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ON A CLASS OF SYSTEMS OF n NEUMANN TWO-POINT BOUNDARY VALUE STURM-LIOUVILLE TYPE EQUATIONS

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ABSTRACT. Employing a three critical points theorem, we prove the existence of multiple solutions for a class of Neumann two-point boundary value Sturm-Liouville type equations. Using a local minimum theorem for differentiable functionals the existence of at least one non-trivial solution is also ensured.

1. Introduction

Consider the following Neumann two-point boundary value Sturm-Liouville type system

(1.1)
$$\begin{cases} -(p_i(x)u_i'(x))' + r_i(x)u_i'(x) + q_i(x)u_i(x) = \lambda F_{u_i}(x, u_1, ..., u_n) \\ x \in (0,1), \\ u_i'(0) = u_i'(1) = 0 \end{cases}$$

for $1 \leq i \leq n$, where $n \geq 1$ is an integer, $p_i \in C^1([0,1)]$, $r_i, q_i \in C^0([0,1)]$ with p_i and q_i positive functions for $1 \leq i \leq n$, λ is a positive parameter, $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a function such that $F(., t_1, ..., t_n)$ is measurable in

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$[0,1]$ for all $(t_1,...,t_n)\in\mathbb{R}^n$, $F(x,.,...,.)$ is C^1 in $\mathbb{R}^n$ for every $x\in[0,1]$ and for every $\varrho>0$,

$$\sup_{|(t_1,...,t_n)|\le\varrho}\sum_{i=1}^{n}|F_{t_i}(x,t_1,...,t_n)|\in L^1([0,1]),$$

and F_{u_i} denotes the partial derivative of F with respect to u_i for $1\le i\le n$.

Based on a three critical point theorem (Theorem 2.1), we establish the existence of at least three weak solutions for the system (1.1) under suitable assumptions on nonlinear term F. Employing a local minimum theorem for differentiable functionals (Theorem 2.2), under appropriate hypotheses on F, we also ensure the existence of at least one non-trivial solution.

Problems of Sturm-Liouville type have been widely investigated. We refer the reader to the papers [3, 5, 6, 18] and the references therein.

For a thorough account on the subject, we also refer the reader to [7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

In the present paper, our motivation comes from the recent paper [5].

2. Preliminaries and basic notations

Our main tools to prove the results are critical point theorems.

First we recall an immediate consequence of [4, Theorem 3.3](see also [1, Theorem 5.2]).

Let X be a nonempty set and $\Phi,\Psi : X\longrightarrow\mathbb{R}$ be two functions. for all $r,r_1,r_2>\inf_X\Phi$, $r_2>r_1$, $r_3>0$, we define

$$\varphi(r):=\inf_{u\in\Phi^{-1}(]-\infty,r[)}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)-\Psi(u)}{r-\Phi(u)}$$

$$\beta(r_1,r_2):=\inf_{u\in\Phi^{-1}(]-\infty,r_1[)}\sup_{v\in\Phi^{-1}([r_1,r_2[)}\frac{\Psi(v)-\Psi(u)}{\Phi(v)-\Phi(u)},$$

$$\gamma(r_2,r_3):=\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2+r_3[)}\Psi(u)}{r_3}$$

and

$$\alpha(r_1,r_2,r_3):=\max\{\varphi(r_1),\varphi(r_2),\gamma(r_2,r_3)\}.$$

Theorem 2.1. [4, Theorem 3.3] *Let X be a reflexive real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

1. $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;
2. *for every u^*, u^{**} such that $\Psi(u^*) \geq 0$ and $\Psi(u^{**}) \geq 0$, one has*

$$\inf_{s\in[0,1]} \Psi(su^* + (1-s)u^{**}) \geq 0.$$

Assume that there are three positive constants r_1, r_2, r_3 with $r_1 < r_2$, such that

$$\alpha(r_1, r_2, r_3) < \beta(r_1, r_2).$$

Then, for each $\lambda \in \left]\frac{1}{\beta(r_1,r_2)}, \frac{1}{\alpha(r_1,r_2,r_3)}\right[$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points u, v, w such that $u \in \Phi^{-1}(]-\infty, r_1[)$, $v \in \Phi^{-1}([r_1, r_2[)$ and $w \in \Phi^{-1}(]-\infty, r_2 + r_3[)$.

Next, we recall a local minimum theorem for differentiable functionals, Theorem 2.5 of [17] as given in [2, Theorem 5.1](see also [2, Proposition 2.1] for related results).

For a given non-empty set X, and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the following functions

$$\delta(r_1, r_2) = \inf_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}$$

and

$$\rho(r_1, r_2) = \sup_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$.

Theorem 2.2. [2, Theorem 5.1] *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^* and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put $I_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that*

$$\delta(r_1, r_2) < \rho(r_1, r_2).$$

Then, for each $\lambda \in]\frac{1}{\rho(r_1,r_2)}, \frac{1}{\delta(r_1,r_2)}[$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ $\forall u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Let us introduce some notations which will be used later.

Set $p_{0i} = \min_{[0,1]} p_i(x)$, $q_{0i} = \min_{[0,1]} q_i(x)$ and $m_i = \min\{p_{0i},\ q_{0i}\}$ for $1 \leq i \leq n$, and set $\underline{m} = \min\{m_i;\ 1 \leq i \leq n\}$.

Here and in the sequel,

$$X := W^{1,2}([0,1]) \times ... \times W^{1,2}([0,1]) = (W^{1,2}([0,1]))^n$$

endowed with the norm

$$\|(u_1, ..., u_n)\|_* = \sum_{i=1}^{n} \|u_i\|$$

where $||u_i|| = \left(\int_0^1 (p_i(x)|u_i'(x)|^2 + \int_0^1 q_i(x)|u_i(x)|^2)dx \right)^{\frac{1}{2}}$.

