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POS-GROUPS WITH SOME CYCLIC SYLOW SUBGROUPS

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ABSTRACT. A finite group G is said to be a POS-group if for each x in G the cardinality of the set $\{y \in G \mid o(y) = o(x)\}$ is a divisor of the order of G. In this paper we study the structure of POS-groups with some cyclic Sylow subgroups.

1. Introduction

Throughout the paper G denotes a finite group, $o(x)$ the order of a group element x, and $|X|$ the cardinality of a set X. Denote by $\pi(G) = \{p \mid p$ is a prime divisor of $|G|\}$. As in [4], the order subset (or, order class) of G determined by an element $x \in G$ is defined to be the set $OS(x) = \{y \in G \mid o(y) = o(x)\}$. Clearly, for every $x \in G$, $OS(x)$ is a disjoint union of some conjugacy classes in G. The group G is said to have perfect order subsets (in short, G is called a POS-group) if $|OS(x)|$ is a divisor of $|G|$ for all $x \in G$. In [4], Finch and Jones first classified abelian POS-groups. Afterwards they continued the study of nonabelian POS-groups and gave some non-solvable POS-groups (see [5],[6]). Recently, Das gave some properties of POS-groups in [2], and Shen classified POS-groups of order $2m$ with $(2, m) = 1$ (see [14]). In this note we study POS-groups with some cyclic Sylow subgroups. In

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section 2, POS-groups with cyclic Sylow 2-subgroups are studied. It is proved that if Sylow 2-subgroups of a POS-group G are cyclic, then 3 divides $|G|$ or G has a self-centralized Sylow 2-subgroup. In the next section, we investigate the structure of POS-groups with cyclic Sylow 2-subgroups of order 4. Finally POS-groups with two prime divisors are studied. If S is a subset of G, denote by $f_S(m)$ the number of elements of order m in S. Let $U(n)$ be the unit group of the ring Z/nZ. Denote by $ord_n(q)$ the order of q in the group $U(n)$. First of all, we consider POS-groups with cyclic Sylow 2-subgroups.

2. **Cyclic Sylow 2-subgroups**

In this part, we study POS-groups with cyclic Sylow 2-subgroups, and prove that if Sylow 2-subgroups of a POS-group G are cyclic, then 3 divides $|G|$ or G has a self-centralized Sylow 2-subgroup. A celebrated theorem of Frobenius asserts that if n is a positive divisor of $|G|$ and $X = \{g \in G| \, g^n = 1\}$, then n divides $|X|$ (see, for example, Theorem 9.1.2 of [9]). This result is used in the sequel frequently. First, we cite some lemmas.

Lemma 2.1. *(Theorem 1,* [10]*). If every element of a finite group G has order which is a power of a prime number and G is solvable, then $|\pi(G)| \leq 2$.*

Recall that G is a 2-Frobenius group if $G = ABC$, where A and AB are normal subgroups of G, AB and BC are Frobenius groups with kernels A and B, and complements B and C respectively. Recall in addition that G is a C_{pp}-group if the centralizer of every non-trivial p-element is a p-group. The following lemma is due to Gruenberg and Kegel (see Corollary of [15]).

Lemma 2.2. *Let G be a solvable C_{pp}-group, then G is a p-group, a Frobenius group or a 2-Frobenius group.*

Lemma 2.3. *(Theorem 3,* [16]*). Let G be a finite group. Then the number of elements whose orders are multiples of n is either zero, or a multiple of the largest divisor of $|G|$ that is prime to n.*

Next we give the following main result.

Theorem 2.4. *If the Sylow 2-subgroups of a POS-group G are cyclic, then 3 is a divisor of $|G|$, or G has a self-centralized Sylow 2-subgroup.*

Proof. Suppose that P_2 is a Sylow 2-subgroup of G and $|P_2| = 2^n$. Since Sylow 2-subgroups of G are cyclic, G is 2-nilpotent. Let the normal 2-complement of G be H. Set $C_G(P_2) = P_2 \times N$, where $N \le H$. Next we will prove that 3 is a divisor of $|N|$ provide P_2 is not self-centralizing. If N has an element of order m, then $f_G(2^n m) = |H/N| \cdot 2^{n-1} \cdot f_N(m)$ is a divisor of $2^n \cdot |H|$. So $f_N(m)$ divides $2|N|$. Note that $|N|$ is odd. It follows that $4 \nmid f_N(m)$. Since $\phi(m)$, the Euler's totient function, divides $f_N(m)$, then we have every order of element of N is a prime power, and thus $|\pi(N)| \le 2$ by Lemma 2.1.

Case I. $\pi(N) = \{p, q\}$. Set $|N| = p^a q^b$. By Lemma 2.2, we have that N is a Frobenius or 2-Frobenius group. If N is Frobenius, without loss of generality, we assume that the order of the kernel of N has divisor q. As p-subgroups are cyclic, then $f_N(p) = (p-1)q^b$ is a divisor of $2|N| = 2p^a q^b$. So $p - 1 = 2$, then $p = 3$. If N is 2-Frobenius, we set $N = ABC$, where A and AB are normal subgroups of N, AB and BC are Frobenius groups with kernels A and B, and complements B and C, respectively. Now let $|A| = p^{a_1}$ and $|C| = p^{a_2}$. Then $f_N(q) = (q-1)p^{a_1}$ divides $2|N| = 2p^a q^b$. Since $p^a \mid f_N(q)$ by Lemma 2.3, it follows that $q = 2p^{a_2} + 1$. Clearly, $q > p$. In addition, $f_N(p) = f_A(p) + (p-1)q^b|A : C_N(c)|$, where c is an element of order p of C. Since $p-1$ and q^b are both divisors of $f_A(p)$ and $(p-1, q) = 1$, we have $(p-1)q^b \mid f_A(p)$. Then $p - 1 \mid 2p^a$, so $p = 3$.

