1}^{q} D_i^{m-q} \prod_{i<j}^{q} (D_i - D_j)(\mathbf{H}_1 d\mathbf{H}_1) \wedge (d\mathbf{D}),$$

where $\mathbf{H}_1 \in \mathcal{V}_{q,m}$, $\mathbf{D}$ is a diagonal matrix with $D_1 > \cdots > D_q > 0$ and

$$(\mathbf{H}_1' d\mathbf{H}_1) = \bigwedge_{i=1}^{m} \bigwedge_{j=i+1}^{q} h_j' dh_i, \quad (d\mathbf{D}) = \bigwedge_{i=1}^{q} dD_i;$$

for alternative expressions of $(d\mathbf{A})$ in terms of other factorizations see [6] and [7].

Now suppose that one (or more) eigenvalue(s) of $\mathbf{A} \in \mathcal{S}_m$ has (have) multiplicity greater than one. Then consider $\mathbf{A} = \mathbf{HDH}'$, where $\mathbf{H} \in \mathcal{O}(m)$, $\mathbf{D}$ is a diagonal matrix with $D_1 \geq \cdots \geq D_m > 0$. Moreover, let $D_{k_1}, \ldots D_{k_l}$ be the l distinct eigenvalues of $\mathbf{A}$, i.e., $D_{k_1} > \cdots > D_{k_l} > 0$, where m_j denotes the repetitions of the eigenvalue D_{k_j}, $j = 1, 2, \ldots, l$, and of course $m_1 + \cdots m_l = m$; finally denote the corresponding set of matrices by $\mathbf{A} \in \mathcal{S}_m^+(m, l)$. It is clear that $\mathbf{A}$ exists in the $m(m+1)/2$-dimensional homogeneous subspace of the symmetric matrices of rank m; more accurately, when there exist multiplicity in the eigenvalues, $\mathbf{A}$ exists in the manifold of dimension $ml - l(l-1)/2$, even exactly for computations we say that $\mathbf{A} \in \mathcal{S}_m^+(l)$. For proving it, consider the matrix $\mathbf{A} \in \mathcal{S}_2^+(2, 1)$, such that $\mathbf{A} = \mathbf{HDH}'$, here $\mathbf{H} \in \mathcal{O}(2)$, and $\mathbf{D}$ is a diagonal matrix with $D_1 \geq D_2 > 0$ where $D_1 = D_2 = \kappa$, then the measure

$$(d\mathbf{A}) = 2^{-2} \prod_{i<j}^{2} (D_i - D_j)(\mathbf{H}'d\mathbf{H}) \wedge d\mathbf{D} = 2^{-2}(\kappa - \kappa)(\mathbf{H}'d\mathbf{H}) \wedge d\mathbf{D} = 0.$$

Also note that, in fact the measure $(d\mathbf{D}) = dD_1 \wedge dD_2 = d\kappa \wedge d\kappa$ is zero. This is analogous to the following situation, to propose for a curve in the space $(\Re^3)$ the Lebesgue measure defined by $dx_1 \wedge dx_2$.

Now, when we consider the factorization of the Lebesgue measure in terms of the spectral decomposition, we do not have $2(2+1)/2 = 3$ but only $2(1)-1(1+1)/2+1 = 2$ mathematically independent elements in $\mathbf{A}$, because in $\mathbf{D}$, $D_1 = D_2 = \kappa$ and then there is only one mathematically independent element.

Also, observe that the space of positive definite $m \times m$ matrices is a subset of the Euclidian space of symmetric $m \times m$ matrices of dimension $m(m+1)/2$, and in fact it forms an open cone described by the following system of inequalities, see [10, p. 61 and p. 77 Problem 2.6]:

$$\mathbf{A} > 0 \Leftrightarrow a_{11} > 0, \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > 0, \cdots, \det(\mathbf{A}) > 0. \tag{3.1}$$

In particular, let $m = 2$, after factorizing the Lebesgue measure in $\mathcal{S}_m$ by the spectral decomposition, the inequalities (3.1) are as follows

$$\mathbf{A} > 0 \Leftrightarrow D_1 > 0, D_2 > 0, D_1 D_2 > 0. \tag{3.2}$$

But if $D_1 = D_2 = \kappa$, (3.2) it reduces to

$$\mathbf{A} > 0 \Leftrightarrow \kappa > 0, \kappa^2 > 0.$$

Which defines a curve (a parabola) in the space, over the line $D_1 = D_2 (= \kappa)$ in the subspace of points (D_1, D_2).

A similar situation appears in the following cases:

- **i):** When we consider multiplicity of the singular values in the singular value decomposition (SVD); such set of matrices will be denoted by $\mathbf{X} \in \mathcal{L}^+_{m,N}(q,l)$ $q \geq l$;
- **ii):** If we consider multiplicity of the eigenvalues; the corresponding set of matrices will be denoted by $\mathbf{A} \in \mathcal{S}^+_m(q,l)$, $q \geq l$;
- **iii):** And if $\mathbf{A}$ is nonpositive definite.