For all $\gamma > 0$ we denote by $K(\gamma)$ the set

$$\left\{ (t_1, ..., t_n) \in \mathbb{R}^n : \ \sum_{i=1}^{n} |t_i| \leq \gamma \right\}. \tag{2.1}$$

3. **Existence of three solutions**

Consider the following Neumann two-point boundary value Sturm-Liouville type system

$$\begin{cases} -(p_i(x)u_i'(x))' + q_i(x)u_i(x) = \lambda F_{u_i}(x, u_1, ..., u_n) \quad x \in (0,1), \\ u_i'(0) = u_i'(1) = 0 \end{cases} \tag{3.1}$$

for $1 \leq i \leq n$, where $n \geq 1$ is an integer, $p_i \in C^1([0,1)]$, $q_i \in C^0([0,1)]$ with p_i and q_i positive functions for $1 \leq i \leq n$, λ is a positive parameter, $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a function such that $F(., t_1, ..., t_n)$ is measurable in $[0,1]$ for all $(t_1, ..., t_n) \in \mathbb{R}^n$, $F(x, ., ..., .)$ is C^1 in $\mathbb{R}^n$ for every $x \in [0,1]$ and for every $\varrho > 0$,

$$\sup_{|(t_1,...,t_n)| \leq \varrho} \sum_{i=1}^{n} |F_{t_i}(x, t_1, ..., t_n)| \in L^1([0,1]).$$

We say that $u = (u_1, ..., u_n)$ is a weak solution to the system (3.1) if $u = (u_1, ..., u_n) \in X$ and

$$\sum_{i=1}^{n} \int_0^1 \left(p_i(x)u_i'(x)v_i'(x) + q_i(x)u_i(x)v_i(x)\right) dx$$
$$-\lambda \sum_{i=1}^{n} \int_0^1 F_{u_i}(x, u_1(x), ..., u_n(x))v_i(x)dx = 0$$

for every $v = (v_1, ..., v_n) \in X$.

For a given positive constant ν, put

$$a(\nu) := \frac{\int_0^1 \sup_{(t_1,...,t_n)\in K(\nu)} F(x, t_1, ..., t_n)dx}{\nu^2}$$

where $K(\nu) = \{(t_1, ..., t_n) \in \mathbb{R}^n : \sum_{i=1}^n |t_i| \leq \nu\}$(see (2.1)).

We formulate the main result of this section as follows:

Theorem 3.1. *Let $F : [0, 1] \times \mathbb{R}^n \to \mathbb{R}$ satisfies the conditions $F(x, t_1, ..., t_n) \geq 0$ for all $(x, t_1, ..., t_n) \in [0, 1] \times (\mathbb{R} \cup \{0\})^n$ and $F(x, 0, ..., 0) = 0$ for every $x \in [0, 1]$. Assume that there exist four positive constants ν_1, ν_2, η and τ with $\nu_1 < n\sqrt{\frac{2}{m}||q_n||_1}\tau < \nu_2 < \eta$ such that*

(A1)

$$\max\left\{a(\nu_1), a(\nu_2), \frac{\eta^2}{\eta^2 - \nu_2^2}a(\eta)\right\}$$

$$< \frac{m}{2n^2||q_n||_1} \frac{\int_0^1 F(x, 0, ..., 0, \tau)dx - \int_0^1 \sup_{(t_1,...,t_n)\in K(\nu_1)} F(x, t_1, ..., t_n)dx}{\tau^2}.$$

Then, for each

$$\lambda \in \left] \frac{\frac{1}{2}||q_n||_1\tau^2}{\int_0^1 \sup_{(t_1,...,t_n)\in K(\nu_1)} F(x, t_1, ..., t_n)dx - \int_0^1 F(x, 0, ..., 0, \tau)dx} \right.$$

$$\left., \frac{m}{4n^2} \min\left\{\frac{1}{a(\nu_1)}, \frac{1}{a(\nu_2)}, \frac{\eta^2 - \nu_2^2}{\eta^2}\frac{1}{a(\eta)}\right\}\right[$$

the system (3.1) admits at least three weak solutions $v^j = (v_1^j, \ldots, v_n^j) \in X$ $(j = 1, 2, 3)$ such that

$$\sum_{i=1}^n |v_i^j(x)| \leq \eta \quad \textit{for each } x \in [a, b], \ j = 1, 2, 3.$$

Proof. In order to apply Theorem 2.1 to our problem, we introduce the functionals Φ, $\Psi : X \to \mathbb{R}$ for each $u = (u_1, ..., u_n) \in X$, as follows

$$\Phi(u) = \sum_{i=1}^n \frac{||u_i||^2}{2}$$

and

$$\Psi(u) = \int_0^1 F(x, u_1(x), ..., u_n(x))dx.$$

It is well known that Φ and Ψ are well defined and continuously differentiable functionals whose derivatives at the point $u=(u_1,...,u_n)\in X$ are the functionals $\Phi'(u),\Psi'(u)\in X^*$, given by

$$\Phi'(u)(v)=\sum_{i=1}^{n}\int_0^1\left(p_i(x)u_i'(x)v_i'(x)+q_i(x)u_i(x)v_i(x)\right)dx$$

and

$$\Psi'(u)(v)=\sum_{i=1}^{n}\int_0^1 F_{u_i}(x,u_1(x),...,u_n(x))v_i(x)dx$$

for every $v=(v_1,...,v_n)\in X$, respectively. Moreover, Φ is sequentially weakly lower semicontinuous and Φ' admits a continuous inverse on X^*, as well as, $\Psi':X\to X^*$ is a compact operator. Obviously, Φ and Ψ satisfy condition 1. of Theorem 2.1. Moreover, since $F(x,t_1,...,t_n)\geq 0$ for all $(x,t_1,...,t_n)\in[0,1]\times(\mathbb{R}\cup\{0\})^n$, for every $u^*,\ u^{**}\in X$ with $\Psi(u^*)\geq 0$ and $\Psi(u^{**})\geq 0$, one has

$$\inf_{s\in[0,1]}\Psi(su^*+(1-s)u^{**})\geq 0.$$

Set $w(x)=(0,...,0,\tau)$, $r_1=\underline{m}(\frac{\nu_1}{2n})^2$, $r_2=\underline{m}(\frac{\nu_2}{2n})^2$ and $r_3=\underline{m}\frac{\eta^2-\nu_2^2}{4n^2}$. One has $\Phi(w)=\frac{1}{2}||q_n||_1\tau^2$. So, bearing the condition $\nu_1<n\sqrt{\frac{2}{\underline{m}}||q_n||_1\tau^2}<\nu_2<\eta$ in mind, we get $r_1<\Phi(w)<r_2$ and $r_3>0$. Since for $1\leq i\leq n$,