Case II. $\pi(N) = \{p\}$. Set $|N| = p^a$. Then by the above discussion we see that $f_N(p) \mid 2p^a$. Since $p \nmid f_N(p)$, we have $p = 3$. □

Note that indeed there exist POS-groups of Theorem 2.4 whose Sylow 2-subgroups are self-centralized and $3 \nmid |G|$. The following is an example of a POS group of order 400 whose Sylow 2-subgroups are cyclic and self-centralizing.

Example 2.5. *Let* $G = \langle a, b \mid a^{25} = b^{16} = 1, a^b = a^{-1}, [a, b^2] = 1\rangle$. *Then* G *is a POS-group with a cyclic Sylow 2-subgroup of order* 2^4.

Finch and Jone formulated a question in [4] whether the order of every POS-group with more than one prime divisor has a divisor 3. Although this question has a negative answer, it seems that orders of most POS-groups have the divisor 3. We put the following problem.

Problem 2.6. *Classify POS-groups whose order has no divisor 3.*

3. **Cyclic Sylow 2-subgroups of order 4**

In this section, we deal with the POS-groups with cyclic Sylow 2-subgroups of order 4. We completely classify such POS-groups whose order has no divisor 3. First, we determine the number of prime divisors of these groups.

Proposition 3.1. *Let G be a POS-group with cyclic Sylow 2-subgroups of order 4. Then $|\pi(G)| \leq 6$.*

Proof. Let $\sigma(G) = \max\{|\pi(o(g))| \mid g \in G\}$, and H be the normal 2-complement of G. Clearly, $\sigma(H) \leq 2$. So we have $|\pi(H)| \leq 5$ by Theorem 1.4(b) of [11]. Therefore, $|\pi(G)| \leq 6$. □

Although such POS-groups have a upper bound of the number of prime divisors, is 6 the actual bound?

Lemma 3.2. *Let $|G| = 2^n p^m$ and the Sylow p-subgroup P be normal in G. If all Sylow subgroups of G are cyclic and G has no element of order $2p^m$, then G is a Frobenius group.*

Proof. Since all Sylow subgroups of G are cyclic, we may see that $G = \langle a,\ b \mid a^{p^m} = b^{2^n} = 1,\ a^b = a^r \rangle$ such that $r^{2^n} \equiv 1 (\mathrm{mod}\, p^m)$ and $(2^n(r-1), p^m) = 1$. By the above condition, we get that 2^n is the order of r in $U(p^m)$. In fact, otherwise if the order $ord_{p^m}(r)(:= o(r))$ of r is less than 2^n, then

$$a^{b^{o(r)}} = a^{r^{o(r)}} = a^1 = a,$$

hence $b^{o(r)} \in C_G(P)$. On the other hand, since $C_G(P) = 1$, $b^{o(r)} = 1$, which contradicts that $o(b) = 2^n$. It is easy to see that the order $ord_{p^i}(r)$ is also 2^n for $1 \leq i \leq n-1$. So the centralizer of every nontrivial p-element is P, and then G is a Frobenius group. □

To complete the proof of Theorem 3.6 and 4.3, we need some conclusions of prime number. We call $r_m(a)$ a *primitive* prime divisor of $a^m - 1$ if $r_m(a) \mid a^m - 1$ but $r_m(a)$ doesn't divide $a^i - 1$ for every $i < m$. Clearly, for primitive prime divisor $p = r_m(a)$, the formula $m | p-1$ always holds. Let $\Phi_n(x)$ be the n^{th} cyclotomic polynomial. It is well known that $x^n - 1$ may be decomposed to the product of all cyclotomic polynomials whose digit is some divisor of n, that is, $x^n - 1 = \prod_{d|n} \Phi_d(x)$. The existence of primitive prime divisor is due to Zsigmondy (see [17]), and the primitive prime divisor is closely connected with the cyclotomic polynomial as follows.

Lemma 3.3. *Primitive prime divisors of $a^m - 1$ exist except if $m = 6$ and $a = 2$, or $m = 2$ and $a = 2^k - 1$.*

Lemma 3.4. *Suppose that $q^n - 1$ has at least one primitive prime divisor and $n \geq 3$. Then $\Phi_n(q) = (P(n), \Phi_n(q)) \cdot Z_n(q)$, where $P(n)$ is the largest prime divisor of n and $Z_n(q)$ the largest divisor of $q^n - 1$ which contains all primitive prime divisors.*

Proof. By page 207 of [13] and Lemma 2.1 of [3], we have $Z_n(q) \bigm| \Phi_n(q)$ and $\Phi_n(q) \bigm| Z_n(q) \cdot P(n)$, and then $\Phi_n(q) = (P(n), \Phi_n(q)) \cdot Z_n(q)$. □

Lemma 3.5. *(Lemma 5, [12]). Suppose that p is an odd primitive prime divisor of $q^k - 1$. Then $p \mid \Phi_f(q)$ if and only if $f = kp^j$ for some $j \geq 0$.*

We also make some preliminaries on the p-adic expansion of any integer. Let p be a prime. So any positive integer can be written in a base p expansion in the form

$$\sum_{i=0}^{n} a_i p^i,$$

where all a_i are integers in $\{0, 1, \cdots, p-1\}$. Moreover for a given positive integer m, the coefficients a_i in such p-adic expansion of m are determined uniquely.

Next we give the structure of POS-groups with cyclic Sylow 2-subgroups of order 4.