As a summary we have the next results, which collect the main conclusions of Section 2 and the present section, the proofs are similar to the proof of Theorem 1 in [9]:

Theorem 3.2. *Consider $\mathbf{Y} \in \mathcal{L}^+_{m,N}(q)$ and $\mathbf{Y} = \mathbf{X}^+$, then*

$$(d\mathbf{Y}) = \prod_{i=1}^{k} \sigma_i^{-2(N+m-k)} (d\mathbf{X})$$

where $\mathbf{X} = \mathbf{H}_1 \mathbf{D}_\sigma \mathbf{P}'_1$ is the nonsingular part of SVD of $\mathbf{X}$, with $\mathbf{H}_1 \in \mathcal{V}_{k,N}$, $\mathbf{P}_1 \in \mathcal{V}_{k,m}$, $\mathbf{D}_\sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_k)$, $\sigma_1 \geq \cdots \geq \sigma_k > 0$, the measure

$(d\mathbf{X})$ *is*

$$(d\mathbf{X}) = 2^{-k}\prod_{i=1}^{k}\sigma_i^{(N+m-2k)}\prod_{i<j}^{k}(\sigma_i^2-\sigma_j^2)(\mathbf{H}_1'd\mathbf{H}_1)\wedge(\mathbf{P}_1'd\mathbf{P}_1)\wedge(d\mathbf{D}_\sigma),$$

and

$$k = \begin{cases} q, & \mathbf{X}\in\mathcal{L}_{m,N}^{+}(q); \\ l, & \mathbf{X}\in\mathcal{L}_{m,N}^{+}(q,l). \end{cases}$$

Similarly, for symmetric matrices we have,

Theorem 3.3. *Let* $\mathbf{V}\in\Re^{m\times m}$ *be a symmetric matrix and let* $\mathbf{W}=\mathbf{V}^{+}$*, then*

(1)

$$(d\mathbf{W}) = \prod_{i=1}^{\beta}|\lambda_i|^{-2m+\beta-1}(d\mathbf{V}),$$

where $\mathbf{V}=\mathbf{H}_1\mathbf{D}_\lambda\mathbf{H}_1'$ *is the nonsingular part of SD of* $\mathbf{V}$*, with* $\mathbf{H}_1\in\mathcal{V}_{\beta,N}$*,* $\mathbf{D}_\lambda=\operatorname{diag}(\lambda_1,\dots,\lambda_\beta)$*,* $|\lambda_1|\geq\cdots\geq|\lambda_\beta|>0$*, the measure* $(d\mathbf{V})$ *is*

$$(d\mathbf{V}) = 2^{-\beta}\prod_{i=1}^{\beta}|\lambda_i|^{m-\beta}\prod_{i<j}^{\beta}(|\lambda_i|-|\lambda_j|)(\mathbf{H}_1d\mathbf{H}_1)\wedge(d\mathbf{D}_\lambda),$$

and

$$\beta = \begin{cases} m, & \mathbf{V}\ \textit{or}\ -\mathbf{V}\in\mathcal{S}_m^{+}; \\ l, & \mathbf{V}\ \textit{or}\ -\mathbf{V}\in\mathcal{S}_m^{+}(m,l); \\ q, & \mathbf{V}\ \textit{or}\ -\mathbf{V}\in\mathcal{S}_m^{+}(q); \\ k, & \mathbf{V}\ \textit{or}\ -\mathbf{V}\in\mathcal{S}_m^{+}(q,k). \end{cases}$$

(2)

$$(d\mathbf{W}) = \prod_{i=1}^{\alpha_1}\lambda_i^{-2(m-\alpha_1/2-\alpha_2+1)}\prod_{j=1}^{\alpha_2}\delta_j^{-2(m-(\alpha-1)/2)}(d\mathbf{V}),$$

where $\alpha=\alpha_1+\alpha_2$*,* $\mathbf{V}=\mathbf{H}_1\mathbf{D}\mathbf{H}_1'$ *is the nonsingular part of SD of* $\mathbf{V}$*, with* $\mathbf{H}_1\in\mathcal{V}_{\alpha,N}$*,* $\mathbf{D}=\operatorname{diag}(\lambda_1,\dots,\lambda_{\alpha_1},-\delta_1,\dots,-\delta_{\alpha_1})$*,* $\lambda_1\geq\cdots\geq\lambda_{\alpha_1}>0$*;* $|\delta_1|\geq\cdots\geq|\delta_{\alpha_2}|>0$*, the measure* $(d\mathbf{V})$ *is*

$$(d\mathbf{V}) = 2^{-\alpha}\prod_{i=1}^{\alpha_1}\lambda_i^{m-\alpha}\prod_{r=1}^{\alpha_2}\delta_r^{m-\alpha}\prod_{i<j}^{\alpha_1}(\lambda_i-\lambda_j)\prod_{r<l}^{\alpha_2}(\delta_r-\delta_l)\prod_{i,r}^{\alpha_1,\alpha_2}(\lambda_i+\delta_r)$$