$$|u_i(x)|\leq\sqrt{\frac{2}{m_i}}||u_i||\ \ \forall\ u_i\in W^{1,2}([0,1]),$$

we obtain

$$\sup_{x\in[0,1]}\sum_{i=1}^{n}|u_i(x)|^2\leq\sum_{i=1}^{n}\frac{2}{m_i}||u_i||^2\leq\frac{2}{\underline{m}}\sum_{i=1}^{n}||u_i||^2 \tag{3.2}$$

for each $u = (u_1, ..., u_n) \in X$, and so from the definition of Φ, we see that

$$\begin{aligned}
\Phi^{-1}(]-\infty, r_1[) &= \{u = (u_1, ..., u_n) \in X; \Phi(u) < r_1\} \\
&= \left\{u \in X; \sum_{i=1}^{n} ||u_i||^2 < 2r_1\right\} \\
&\subseteq \left\{u \in X; \sum_{i=1}^{n} |u_i(x)|^2 < \frac{\nu_1^2}{n^2} \text{ for all } x \in [0,1]\right\} \\
&\subseteq \left\{u \in X; \sum_{i=1}^{n} |u_i(x)| \leq \nu_1 \text{ for all } x \in [0,1]\right\},
\end{aligned}$$

which follows

$$\begin{aligned}
&\sup_{(u_1,...,u_n)\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u) \\
&= \sup_{(u_1,...,u_n)\in\Phi^{-1}(]-\infty,r_1[)} \int_0^1 F(x, u_1(x), ..., u_n(x))dx \\
&\leq \int_0^1 \sup_{(t_1,...,t_n)\in K(\nu_1)} F(x, t_1, ..., t_n)dx.
\end{aligned}$$

In a similar way, we get

$$\begin{aligned}
&\sup_{(u_1,...,u_n)\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) \\
&= \sup_{(u_1,...,u_n)\in\Phi^{-1}(]-\infty,r_2[)} \int_0^1 F(x, u_1(x), ..., u_n(x))dx \\
&\leq \int_0^1 \sup_{(t_1,...,t_n)\in K(\nu_2)} F(x, t_1, ..., t_n)dx
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{(u_1,...,u_n)\in\Phi^{-1}(]-\infty,r_2+r_3[)} \Psi(u) \\
&= \sup_{(u_1,...,u_n)\in\Phi^{-1}(]-\infty,r_2+r_3[)} \int_0^1 F(x, u_1(x), ..., u_n(x))dx \\
&\leq \int_0^1 \sup_{(t_1,...,t_n)\in K(\eta)} F(x, t_1, ..., t_n)dx.
\end{aligned}$$

So, taking into account that $0 \in \Phi^{-1}(]-\infty, r_1[)$, one has

$$\varphi(r_1) \leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} \leq \frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu_1)} F(x,t_1,...,t_n)dx}{\underline{m}(\frac{\nu_1}{2n})^2},$$

$$\varphi(r_2) \leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} \leq \frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu_1)} F(x,t_1,...,t_n)dx}{\underline{m}(\frac{\nu_2}{2n})^2}$$

and

$$\begin{aligned}\gamma(r_1,r_2) &\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2+r_3[)} \Psi(u)}{r_3}\\ &\leq \frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\eta)} F(x,t_1,...,t_n)dx}{\underline{m}\frac{\eta^2-\nu_2^2}{4n^2}}\end{aligned}$$

On the other hand, for each $u \in \Phi^{-1}(]-\infty, r_1[)$, one has

$$\begin{aligned}\beta(r_1,r_2) &\geq \frac{\int_0^1 F(x,w_1(x),...,w_n(x))dx - \int_0^1 F(x,u_1(x),...,u_n(x))dx}{\sum_{i=1}^n \frac{\|w_i\|^2}{2} - \sum_{i=1}^n \frac{\|u_i\|^2}{2}}\\ &\geq \frac{\int_0^1 F(x,w_1(x),...,w_n(x))dx - \int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu_1)} F(x,t_1,...,t_n)dx}{\frac{1}{2}\|q_n\|_1\tau^2}\\ &= \frac{2}{\|q_n\|_1}\frac{\int_0^1 F(x,0,...,0,\tau)dx - \int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu_1)} F(x,t_1,...,t_n)dx}{\tau^2}.\end{aligned}$$

Thanks to assumption (A1) we observe that

$$\alpha(r_1,r_2,r_3) < \beta(r_1,r_2).$$

Therefore, taking into account that the weak solutions of the system (3.1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$, and recalling (3.2), Theorem 2.1 follows the conclusion. □

Remark 3.2. *The weak solutions of the system* (3.1) *where F is a C^1-function, by using standard methods, belong to $C^2([0,1])$. Namely, in this case, the classical and the weak solutions of the system* (3.1) *coincide.*

Now, we point out the following direct consequence of Theorem 3.1.

Theorem 3.3. *Let $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ satisfies the conditions $F(x,t_1,...,t_n) \geq 0$ for all $(x,t_1,...,t_n) \in [0,1] \times (\mathbb{R} \cup \{0\})^n$ and $F(x,0,...,0) = 0$ for every $x \in [0,1]$. Assume that there exist three positive constants ν_1, κ and τ with $\nu_1 < n\sqrt{\frac{2}{\underline{m}}\|q_n\|_1}\tau < \frac{1}{\sqrt{2}}\kappa$ such that*

(A2) $a(\nu_1) < \frac{(\frac{m}{2n^2||q_n||_1})^2}{1+\frac{m}{2n^2||q_n||_1}} \frac{\int_0^1 F(x,0,...,0,\tau)dx}{\tau^2}$;

(A3) $a(\kappa) < \frac{1}{2} \frac{\frac{m}{2n^2||q_n||_1}}{1+\frac{m}{2n^2||q_n||_1}} \frac{\int_0^1 F(x,0,...,0,\tau)dx}{\tau^2}$

Then, for each

$$\lambda \in \left] \frac{1+\frac{m}{2n^2||q_n||_1}}{\frac{m}{2n^2||q_n||_1}} \frac{\frac{1}{2}||q_n||_1\tau^2}{\int_0^1 F(x,0,...,0,\tau)dx}, \ \frac{m}{4n^2} \min\left\{\frac{1}{a(\nu_1)}, \frac{1}{2a(\kappa)}\right\}\right[$$

the system (3.1) admits at least three weak solutions $v^j = (v_1^j, \ldots, v_n^j) \in X$ $(j = 1, 2, 3)$ *such that*

$$\sum_{i=1}^n |v_i^j(x)| \le \kappa \quad \textit{for each } x \in [a, b], \ \ j = 1, 2, 3.$$