Theorem 3.6. *Let G be a POS-group with a cyclic Sylow 2-subgroup of order 4. Then 3 is a divisor of $|G|$, or G is one of the following groups:*

(*a*) *the cyclic group of order 4;*
(*b*) *Frobenius groups $Z_{5^m} : Z_4$;*
(*c*) *quasi-dihedral groups $\langle a, b \mid a^{5^m} = b^4 = 1, a^b = a^{-1} \rangle$.*

Proof. Let H be the normal 2-complement. If $4 \nmid p - 1$ for every $p \in \pi(H)$, then $3 \in \pi(H)$ since the smallest prime number in $\pi(H)$ is a Fermat prime. Next we assume that $\pi(H)$ has a prime p such that $4 | p-1$. Then H is a C_{pp}-group, and hence H is Frobenius or 2-Frobenius by Lemma 2.2. Note that $\pi(H)$ has only such prime p (see [11]). We should consider the following three cases.

Case I. H is Frobenius. Let $H = K : L$ with the kernel K and the complement L. If L is a p-group, then L is cyclic. Since $f_H(p) = (p-1)|K|$ divides $4|K| \cdot |L|$, we have $p = 5$. In addition, since K is nilpotent, K has at least two prime divisors. If $|\pi(K)| = 1$, clearly

$3 \in \pi(H)$. Next let $\pi(K) = \{p_1, p_2\}$. Suppose that P_i is a Sylow p_i-subgroup of K for $i = 1, 2$. Then $f_H(p_i)$ divides $4|P_1|\cdot|P_2|\cdot|L|$. If $3 \nmid |H|$, then we assume that $5 < p_1 < p_2$. Note that $p_i \nmid f_H(p_i)$, so we have $p_1 = 2 \cdot 5^k + 1$ with $k \geq 1$. Set $p_2 = 2 \cdot 5^{k_1} p_1^{k_2} + 1$. Let u be an element of order 4. By Theorem 2.4, u is a fixed-point-free automorphism of H. Next we will prove that K is abelian. Otherwise, we assume that $H_0 = K_0 : L_0$ is the minimal counterexample. Thus K_0 is a p_1-group. Set $\Phi(K_0)$ the Frattini subgroup of K_0. Clearly $\Phi(K_0) > 1$. Since $\Phi(K_0)$ is a characteristic subgroup of K_0, $K_0/\Phi(K_0) : L_0$ has a fixed-point-freely automorphism of order 4. So $K_0/\Phi(K_0)$ is abelian, and then K_0 is abelian, a contradiction. Thus K is abelian. Let $|G| = 4 \cdot 5^m p_1^a p_2^b$. Since K is abelian, we may assume that $f_H(p_i) = p_i^{s_i} - 1$ for $i = 1, 2$. In addition, since $f_H(p_i)$ divides $|G|$, we get a Diophantine equation

$$p_i^{s_i} - 1 = 2^u 5^j p_1^s p_2^t \tag{3.1}$$

where $j, s, t \geq 0$ and $1 \leq u \leq 2$. Next we will prove that $s_i = 1$ for $i = 1, 2$. The following two subcases, should be studied.

Subcase I.I $i = 1$. Then clearly $s = 0$. In view of Lemma 3.4, $p_1^{s_1} - 1$ has a primitive prime divisor except if p_1 is a Mersenne prime and $s_1 = 2$. If $t = 0$, since $\pi(p_1 - 1) = \{2, 5\}$, we have $s_1 = 1$ or 2 by Lemma 3.4. Now if $s_1 = 2$, then p_1 is a Mersenne prime, say $2^l - 1$. So $s_1^2 - 1 = 2^{l+1}(2^{l-1} + 1) = 2^u 5^j$, then $l = 1$ since $u \leq 2$, a contradiction. When $t > 0$, the equation (3.1) becomes

$$p_1^{s_1} - 1 = 2^u 5^j p_2^t. \tag{3.2}$$

Similarly, since $\pi(p_1 - 1) = \{2, 5\}$, then s_1 is equal to 1 or a prime by Lemma 3.4. Since p_2 is a primitive prime divisor of $p_1^{s_1} - 1$, $s_1 | p_2 - 1 = 2 \cdot 5^{k_1} p_1^{k_2}$. So $s_1 = 1, 2, 5$ or p_1. If $s_1 = 2$, it is easy to see $8 \mid p_1^2 - 1 = f_H(p_1)$, a contradiction. If $s_1 = 5$, by Lemmas 3.5 and 3.6, then (3.2) becomes

$$\frac{p_1^5 - 1}{5(p_1 - 1)} = p_2^t. \tag{3.3}$$

If $k_2 > 0$, then (3.3) becomes

$$16 \cdot 5^{4k-1} + 8 \cdot 5^{3k} + 8 \cdot 5^{2k} + 4 \cdot 5^k + 1 = (2 \cdot 5^{k_1}(2 \cdot 5^k + 1)^{k_2} + 1)^t. \tag{3.4}$$

(3.4) is the expansion of p_2^t, if $k > 0$ in base 5. The left hand side modulo 5^{2k} is $4.5^k + 1$. Clearly, $t < 5$. So $k = k_1$ and $t = 2$. Moreover, the 5-adic expansion of the left term of (3.4) is $3 \cdot 5^{4k} + 5^{4k-1} + 5^{3k-1} + 3 \cdot 5^{3k} + 5^{2k+1} + 3 \cdot 5^{2k} + 4 \cdot 5^k + 1$. But the largest digit of one of the right term is more than or equal to $2k(k_2 + 1)$, so $2k(k_2 + 1) \leq 4k$, then $k_2 = 1$. It follows that the right term of (3.4) is equal to $16\cdot 5^{4k}+16\cdot 5^{3k}+12\cdot 5^{2k}+4\cdot 5^k+1$, a contradiction.