$$\wedge(\mathbf{H}_1'd\mathbf{H}_1)\wedge(d\mathbf{D}).$$

and

$$\alpha = \begin{cases} m, & \mathbf{V} \in \mathcal{S}_m^{\pm}(m_1, m_2); \\ l, & \mathbf{V} \in \mathcal{S}_m^{\pm}(l_1, l_2); \\ q, & \mathbf{V} \in \mathcal{S}_m^{\pm}(q, q_1, q_2); \\ k, & \mathbf{V} \in \mathcal{S}_m^{\pm}(q, k_1, k_2), \end{cases}$$

and $\mathbf{V} \in \mathcal{S}_m^{\pm}(l_1, l_2)$, $l_1 + l_2 = l \leq m$ *denotes a nonsingular indefinite matrix with repeated eigenvalues and* $\mathbf{V} \in \mathcal{S}_m^{\pm}(q, k_1, k_2)$, $k_1 + k_2 = k \leq q \leq m$ *denotes a singular indefinite matrix with repeated eigenvalues.*

4. **Conclusions**

This work determines the Jacobians of the SVD and the SD under multiplicity of the singular values and eigenvalues, respectively. For the SD case, we compute the Jacobian for nonsingular and singular indefinite matrices with and without repeated eigenvalues. Also, we calculate the Jacobians for a general matrix and its Moore-Penrose inverse, and for a symmetric matrix with all its variants (nonpositive, nonnegative and indefinite). In every case we specify the measures of Hausdorff which support the Jacobian computations. We highlight that the results of the paper is the foundations of an explored problem in literature: the test criteria in MANOVA when there exist multiplicities in: the matrix of sum of squares and sum of products, due to the hypothesis $\mathbf{S}_H$; the matrix of sum of square and sum of products, due to the error $\mathbf{S}_E$; the matrices $\mathbf{S}_H\mathbf{S}_E^{-1}$; $(\mathbf{S}_H + \mathbf{S}_E)^{-1}\mathbf{S}_H$; see [3].

Acknowledgments

The research was partially supported by IDI-Spain, grants FQM2006 - 2271 and MTM2008 - 05785, and CONACYT-México, research grant no. 81512. The paper was written during J. A. Díaz- García's stay as a visiting professor at the Department of Statistics and O. R. of the University of Granada, Spain. The author wish to express his gratitude to the editor and referees for their helpful comments and suggestions.

References

[1] P. Billingsley, Probability and Measure, 2nd Edition, John Wiley & Sons, Inc., New York, 1986.

[2] J. A. Díaz-García, A note about measures and Jacobians of singular random matrices, *J. Multivariate Anal.* **98** (2007), no. 5, 960–969.

[3] J. A. Díaz-García, On generalized multivariate analysis of variance, *Braz. J. Probab. Stat.* **25** (2010), no. 1, 1–13.
[4] J. A. Díaz-García, R. Gutiérrez-Jáimez, and K. V. Mardia, Wishart and Pseudo-Wishart distributions and some applications to shape theory, *J. Multivariate Anal.* **63** (1997), no. 1, 73–87.
[5] J. A. Díaz-García, and R. Gutiérrez-Jáimez, Proof of the conjectures of H. Uhlig on the singular multivariate beta and the jacobian of a certain matrix transformation, *Ann. Statist.* **25** (1997), no. 5, 2018–2023.
[6] J. A. Díaz-García, and G. González-Farías, Singular random matrix decompositions: Jacobians, *J. Multivariate Anal.* **93** (2005), no. 2, 196–212.
[7] J. A. Díaz-García, and G. González-Farías, Singular random matrix decompositions: Distributions, *J. Multivariate Anal.* **94** (2005), no. 1, 109–122.
[8] J. A. Díaz-García, and R. Gutiérrez-Jáimez, Functions of singular random matrices with applications, *Test* **14** (2005), no. 2, 475–487.
[9] J. A. Díaz-García, and R. Gutiérrez-Jáimez, Distribution of the generalised inverse of a random matrix and its applications, *J. Statist. Plann. Inference* **136** (2006), no. 1, 183–192.
[10] R. J. Muirhead, Aspects of Multivariate Statistical Theory, John Wiley & Sons, Inc., New York, 1982.
[11] I. Olkin, The density of the inverse and pseudo-inverse of a random matrix, *Statist. Probab. Lett.* **38** (1998), no. 2, 131–135.
[12] C. R. Rao, Linear Statistical Inference and its Applications, 2nd Editon, John Wiley & Sons, Inc., New York-London-Sydney, 1973.
[13] S. M. Srivastava, Singular Wishart and multivariate beta distributions, *Ann. Statist.* **31** (2003), no. 5, 1537–1560.
[14] H. Uhlig, On singular Wishart and singular multivariate beta distributions, *Ann. Statist.* **22** (1994), no. 1, 395–405.

José A. Díaz-García
Department of Statistics and Computation, Universidad Autonoma Agraria Antonio Narro, 25315 Buenavista, Saltillo, Coahuila, Mexico
Email: jadiaz@uaaan.mx; jadiazg@ugr.es