Proof. Since $\frac{m}{2n^2||q_n||_1} < 1$, from (A2) one has

$$\begin{aligned} a(\nu_1) &= \frac{\int_0^1 \sup_{(t_1,...,t_n)\in K(\nu_1)} F(x,t_1,...,t_n)dx}{\nu_1^2} \\ &< \frac{\int_0^1 \sup_{(t_1,...,t_n)\in K(\nu_1)} F(x,t_1,...,t_n)dx}{\frac{m}{2n^2||q_n||_1}\nu_1^2} \\ &< \frac{\frac{m}{2n^2||q_n||_1}}{1+\frac{m}{2n^2||q_n||_1}} \frac{\int_0^1 F(x,0,...,0,\tau)dx}{\tau^2}. \end{aligned}$$

By choosing $\nu_2 = \frac{1}{\sqrt{2}}\kappa$ and $\eta = \kappa$, from (A3) one has

$$a(\nu_2) = a(\frac{1}{\sqrt{2}}\kappa) \le 2a(\kappa) < \frac{\frac{m}{2n^2||q_n||_1}}{1+\frac{m}{2n^2||q_n||_1}} \frac{\int_0^1 F(x,0,...,0,\tau)dx}{\tau^2} \tag{3.3}$$

and

$$\frac{\eta^2}{\eta^2-\nu_2^2} a(\eta) = 2a(\kappa) < \frac{\frac{m}{2n^2||q_n||_1}}{1+\frac{m}{2n^2||q_n||_1}} \frac{\int_0^1 F(x,0,...,0,\tau)dx}{\tau^2}. \tag{3.4}$$

Moreover, bearing the condition $\nu_1 < n\sqrt{\frac{2}{m}||q_n||_1}\tau$ in mind, from (A2) we deduce

$$\frac{m}{2n^2||q_n||_1} \frac{\int_0^1 F(x,0,...,0,\tau)dx - \int_0^1 \sup_{(t_1,...,t_n)\in K(\nu_1)} F(x,t_1,...,t_n)dx}{\tau^2}$$

$$> \frac{m}{2n^2||q_n||_1}\frac{\int_0^1 F(x,0,...,0,\tau)dx}{\tau^2} - \frac{\int_0^1 \sup_{(t_1,...,t_n)\in K(\nu_1)} F(x,t_1,...,t_n)dx}{\nu_1^2}$$

$$> \Big(\frac{m}{2n^2||q_n||_1} - \frac{(\frac{m}{2n^2||q_n||_1})^2}{1+\frac{m}{2n^2||q_n||_1}}\Big)\frac{\int_0^1 F(x,0,...,0,\tau)dx}{\tau^2}$$

$$= \frac{\frac{m}{2n^2||q_n||_1}}{1+\frac{m}{2n^2||q_n||_1}}\frac{\int_0^1 F(x,0,...,0,\tau)dx}{\tau^2}.$$

Hence, using again Assumptions (A2) and (A3), owing to (3.3) and (3.4), we observe that all assumptions of Theorem 3.1 are satisfied. So, by applying Theorem 3.1 we have the conclusion. □

We here want to point out a simple version of Theorem 3.3 when $n = 1$.

Let $p_1 = p$, $q_1 = q$ and $m_1 = m$. Let $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ be an L^1-Carathéodory function. Let F be the function defined by $F(x,t) = \int_0^t f(x,s)ds$ for each $(x,t)\in[0,1]\times\mathbb{R}$. Then, we have the following existence result.

Theorem 3.4. *Let the function f satisfies the condition $f(x,t)\geq 0$ for all $x\in[0,1]$ and for all $t\geq 0$. Assume that there exist three positive constants ν_1, κ and τ with $\nu_1 < \sqrt{\frac{2}{m}||q||_1}\tau < \frac{1}{\sqrt{2}}\kappa$ such that*

(A4) $\frac{\int_0^1 \sup_{|t|\leq\nu_1} F(x,t)dx}{\nu_1^2} < \frac{(\frac{m}{2||q||_1})^2}{1+\frac{m}{2||q||_1}}\frac{\int_0^1 F(x,\tau)dx}{\tau^2}$;

(A5) $\frac{\int_0^1 \sup_{|t|\leq\kappa} F(x,t)dx}{\kappa^2} < \frac{1}{2}\frac{\frac{m}{2||q||_1}}{1+\frac{m}{2||q||_1}}\frac{\int_0^1 F(x,\tau)dx}{\tau^2}$.

Then, for each

$$\lambda\in\Big]\frac{1+\frac{m}{2||q||_1}}{\frac{m}{2||q||_1}}\frac{\frac{1}{2}||q||_1\tau^2}{\int_0^1 F(x,\tau)dx}, \frac{m}{4}\min\Big\{\frac{\nu_1^2}{\int_0^1\sup_{|t|\leq\nu_1}F(x,t)dx}, \frac{\kappa^2}{2\int_0^1\sup_{|t|\leq\kappa}F(x,t)dx}\Big\}\Big[$$

the problem

$$\begin{cases} -(p(x)u'(x))' + q(x)u'(x) = \lambda f(x,u) & x\in(0,1),\\ u'(0) = u'(1) = 0 \end{cases} \tag{3.5}$$

admits at least three weak solutions $v^j\in W^{1,2}([0,1])$ $(j = 1,2,3)$ such that

$$|v^j(x)|\leq\kappa \quad \text{for each } x\in[a,b],\ j = 1,2,3.$$

The following result gives the existence of at least three weak solutions in $W^{1,2}([0,1])$ to the problem (3.5) in the autonomous case.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(t) \geq 0$ for every $t \geq 0$, and put $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$. We have the following result as a straightforward consequence of Theorem 3.4.