If $k_2=0$, then (3.3) becomes

$$16\cdot 5^{4k-1}+8\cdot 5^{3k}+8\cdot 5^{2k}+4\cdot 5^k+1=(2\cdot 5^{k_1}+1)^t. \tag{3.5}$$

(3.5) is the expansion of p_2^t in base (5) if $k_2=0$, then we may see $k=k_1$. Comparing the largest digits of the formulas of 5-adic expansion of both sides of (3.5), we got that $t\le 4$. It is easy to check that for every such t the equation (3.5) does not hold.

If $s_1=p_1$, then, by Lemmas 3.5 and 3.6, (3.2) becomes

$$\frac{p_1^{p_1}-1}{p_1-1}=p_2^t. \tag{3.6}$$

If $k_2=0$, then p_2 is a primitive prime divisor of $p_1^{p_1}-1$ since $p_1<p_2$. So $p_1\mid p_2-1=2\cdot 5^{k_1}$, and then $p_1=2$ or 5. By (3.6), we have $p_2^t=3$ or 781. Since $781=11\cdot 71$ has two prime divisors, it follows that $p_2=3$, which contradicts that $p_2>5$.

When $k_2>0$, we extend the number of (3.6) into the p_1-adic expansion, in which the first and second digits of the left and right are p_1+1 and $l\cdot p_1^{k_2}+1$ and $p_1>l\equiv 2t\cdot 5^k(\mathrm{mod} p_1)$. So $k_2=1$ and $2t\cdot 5^k\equiv 1(\bmod p_1)$. It follows that $p_1\mid 2t\cdot 5^k-1+p_1=2\cdot 5^k(t+1)$, and then $t+1\equiv 0(\bmod p_1)$. On the other hand, $t<p_1$, we have $t=p_1-1$. Thus (3.6) is changed into

$$\frac{p_1^{p_1}-1}{p_1-1}=(2\cdot 5^k p_1+1)^{p_1-1}. \tag{3.7}$$

It is not hard to see that the largest digit of right hand of (3.7) is more than one of left term in light of the form of the p_1-adic expansion of both side of (3.7), a contradiction. Thus $s_1=1$.

Subcase I.II. $i=2$. Then (3.1) becomes

$$p_2^{s_2}-1=2^u 5^j p_1^s. \tag{3.8}$$

Clearly, $s_2\neq 2$ (otherwise $8\mid p_2^2-1$). If k_1 and k_2 are both more than 0, since $\pi(p_2-1)=\{2,5,p_1\}$, we have that $s_2=1$ by Lemma 3.4. If $k_1=0$, then 5 is a primitive prime divisor of $p_2^{s_2}-1$. So $s_2\,\big|\,5-1=4$, hence $s_2=1$ or 4. But if $s_2=4$, gives that $8\mid p_2^4-1$, a contradiction.

$k_2=0$, gives that $s_2\,\big|\,p_1-1=2\cdot 5^k$. So $s_2=5$. Next we may only consider following Diophantine equation by Lemmas 3.5 and 3.6 that is

$$\frac{p_2^5 - 1}{p_2 - 1} = 5p_1^s. \tag{3.9}$$

Using the same method as one of (3.3), we can get that the (3.9) also has no solution. Thus s_2 is also equal to 1. Therefore, K is cyclic. Since 5^m divides $f_H(p_i)$, we have $p_1 = 2 \cdot 5^m + 1$ and $p_2 = 2 \cdot 5^m \cdot p_1^{k_2} + 1$ with $k_2 \geq 0$. Now note that $f_H(p_1^a p_2^b) = \phi(p_1^a p_2^b) = 4 \cdot 5^{2m} p_1^{k_2+a-1} p_2^{b-1}$ divides $|G| = 4 \cdot 5^m p_1^a p_2^b$, a contradiction.

If K is a p-group, then $f_H(q)$ divides $4|L|$ for every $q \in \pi(L)$. But since $4 \nmid q - 1$ and the smallest prime in $\pi(L)$ is a Fermat one, we have $3 \in \pi(L)$.

Case II. H is 2-Frobenius. Similarly, assume that $H = ABC$, where A, B, C are the same as above. Clearly, the commutator subgroup $H' = AB$. If $3 \nmid |H|$, by Theorem 2.4, H admits a fixed-point-freely automorphism of order 4. Then H' is nilpotent (see Exercises 1, Chap. 10, [7]), a contradiction.

Case III. H is a p-group. Certainly, $p = 5$. By Proposition 2.8 of [2], the Sylow 5-subgroup, that is H, is cyclic. Let $|H| = 5^m$. By Theorem 2.4 we have that the Sylow 2-subgroup of G is self-centralized. So G is not cyclic. Then $G = \langle a, b \mid a^{5^m} = b^4 = 1, a^b = a^r \rangle$ such that $r^4 \equiv 1 (\operatorname{mod} 5^m)$ and $(r - 1, 5^m) = 1$. If the centralizer of a is $\langle a \rangle$, then G is Frobenius by Lemma 3.3. If $|C_G(a)| = 2 \cdot 5^m$, then $r^2 \equiv 1 (\operatorname{mod} 5^m)$. Since $(r - 1, 5^m) = 1$, we have $r \equiv -1 (\operatorname{mod} 5^m)$. Therefore, $G = \langle a, b \mid a^{5^m} = b^4 = 1, a^b = a^{-1} \rangle$. $\square$

4. POS-groups with two prime divisors

In this section, assume that $|G| = 2^n p^m$ with p an odd prime number. If G is a POS-group, then p is a Fermat prime, say $2^{2^k} + 1$. By Proposition 3.1 of [2], we know that if $2^n < (p - 1)^3$, i. e., $n < 3 \cdot 2^k$, then the Sylow p-subgroup is cyclic and normal. Certainly there exists a POS-group with non-normal cyclic Sylow subgroups, such as $SL_2(3)$ of order 24. In this section, we give the structure of G with cyclic Sylow p-subgroups. First we cite some lemmas.

Lemma 4.1. *(Theorem 1, [1]). Let G be a 2-group of order 2^n and $\exp(G) = 2^e > 2$. Then the number of elements of order 2^i is a multiple of 2^i for $2 \leq i \leq e$ except in the following cases:*

(*a*) *the cyclic 2-group;*

(*b*) *the dihedral 2-group* $\langle a, b \mid a^{2^{n-1}} = b^2 = 1, a^b = a^{-1}\rangle$;
(*c*) *the semi-dihedral 2-group* $\langle a, b \mid a^{2^{n-1}} = b^2 = 1, a^b = a^{2^{n-2}-1}\rangle$;
(*d*) *the generalized quaternion 2-group* $\langle a, b \mid a^{2^{n-1}} = 1, a^{2^{n-2}} = b^2, a^b = a^{-1}\rangle$.

Lemma 4.2. *Let G be a finite group with a normal subgroup N. If $x \in G\backslash N$ has order m, then $f_{Nx}(m) = f_{Ny}(m)$ for all cosets Ny which are G/N-conjugate to Nx.*

Proof. Suppose that Ny is G/N-conjugate to Nx, so $Ny = Ng^{-1}xg$ for some $g \in G$. Then the map $\varphi : Nx \mapsto Ny$, defined by $nx \mapsto g^{-1}nxg$, induces a bijection between the subset of elements of order m in Nx and the corresponding subset of Ny. □

In the following we give the structure of POS-groups with two prime divisors and cyclic Sylow p-subgroups.

Theorem 4.3. *Let G be a POS-group with a cyclic Sylow p-subgroup P and $|\pi(G)| = 2$. Then G is a Frobenius group $Z_{p^m} : Z_{2^{2^k}}$, where $p = 1 + 2^{2^k}$ is a Fermat prime and $m > 0$ arbitrary, or satisfies one of the following conditions:*
(*a*) $p = 3$, $C_G(P) \cong P \times Z_2 \times Z_2$ *and* $N_G(P) \cong P : (Z_2 \times Z_4)$;
(*b*) *the number of elements of order 2 of G is 1;*
(*c*) *G is p-nilpotent.*

Proof. Suppose that $P = \langle x\rangle$ is a Sylow p-subgroup and the number of Sylow p-subgroups is $|G : N_G(P)| = 2^t$. By Zassenhaus's theorem we may let $N = N_G(P) = P : K$ and $C = C_G(P) = P \times U$, where K and U are 2-subgroups of G. Since $C_G(x^g) = C_G(x)^g$ and $C_G(x) = C_G(\langle x\rangle)$, we have that the number $f_G(2p^m)$ of elements of order $2p^m$ is

$$2^t\phi(p^m)f_{C_{G(x)}}(2) = 2^{t+2^k}p^{m-1}f_{C_{G(x)}}(2),$$

where ϕ is Euler totient function. Let $|G| = 2^np^m$, where $p = 2^{2^k} + 1$ is Fermat. Since $f_G(2p^m)$ is a divisor of $|G| = 2^np^m$ and $f_{C_{G(x)}}(2)$ is odd or 0, it leads to $f_{C_{G(x)}}(2) = 0, 1$ or p. Now note that

$$K/U \cong N/C \lesssim Aut(P) \cong Z_{2^{2^k}p^{m-1}},$$

so U/K is cyclic. If $K/U = 1$, that is $N = C$, then G is p-nilpotent by the well-known Burnside's theorem. If $K/U \neq 1$, then $N/U \cong P : K/U$ has no element of order $2p^m$. In fact, otherwise we may choose an element $y \in K\backslash U$ such that $xU^{yU} = xU$, and then $x^{-1}x^y \in U$. Since

$\langle x\rangle = P \lhd G$, we have $x^{-1}x^y \in P$. So $x^{-1}x^y \in U \cap P = 1$, then $x^y = x$. Hence $y \in U$, a contradiction. Therefore N/U is a Frobenius group by Lemma 3.3. If $U \neq 1$, that is $f_U(2) \neq 0$, then we have

$$f_G(2) = f_U(2) + f_{N\setminus U}(2) + f_{G\setminus N}(2) = f_U(2) + p^m f_{yU}(2) + f_{G\setminus N}(2)$$

by Lemma 4.2, where $o(yU) = 2$. If $f_{yU}(2) \neq 0$, then $f_G(2) > p^m$, which contradicts that $f_G(2) \mid p^m$. So $f_G(2) = f_U(2) + f_{G\setminus N}(2) = f_{\bigcup_{g\in G} U^g}(2) + f_{G\setminus \bigcup_{g\in G} K^g}(2)$. If $f_U(2) = 1$, denote by z the unique element of order 2 in U, then $f_{\bigcup_{g\in G} U^g}(2) = |G : C_G(z)|$. So we may assume that $f_{\bigcup_{g\in G} U^g}(2) = 2^s$. Now we choose any element a of order 2 in $G\setminus \bigcup_{g\in G} K^g$, we see that $p^m \nmid |C_G(a)|$, hence $p \mid f_{G\setminus \bigcup_{g\in G} K^g}(2)$. We set $f_{G\setminus \bigcup_{g\in G} K^g}(2) = p\cdot l$. Now use class equation of G, we can obtain $f_G(2) = 2^s + p\cdot l \mid p^m$. So $s = l = 0$, and then $f_G(2) = 1$. Next we consider the case of $f_U(2) = p$. Now assume that $|U| = 2^u$.