Theorem 3.5. *Assume that there exist three positive constants ν_1, κ and τ with $\nu_1 < \sqrt{\frac{2}{m}||q||_1}\tau < \frac{1}{\sqrt{2}}\kappa$ such that*

(A6) $f(t) \geq 0$ for each $t \in [-\kappa,\ 0]$;
(A7) $\frac{F(\nu_1)}{\nu_1^2} < \frac{(\frac{m}{2||q||_1})^2}{1+\frac{m}{2||q||_1}} \frac{F(\tau)}{\tau^2}$;
(A8) $\frac{F(\kappa)}{\kappa^2} < \frac{1}{2} \frac{\frac{m}{2||q||_1}}{1+\frac{m}{2||q||_1}} \frac{F(\tau)}{\tau^2}$.
Then, for each

$$\lambda \in \left] \frac{2||q||_1^2 + ||q||_1 m}{2m} \frac{\tau^2}{F(\tau)},\ \frac{m}{4} \min\left\{ \frac{\nu_1^2}{F(\nu_1)}, \frac{1}{2} \frac{\kappa^2}{F(\kappa)} \right\} \right[$$

the problem

$$\text{(3.6)} \qquad \begin{cases} -(p(x)u'(x))' + q(x)u'(x) = \lambda f(u) & x \in (0,1), \\ u'(0) = u'(1) = 0 \end{cases}$$

admits at least three classical solutions $v^j \in W^{1,2}([0,1])$ $(j = 1,2,3)$ such that

$$|v^j(x)| \leq \kappa \ \text{ for each } x \in [a,b],\ j = 1,2,3.$$

As an example of the results, the following consequence, ensures the existence of at least two non-trivial classical solutions to the problem (3.6).

Theorem 3.6. *Let $f : \mathbb{R} \to]0,+\infty[$ be a continuous function such that*

$$\lim_{t \to 0^+} \frac{f(t)}{t} = \lim_{t \to +\infty} \frac{f(t)}{t} = 0.$$

Then, for every $\lambda > \inf\left\{ \frac{2||q||_1^2+||q||_1 m}{2m} \frac{\tau^2}{F(\tau)} \ : \ \tau > 0,\ F(\tau) > 0 \right\}$, the problem (3.6) admits at least two non-trivial classical solutions.

Proof. Fix $\lambda > \inf\left\{ \frac{2||q||_1^2+||q||_1 m}{2m} \frac{\tau^2}{F(\tau)} \ : \ \tau > 0,\ F(\tau) > 0 \right\}$, and let τ be a positive constant such that $F(\tau) > 0$ and $\lambda > \frac{2||q||_1^2+||q||_1 m}{2m} \frac{\tau^2}{F(\tau)}$. Since $\lim_{t \to 0^+} \frac{f(t)}{t} = 0$, there is a positive constant ν_1 with $\nu_1 < \sqrt{\frac{2}{m}||q||_1}\tau$

such that $\frac{F(\nu_1)}{\nu_1^2} < \frac{m}{4\lambda}$, and since $\lim_{t\to+\infty} \frac{f(t)}{t} = 0$, there is a positive constant κ with $\sqrt{\frac{2}{m}||q||_1}\tau < \frac{1}{\sqrt{2}}\kappa$ such that $\frac{F(\kappa)}{\kappa^2} < \frac{m}{8\lambda}$. Therefore, Theorem 3.5 follows the conclusion. □

We end this section by giving the following result which provides the existence of at least three weak solutions for the system (1.1).

Put $m_i' = \min\Big\{\min_{[0,1]} e^{-R_i}p_i,\ \min_{[0,1]} e^{-R_i}q_i\Big\}$ where R_i is a primitive of $\frac{r_i}{p_i}$ for $1 \leq i \leq n$, and put $\underline{m}' = \min\{m_i';\ 1 \leq i \leq n\}$. Then, we have the following result.

Theorem 3.7. *Let $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ satisfy the conditions $F(x, t_1, ..., t_n) \geq 0$ for all $(x, t_1, ..., t_n) \in [0,1] \times (\mathbb{R} \cup \{0\})^n$ and $F(x, 0, ..., 0) = 0$ for every $x \in [0,1]$. Assume that there exist four positive constants ν_1, ν_2, η and τ with $\nu_1 < n\sqrt{\frac{2}{\underline{m}'}||e^{-R_n}q_n||_1}\tau < \nu_2 < \eta$ where $||e^{-R_n}q_n||_1 = \int_0^1 e^{-R_n(x)}q_n(x)dx$ such that*

(A9)

$$\max\Big\{a(\nu_1), a(\nu_2), \frac{\eta^2}{\eta^2-\nu_2^2}a(\eta)\Big\}$$

$$< \underline{m}'\frac{\int_0^1 F(x,0,...,0,\tau)dx - \int_0^1 \sup_{(t_1,...,t_n)\in K(\nu_1)} F(x,t_1,...,t_n)dx}{2n^2||e^{-R_n}q_n||_1\tau^2}.$$

Then, for each

$$\lambda \in \Big]\frac{\frac{1}{2}||e^{-R_n}q_n||_1\tau^2}{\max\{||e^{-R_i}||_1;\ 1 \leq i \leq n\}}$$

$$\times\frac{1}{\Big(\int_0^1 \sup_{(t_1,...,t_n)\in K(\nu_1)} F(x,t_1,...,t_n)dx - \int_0^1 F(x,0,...,0,\tau)dx\Big)}$$

$$,\ \frac{m}{4n^2}\frac{1}{\min\{||e^{-R_i}||_1;\ 1 \leq i \leq n\}}\min\Big\{\frac{1}{a(\nu_1)}, \frac{1}{a(\nu_2)}, \frac{\eta^2-\nu_2^2}{\eta^2}\frac{1}{a(\eta)}\Big\}\Big[$$

the system (1.1) admits at least three weak solutions $v^j = (v_1^j, \ldots, v_n^j) \in X$ $(j = 1,2,3)$ such that

$$\sum_{i=1}^n |v_i^j(x)| \leq \eta \ \ \text{for each } x \in [a,b],\ j = 1,2,3.$$

Proof. Taking into account that the solutions of the system

$$\begin{cases} -(e^{-R_i}p_i(x)u_i'(x))' + e^{-R_i}q_i(x)u_i(x) = \lambda e^{-R_i}F_{u_i}(x, u_1, ..., u_n) \\ x \in (0,1), \\ u_i'(0) = u_i'(1) = 0 \end{cases}$$

and the solutions of the system (1.1) coincide, Theorem 3.1 follows the conclusion. □

4. **Existence of a non-trivial solution**

In this section, by the use of Theorem 2.2, we prove that under appropriate assumptions on F, the system (1.1) admits at least one non-trivial weak solution.