If $f_U(4) = 0$, then U is an elementary abelian 2-group. So $f_U(2) = 2^u - 1 = p$. It follows that $u = 2$ and $k = 0$. By Lemma 2.3, we have $2^n \mid f_G(p^m) + f_G(2p^m) = 2^{t+2^k}p^{m-1}(1+p) = 2^{t+3}p^{m-1}$, and then $n \leq t+3$. So $|K| = 2^{n-t} \leq 2^3$. It is easy to see that K is $Z_2 \times Z_4$.

If $f_U(4) \neq 0$, then $f_G(4p^m) = 2^{t+2^k}p^m f_U(4) \mid 2^n p^m$. So $f_U(4) \mid 2^{n-t-2^k}p$. On the other hand, since

$$4 \mid 1 + f_U(2) + f_U(4) = 2 + 2^{2^k} + f_U(4),$$

and $f_U(4) \mid 2p$. By Lemma 4.1, we have that U is a dihedral or semi-dihedral 2-group. If U is a semi-dihedral one, then $f_U(2) = 1 + 2^{u-2}$ and $f_U(4) = 2 + 2^{u-2}$, which contradicts that $f_U(2) = p$ and $f_U(4) \mid 2p$. If U is a dihedral one, then $f_U(2) = 2^{u-1} + 1 = p$ and $f_U(2^i) = \phi(2^i) = 2^{i-1}$ for $1 \leq i \leq u-1$. Certainly, $u = 1 + 2^{2^k}$ since $2^{u-1} + 1 = p$, we get that

$$|U| = 2^{2^k+1}. \tag{4.1}$$

Moreover, since K/U is a cyclic 2-group, we have that there exists $L \lhd K$ such that $L/U \cong Z_2$. By the above discussion, we may see that L has no element of order 2. In addition, $L\setminus U$ has an element of order 4. In fact, otherwise $f_L(4) = 2$, by Lemma 4.1, L is a cyclic, dihedral, semi-dihedral or generalized quaternion 2-group, which have an element of order 4 in $L\setminus U$, a contradiction. So $f_G(4) > 2$. Next we discuss the number of elements of order 4 in G. Clearly,

$$f_G(4) = f_{\bigcup_{g\in G} U^g}(4) + f_{G\setminus \bigcup_{g\in G} U^g}(4).$$

Note that two elements of order 4 in U are conjugate, so the elements of order 4 in $\bigcup_{g\in G} U^g$ make one congugacy class of G. Thus

$$f_{\bigcup_{g\in G} U^g}(4) = |G : C_G(w)|,$$

where w is of order 4 in U. Since $p^m \mid |C_G(w)|$, we may set $f_{\bigcup_{g\in G} U^g}(4) = 2^s$. In addition, obviously $p \mid f_{G\setminus \bigcup_{g\in G} K^g}(4)$. Assume that $f_{G\setminus \bigcup_{g\in G} K^g}(4) = p \cdot l$. Then we have $f_G(4) = 2^s + p \cdot l \mid 2^n p^m$, hence $f_G(4) = 2^j$ for $1 \leq j \leq n$. Since $f_G(2) \mid p^m$, we may set $f_G(2) = p^i$. By Frobenius's theorem, we have

$$4 \mid 1 + f_G(2) + f_G(4) = 1 + 2^j + p^i.$$

Since $f_G(4) > 2$, we have $4 \mid 1 + p^i$, and thus $k = 0$ and i is odd. By Lemma 2.3 we have $2^n \mid f_G(p^m) + f_G(2p^m) + \cdots + f_G(2^{u-1}p^m) = 2^{t+1}p^{m-1}(1 + p + f_U(4)) + \cdots + f_U(2^{u-1})) = 2^{t+2}p^{m-1}(1 + 2^{u-2})$, and so $n \leq t + 2$. It leads to

$$|K| = 2^{n-t} \leq 2^2. \tag{4.2}$$

By (4.1) we may get $U = K$, which contradicts that $U \neq K$.

If $U = 1$, then $N = P : K$ is a Frobenius group and K is cyclic. Then the number $f_N(2)$ of order 2 in N is equal to $p^m f_K(2)$. Since $f_K(2) \geq 1$, we have $f_G(2) \geq f_N(2) \geq p^m$. On the other hand, $f_G(2) \mid p^m$. Hence $f_G(2) = p^m$. By Lemma 2.3, we have $2^n \mid f_G(p^m)$, and $f_G(p^m) = 2^t \cdot \phi(p^m)$. Hence $t = n - 2^k$. Next we use induction to prove that $f_G(2^i) = 2^{i-1}p^m$ for $1 \leq i \leq 2^k$. Clearly when $i = 1$, it is true. Assume that it is true for $i = j - 1$. Now we deal with the case of $i = j$. Since

$$2^j \mid 1 + f_G(2) + \cdots + f_G(2^j) = 1 + p^m + \cdots + 2^{j-2}p^m + 2^{j-1}p^m + f_{G\setminus N}(2^j)$$

and $p^m \equiv 1 (\operatorname{mod} 2^j)$, we have $2^j \mid f_{G\setminus N}(2^j)$. On the other hand, since $f_G(2^j) = 2^{j-1}p^m + f_{G\setminus N}(2^j) \mid 2^n p^m$, we have $2^j \nmid f_G(2^j)$. Hence $f_G(2^j) = 2^{j-1}p^m$. Since every Sylow 2-subgroup of G has at most one subgroup of order 2 of N (otherwise the generated subgroup by some two elements of order 2 of N is a Frobenius group, which is not a 2-group), we have that the number of Sylow 2-subgroups is p^m, and then the intersection of every pair Sylow 2-subgroups is trivial. Thus G is a Frobenius group. It leads to $t = 0$, that is $n = 2^k$, hence $G \cong Z_{p^m} : Z_{2^{2^k}}$ is a Frobenius group. □

Note that there exist groups satisfying the condition (a) of Theorem 4.3. We give an example as follows.