For given a nonnegative constant ν and a positive constant τ with $\underline{m}(\frac{\nu}{n})^2 \neq 2\tau^2||q_n||_1$, put

$$b_\tau(\nu) := \frac{\int_0^1 \sup_{(t_1,...,t_n)\in K(\nu)} F(x, t_1, ..., t_n)dx - \int_0^1 F(x, 0, ..., 0, \tau)dx}{\frac{\underline{m}}{2}(\frac{\nu}{n})^2 - \tau^2||q_n||_1}$$

where $K(\nu) = \{(t_1, ..., t_n) \in \mathbb{R}^n : \sum_{i=1}^n |t_i| \leq \nu\}$ (see (2.1)) and $||q_n||_1 = \int_0^1 q_n(x)dx$.

We formulate the main result of this section as follows:

Theorem 4.1. *Assume that there exist a non-negative constant ν_1 and two positive constants ν_2 and τ with $\nu_1 < n\sqrt{\frac{2}{\underline{m}}||q_n||_1}\tau < \nu_2$ such that*

(B1) $b_\tau(\nu_2) < b_\tau(\nu_1)$.

Then, for each $\lambda \in \left]\frac{1}{2}\frac{1}{b_\tau(\nu_1)}, \ \frac{1}{2}\frac{1}{b_\tau(\nu_2)}\right[$ the system (3.1) admits at least one non-trivial weak solution $u_0 = (u_{01}, \dots, u_{0n}) \in X$ such that $\frac{\underline{m}}{2}(\frac{\nu_1}{n})^2 < \sum_{i=1}^n ||u_{0i}||^2 < \frac{\underline{m}}{2}(\frac{\nu_2}{n})^2$.

Proof. In order to apply Theorem 2.2 to our problem, we take Φ and Ψ as in the proof of Theorem 3.1. Set $w(x) = (0, ..., 0, \tau)$, $r_1 = \underline{m}(\frac{\nu_1}{2n})^2$ and $r_2 = \underline{m}(\frac{\nu_2}{2n})^2$. One has $\Phi(w) = \frac{1}{2}||q_n||_1\tau^2$. So, bearing the condition $\nu_1 < n\sqrt{\frac{2}{\underline{m}}||q_n||_1}\tau^2 < \nu_2$ in mind, we get

$$r_1 < \Phi(w) < r_2.$$

From the definition of Φ and recalling (3.2), we see that

$$\begin{aligned}
\Phi^{-1}(]-\infty, r_2[) &= \{u = (u_1, ..., u_n) \in X; \Phi(u) < r_2\} \\
&= \left\{ u \in X; \sum_{i=1}^{n} ||u_i||^2 < 2r_2 \right\} \\
&\subseteq \left\{ u \in X; \sum_{i=1}^{n} |u_i(x)|^2 < \frac{\nu_2^2}{n^2} \text{ for all } x \in [0,1] \right\} \\
&\subseteq \left\{ u \in X; \sum_{i=1}^{n} |u_i(x)| \leq \nu_2 \text{ for all } x \in [0,1] \right\},
\end{aligned}$$

which follows

$$\begin{aligned}
&\sup_{(u_1,...,u_n)\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) \\
&= \sup_{(u_1,...,u_n)\in\Phi^{-1}(]-\infty,r_2[)} \int_0^1 F(x, u_1(x), ..., u_n(x))dx \\
&\leq \int_0^1 \sup_{(t_1,...,t_n)\in K(\nu_2)} F(x, t_1, ..., t_n)dx.
\end{aligned}$$

So, one has

$$\begin{aligned}
\delta(r_1, r_2) &\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \\
&\leq \frac{\int_0^1 \sup_{(t_1,...,t_n)\in K(\nu_2)} F(x, t_1, ..., t_n)dx - \Psi(w)}{r_2 - \Phi(w)} \\
&\leq 2b_\tau(\nu_2).
\end{aligned}$$

On the other hand, by similar reasoning as before, one has

$$\begin{aligned}
\rho(r_1, r_2) &\geq \frac{\Psi(w) - \sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{\Phi(w) - r_1} \\
&\geq \frac{\Psi(w) - \int_0^1 \sup_{(t_1,...,t_n)\in K(\nu_1)} F(x, t_1, ..., t_n)dx}{\Phi(w) - r_1} \\
&\geq 2b_\tau(\nu_1).
\end{aligned}$$

Hence, from Assumption (B1), one has $\delta(r_1, r_2) < \rho(r_1, r_2)$. Therefore, applying Theorem 2.2, taking into account that the weak solution of the system (3.1) are exactly the solution of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$, we have the conclusion. □

Now we point out the following consequence of Theorem 4.1.

Theorem 4.2. *Assume that there exist two positive constants* ν *and* τ *with* $n\sqrt{\frac{2}{\underline{m}}||q_n||_1}\tau < \nu$ *such that*

(B2) $\frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu)} F(x,t_1,\dots,t_n)dx}{\nu^2} < \frac{\underline{m}}{2n^2||q_n||_1}\frac{\int_0^1 F(x,0,\dots,0,\tau)dx}{\tau^2}$;
(B3) $F(x,0,...,0) = 0$ *for every* $x \in [0,1]$.

Then, for each

$$\lambda \in \left]\frac{\frac{1}{2}||q_n||_1\tau^2}{\int_0^1 F(x,0,...,0,\tau)dx}, \frac{\underline{m}(\frac{\nu}{2n})^2}{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu)} F(x,t_1,...,t_n)dx}\right[$$

the system (3.1) admits at least one non-trivial weak solution $u_0 = (u_{01},\dots,u_{0n}) \in X$ *such that* $\sum_{i=1}^n ||u_i||_\infty < \nu$.