Example 4.4. *$\langle a, b, c \mid a^{3^m} = b^2 = c^4 = 1, a^b = a, a^{c^2} = a, a^c = a^{-1}, [b, c] = 1\rangle$ is a POS-group of order $8 \cdot 3^m$ with cyclic Sylow 3-subgroups.*

Using the GAP software [8], it seems that the group satisfying the condition (a) has not been found except those of Example 4.4. We put the following conjecture.

Conjecture 4.5. *POS-groups satisfying the condition (a) of Theorem 4.3 are those of Example 4.4.*

It is not hard to determine groups satisfying the condition (b) of Theorem 4.3. Since the number of elements of order 2 of G is 1, so is for a Sylow 2-subgroup P_2 of G. Thus P_2 is cyclic or a generalized quaternion group. Those groups were classified by Zassenhaus in [18]. The Table 1 lists all such groups with two prime divisors.

Table 1.

Type	*Order*	*Generators*	*Relations*	*Conditions*
I	$2^n p^m$		*cyclic group*	
II	$2^n p^m$	a, b	$a^{p^m} = b^{2^n} = 1,$ $a^b = a^r$	$(r-1, p^m) = 1,$ $r^{2^n} \equiv 1 (\operatorname{mod} p^m)$
III	$2^{n+1} p^m$	a, b, c	$b^{2^{n-1}} = c^2,$ $a^c = a^s, b^c = b^{-1}$	$n \geq 2,$ $s^2 \equiv 1 (\operatorname{mod} p^m)$

We now give the following result. Note that $p = 2^{2^k} + 1$ is a Fermat prime.

Theorem 4.6. *Let G be a POS-group with a cyclic Sylow p-subgroup and $|\pi(G)| = 2$. If the number of elements of order 2 of G is 1, then G is one of the following groups:*

(a) cyclic groups $Z_{2^n 3^m}$;

(b) groups $\langle a, b \mid a^{p^m} = b^{2^n} = 1, a^b = a^r\rangle$, where $ord_{p^m}(r) \geq 2^k$;

(c) groups $\langle a, b \mid a^{p^m} = b^{2^{2^k+1}} = 1, b^{2^{2^k}} = c^2, a^b = a^r, a^c = a^{-1}, b^c = b^{-1}\rangle$, where $ord_{p^m}(r) \geq 2^k$.

Proof. Clearly if G is cyclic, then $G \cong Z_{2^n 3^m}$. Let $ord_{p^m}(r) = o(r)$. Assume that G is of Type II. Since $a^b = a^r$ and $o(r) = ord_{p^t}(r)$ for

$1 \le t \le m$, we have $(a^{p^i})^{b^{o(r)}} = (a^{p^i})^{r^{o(r)}} = a^{p^i}$ with $1 \le i \le m-1$. So G has an element of order $2^{n-o(r)}p^{m-i}$ and the number of these elements is $\phi(2^{n-o(r)}p^{m-i}) = 2^{2^k+n-o(r)-1}p^{m-i-1}$. Then $2^k + n - o(r) - 1 \le n$, that is $o(r) \ge 2^k - 1$. Clearly $o(r)|2^{2^k}$, thus $o(r) \ge 2^k$. For other order 2^i of the element $x_i \in G$ for $1 \le i \le n$, the number of these elements is $\phi(2^i) \cdot |G : N_G(\langle x_i\rangle)|$, which is a divisor of $|G|$.

Assume that G is of Type III. If $s \equiv 1(\mathrm{mod}\, p^m)$, then $f_G(4) = 2|\langle a,b\rangle : N_{\langle a,b\rangle}(\langle x\rangle)| + (2^n p^m - p^m + 1)$, where $x \in \langle a,b\rangle$ is of order 4. We may assume that $|\langle a,b\rangle : N_{\langle a,b\rangle}(\langle x\rangle)| = p^t$. Then $f_G(4) = 2p^t + (2^n - 1)p^m + 1$ is a divisor of $2^{n+1}p^m$. So $t = 0$ and $p = 3$, we may get a Diophantine equation

$$1 + (2^n - 1) \cdot 3^{m-1} = 2^i. \tag{4.3}$$

Clearly, $n|i$. So

$$\frac{2^i - 1}{2^n - 1} = 3^{m-1}. \tag{4.4}$$

By Lemma 3.4, we have that 3 is a primitive prime divisor of $2^2 - 1$. It follows that $i = 2$ or 6. Thus $n = 1$ and $m = 2$, or $n = 3$ and $m = 3$. Since $n \ge 2$, we have $n = 3$ and $m = 3$, that is $G = \langle a, b, c \,|a^{27} = b^8 = 1, a^b = a^r, b^4 = c^2, a^c = a, b^c = b^{-1}\rangle$. Using the GAP [8], we checked $r = 1$ or -1, G is not a POS-group.