Proof. By the use of Theorem 4.1 and taking $\nu_1 = 0$, $\nu_2 = \nu$ we get the conclusion. Indeed, owing to our assumptions, one has

$$\begin{aligned} b_\tau(\nu_2) &< \frac{\left(1-\frac{2n^2||q_n||_1\tau^2}{\underline{m}\nu^2}\right)\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu)} F(x,t_1,...,t_n)dx}{\frac{\underline{m}}{2}(\frac{\nu}{n})^2 - \tau^2||q_n||_1} \\ &= \frac{\left(1-\frac{||q_n||_1\tau^2}{\frac{\underline{m}}{2}(\frac{\nu}{n})^2}\right)\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu)} F(x,t_1,...,t_n)dx}{\frac{\underline{m}}{2}(\frac{\nu}{n})^2 - \tau^2||q_n||_1} \\ &= \frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu)} F(x,t_1,...,t_n)dx}{\frac{\underline{m}}{2}(\frac{\nu}{n})^2}. \end{aligned}$$

On the other hand, taking Assumption (B3) into account, one has

$$\frac{\int_0^1 F(x,0,...,0,\tau)dx}{||q_n||_1\tau^2} = b_\tau(\nu_1).$$

Moreover, taking (3.2) into account, $\sum_{i=1}^n ||u_i||_\infty < \nu$ whenever $\Phi(u) < r_2$. Now, owing to assumption (B2), it is sufficient to invoke Theorem 4.1 for concluding the proof. □

Here we want to point out a simple version of Theorem 4.1 when $n = 1$.

Let $p_1 = p$, $q_1 = q$ and $m_1 = m$. Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Let F be the function defined by $F(x,t) =$

$\int_0^t f(x,s)ds$ for each $(x,t) \in [0,1] \times \mathbb{R}$. For given nonnegative constant ν and a positive constant τ with $m\nu^2 \neq 2\tau^2||q||_1$, put

$$c_\tau(\nu) := \frac{\int_0^1 \sup_{|t|\leq\nu} F(x,t)dx - \int_0^1 F(x,\tau)dx}{\frac{m}{2}\nu^2 - \tau^2||q||_1}.$$

We have the following result.

Theorem 4.3. *Assume that there exist a non-negative constant ν_1 and two positive constants ν_2 and τ with $\nu_1 < \sqrt{\frac{2}{m}||q||_1}\tau < \nu_2$ such that*

(B4) $c_\tau(\nu_2) < c_\tau(\nu_1)$.

Then, for each $\lambda \in \left]\frac{1}{2}\frac{1}{c_\tau(\nu_1)},\ \frac{1}{2}\frac{1}{c_\tau(\nu_2)}\right[$ the problem (3.5) *admits at least one non-trivial weak solution $u_0 \in W^{1,2}([0,1])$ such that*

$$\frac{m}{2}\nu_1^2 < \sum_{i=1}^n ||u_{0i}||^2 < \frac{m}{2}\nu_2^2.$$

The last result gives the existence of at least one non-trivial weak solution in $W^{1,2}([0,1])$ to the problem (3.5) in the autonomous case.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and put $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$. We have the following result as a direct consequence of Theorem 4.3.

Theorem 4.4. *Assume that there exist a non-negative constant ν_1 and two positive constants ν_2 and τ with $\nu_1 < \sqrt{\frac{2}{m}||q||_1}\tau < \nu_2$ such that*

(B5) $f(t) \geq 0$ for each $t \in [-\nu_2,\ \max\{\nu_2,\tau\}]$;
(B6) $\frac{F(\nu_2)-F(\tau)}{\frac{m}{2}(\frac{\nu}{n})^2-\tau^2||q||_1} < \frac{F(\nu_1)-F(\tau)}{\frac{m}{2}(\frac{\nu}{n})^2-\tau^2||q||_1}$.

Then, for each $\lambda \in \left]\frac{1}{2}\frac{\frac{m}{2}(\frac{\nu}{n})^2-\tau^2||q||_1}{F(\nu_1)-F(\tau)},\ \frac{1}{2}\frac{\frac{m}{2}(\frac{\nu}{n})^2-\tau^2||q||_1}{F(\nu_2)-F(\tau)}\right[$ the problem (3.6) *admits at least one non-trivial classical solution $u_0 \in W^{1,2}([0,1])$ such that*

$$\frac{m}{2}\nu_1^2 < \sum_{i=1}^n ||u_{0i}||^2 < \frac{m}{2}\nu_2^2.$$

As an example, we point out the following special case of the main result.

Theorem 4.5. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative function such that $\lim_{t\to 0^+} \frac{g(t)}{t} = +\infty$.*
Then, for each $\lambda \in \left]0, \frac{m}{4} \sup_{\nu>0} \frac{\nu^2}{\int_0^\nu g(\xi)d\xi}\right[$, the problem

$$\begin{cases} -(p(x)u'(x))' + q(x)u'(x) = \lambda g(u) & x \in (0,1), \\ u'(0) = u'(1) = 0 \end{cases}$$

admits at least one non-trivial classical solution in $W^{1,2}([0,1])$.

Proof. For fixed λ as in the conclusion, there exists a positive constant ν such that

$$\lambda < \frac{m}{4} \frac{\nu^2}{\int_0^\nu g(\xi)d\xi}.$$

Moreover, the condition $\lim_{t\to 0^+} \frac{g(t)}{t} = +\infty$ implies $\lim_{t\to 0^+} \frac{\int_0^t g(\xi)d\xi}{t^2} = +\infty$. Therefore, a positive constant τ satisfying $\tau < \frac{\nu}{\sqrt{\frac{2}{m}||q||_1}}$ can be chosen such that

$$\frac{1}{\lambda}(\frac{||q||_1}{2}) < \frac{\int_0^\tau g(\xi)d\xi}{\tau^2}.$$

Hence, the conclusion follows from Theorem 4.4 with $\nu_1 = 0$, $\nu_2 = \nu$ and $f(t) = g(t)$ for every $t \in \mathbb{R}$. □

Remark 4.6. *For fixed ρ put $\lambda_\rho := \frac{m}{4} \sup_{\nu\in]0,\rho[} \frac{\nu^2}{\int_0^\nu g(\xi)d\xi}$. The result of Theorem 4.5 for every $\lambda \in]0, \lambda_\rho[$ holds with $|u_0(x)| < \rho$ for all $x \in [0,1]$ where u_0 is the ensured non-trivial classical solution in $W^{1,2}([0,1])$ (see* [7, Remark 4.3]*).*

We present the following example to illustrate the result.

Example 1. *Consider the problem*

$$\begin{cases} -(e^x u')' + e^x u = \lambda(1 + e^{-u^+}(u^+)^2(3 - u^+)) & x \in (0,1), \\ u'(0) = u'(1) = 0. \end{cases} \tag{4.1}$$

where $u^+ = \max\{u, 0\}$. Let

$$g(t) = 1 + e^{-t^+}(t^+)^2(3 - t^+)$$

for all $t \in \mathbb{R}$ where $t^+ = \max\{t, 0\}$. It is clear that $\lim_{t\to 0^+} \frac{g(t)}{t} = +\infty$. Hence, by applying Theorem 4.5, since $m = 1$, for every $\lambda \in]0, \frac{e}{4(1+e)}[$ the problem (4.1) *has at least one non-trivial classical solution $u_0 \in W^{1,2}([0,1])$ such that $||u_0||_\infty < 1$.*

Finally, we give the following existence property of the system (1.1).

Put $m'_i = \min\left\{\min_{[0,1]} e^{-R_i}p_i,\ \min_{[0,1]} e^{-R_i}q_i\right\}$ where R_i is a primitive of $\frac{r_i}{p_i}$ for $1 \le i \le n$, and put $\underline{m}' = \min\{m'_i;\ 1 \le i \le n\}$.

For given a nonnegative constant ν and a positive constant τ with $\underline{m}'(\frac{\nu}{n})^2 \neq 2\tau^2||q_n||_1$, put

$$d_\tau(\nu) := \frac{\int_0^1 \sup_{(t_1,...,t_n)\in K(\nu)} F(x,t_1,...,t_n)dx - \int_0^1 F(x,0,...,0,\tau)dx}{\frac{\underline{m}'}{2}(\frac{\nu}{n})^2 - \tau^2||e^{-R_n}q_n||_1}$$

where $K(\nu) = \{(t_1,...,t_n) \in \mathbb{R}^n :\ \sum_{i=1}^n |t_i| \le \nu\}$ (see (2.1)) and $||e^{-R_n}q_n||_1 = \int_0^1 e^{-R_n(x)}q_n(x)dx$.

Then, we have the following result.

Theorem 4.7. *Assume that there exist a non-negative constant ν_1 and two positive constants ν_2 and τ with $\nu_1 < n\sqrt{\frac{2}{\underline{m}'}||e^{-R_n}q_n||_1}\tau < \nu_2$ such that*

(B7) $d_\tau(\nu_2) < d_\tau(\nu_1)$.

Then, for each $\lambda \in \left]\frac{1}{2\max\{||e^{-R_i}||_1;\ 1\le i\le n\}}\frac{1}{d_\tau(\nu_1)},\ \frac{1}{2\min\{||e^{-R_i}||_1;\ 1\le i\le n\}}\frac{1}{d_\tau(\nu_2)}\right[$ the system (1.1) admits at least one non-trivial weak solution $u_0 = (u_{01},\ldots,u_{0n}) \in X$ such that $\frac{\underline{m}'}{2}(\frac{\nu_1}{n})^2 < \sum_{i=1}^n ||u_{0i}||^2 < \frac{\underline{m}'}{2}(\frac{\nu_2}{n})^2$.

Proof. By the same reasoning as in the proof of Theorem 3.7, Theorem 4.1 follows the conclusion. □

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References

[1] D. Averna and G. Bonanno, A mountain pass theorem for a suitable class of functions, *Rocky Mountain J. Math.* **39** (2009), no. 3, 707–727.

[2] G. Bonanno, A critical point theorem via the Ekeland variational principle, *Nonlinear Anal.* **75** (2012), no. 5, 2992–3007.

[3] G. Bonanno and S. M. Buccellato, Two point boundary value problems for the Sturm-Liouville equation with highly discontinuous nonlinearities, *Taiwanese J. Math.* **14** (2010), no. 5, 2059–2072.

[4] G. Bonanno and P. Candito, Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities, *J. Differential Equations* **244** (2008), no. 12, 3031–3059.

[5] G. Bonanno and G. D'Aguì, A Neumann boundary value problem for the Sturm-Liouville equation, *Appl. Math. Comput.* **208** (2009), no. 2, 318–327.

[6] G. Bonanno and B. Di Bella, A fourth-order boundary value problem for a Sturm-Liouville type equation, *Appl. Math. Comput.* **217** (2010), no. 8, 3635–3640.

[7] G. Bonanno, B. Di Bella and D. O'Regan, Non-trivial solutions for nonlinear fourth-order elastic beam equations, *Comput. Math. Appl.* **62** (2011), no. 4, 1862–1869.

[8] G. Bonanno and G. Molica Bisci, A remark on perturbed elliptic Neumann problems, *Stud. Univ. Babe-Bolyai Math.* **55** (2010), no. 4, 17–25.

[9] G. Bonanno, G. Molica Bisci and V. Rădulescu, Existence of three solutions for a non-homogeneous Neumann problem through Orlicz-Sobolev spaces, *Nonlinear Anal.* **74** (2011), no. 14, 4785–4795.

[10] G. Bonanno and P. F. Pizzimenti, Neumann boundary value problems with not coercive potential, *Mediterr. J. Math.* **9** (2012), no. 4, 601–609.

[11] G. Bonanno and G. Riccobono, Multiplicity results for Sturm-Liouville boundary value problems, *Appl. Math. Comput.* **210** (2009), no. 2, 294–297.

[12] G. D'Aguì and G. Molica Bisci, Three non-zero solutions for elliptic Neumann problems, *Anal. Appl.* **9** (2011), no. 4, 383–394.

[13] J. R. Graef, S. Heidarkhani and L. Kong, A critical points approach for the existence of multiple solutions of a Dirichlet quasilinear system, *J. Math. Anal. Appl.* **388** (2012), no. 2, 1268–1278.

[14] J. R. Graef, S. Heidarkhani and L. Kong, A critical points approach to multiplicity results for multi-point boundary value problems, *Appl. Anal.* **90** (2011), no. 12, 1909–1925.

[15] S. Heidarkhani, Non-trivial solutions for a class of $(p_1, ..., p_n)$-biharmonic systems with Navier boundary conditions, *Ann. Polon. Math.* **105** (2012), no. 1, 65–76.

[16] S. Heidarkhani, Multiple solutions for a class of multipoint boundary value systems driven by a one-dimensional $(p_1, ..., p_n)$-Laplacian operator, *Abstr. Appl. Anal.* (2012), Article ID 389530, 15 pages.

[17] B. Ricceri, A general variational principle and some of its applications, *J. Comput. Appl. Math.* **113** (2000), no. 1-2, 401–410.

[18] Y. Tian and W. Ge, Second-order Sturm-Liouville boundary value problem involving the one- dimensional p-Laplacian, *Rocky Mountain J. Math.* **38** (2008), no. 1, 309–327.

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