If $s \equiv -1(\mathrm{mod}\, p^m)$, then all elements of $G \backslash \langle a,b\rangle$ are of order 4. So $f_G(4) = 2p^t + 2^n p^m$, where $2p^t$ is the number of elements of order 4 in $\langle a,b\rangle$. Then we can get an equation as follows, that is

$$2p^t + 2^n p^m = 2^i p^j. \tag{4.5}$$

Clearly, $i = 1$. Then (4.5) becomes $p^t + 2^{n-1}p^m = p^j$. So $j > t$. Thus the (4.5) becomes

$$1 + 2^{n-1}p^{m-t} = p^{j-t}. \tag{4.6}$$

Since $j - t > 0$, we have $m = t$. So the (4.6) becomes

$$2^{n-1} = p^{j-t} - 1. \tag{4.7}$$

Since 2 is a primitive prime divisor of $p - 1$, by Lemma 3.4 we have $j - t = 1$ in (4.7). Then $n = 2^k + 1$. Since $\langle a,b\rangle$ is same as one of Type II, $f_G(2^{n-o(r)}p^{m-i}) = \phi(2^{n-o(r)}p^{m-i}) = 2^{2^k+n-o(r)-1}p^{m-i-1}$. So $2^k + n - o(r) - 1 \le n + 1$, then $o(r) \ge 2^k$. Thus $G = \langle a, b \mid a^{p^m} = b^{2^{2^k+1}} = 1, a^b = a^r, b^{2^{2^k}} = c^2, a^c = a^{-1}, b^c = b^{-1}\rangle$. $\square$

For the remain part (c) of Theorem 4.3, we may get the following result.

Theorem 4.7. *Let G be a POS-group with a cyclic Sylow p-subgroup P and $|\pi(G)| = 2$. If G is p-nilpotent, then $G \cong Z_{2^n 3^m}$, $D_8 \times Z_{5^m}$, $Q_{2^{2^k+2}} \times Z_{p^m}$, where $p = 2^{2^k}+1$ a Fermat prime, or satisfies the condition that $p = 3$ and $N_G(P) = C_G(P) \cong Z_2 \times Z_2 \times P$.*

Proof. By the proof of Theorem 4.3 we see that $N_G(P) = C_G(P)$. Let $N = N_G(P) = P \times U$, $|G : N| = 2^t$ and P_2 be the Sylow 2-subgroup. Then $f_G(2p^m) = 2^{t+2^k} p^{m-1} f_U(2)$, so $f_U(2) = 1$ or p. We divide into two cases.

Case (a). $f_U(2) = 1$.

Then U is cyclic or a generalized quaternion group. If U is cyclic, then

$$f_G(2^{n-t}p^m) = 2^{t+2^k} p^{m-1} 2^{n-t-1} = 2^{2^k+n-1} \mid 2^n p^m,$$

and so $2^k \leq 1$, that is $p = 3$. Since we may assume that $f_{\bigcup_{g\in G} U^g}(4) = 2^i$. Also

$$f_G(4) = f_{\bigcup_{g\in G} U^g}(4) + f_{P_2 \setminus \bigcup_{g\in G} U^g}(4).$$

Since $3 \mid f_{P_2 \setminus \bigcup_{g\in G} U^g}(4)$, we have $f_G(4)$ is a power of 2, say 2^j. Let $f_G(2) = 3^h$ for $h \geq 0$. Now assume that P acts on the set Ω of elements of order 2 of G, we have

$$f_G(2) = 3^h \equiv |C_\Omega(P)| = 1(\mathrm{mod} 3).$$

Thus $h = 0$, i.e. $f_G(2) = 1$. On the other hand, by Frobenius's theorem we can get that $4 \mid 1 + f_G(2) + f_G(4) = 2 + 2^j$, and then $f_G(4) = 2$. By Lemma 4.1, P_2 is also cyclic. Therefore, G is 2-nilpotent, that is $G \cong Z_{2^n 3^m}$.

If U is a generalized quaternion group, then $f_U(4) = 2^{n-t-1} + 2$. So

$$f_G(4p^m) = 2^{t+2^k+1} p^{m-1} (2^{n-t-2} + 1) \mid 2^n p^m,$$

then $p = 2^{n-t-2} + 1$. Thus $n - t = 2^k + 2$. Let $f_G(2) = p^h$ for $h \geq 0$. Similarly, assume that P acts on the set Ω of elements of order 2 of G, we have

$$f_G(2) = p^h \equiv |C_\Omega(P)| = 1(\mathrm{mod} p).$$

Thus $h = 0$, i.e., $f_G(2) = 1$. By Lemma 4.1, P_2 is also a generalized quaternion group. Choose an element x of order 2^{n-1}, then $\langle x\rangle$ is characteristic in P_2. So $\langle x\rangle$ is normal in G. Apply the N/C-theorem, we have $N_G(\langle x\rangle)/C_G(x) \lesssim Aut(\langle x\rangle)$, then $C_G(x) = \langle x\rangle \times P$. It leads to $x \in U$, so $P_2 = U$. Therefore $G \cong Q_{2^{2^k+2}} \times Z_{p^m}$, where $p = 2^{2^k} + 1$.

Case (b). $f_U(2) = p$.

Similarly, if $f_U(4) \neq 0$, then $f_U(4) \mid 2^n p$. On the other hand, since $4 \mid 1 + f_U(2) + f_U(4)$, it follows that $f_U(4)$ =2 or $2p$. By Lemma 4.1, U is a dihedral or semi-dihedral group. If U is a dihedral one, then $f_U(2) = 1 + 2^{n-t-1}$ and $f_U(4) = 2$. So $n - t = 2^k + 1$. Also U has an element of order 2^{n-t-1}, so $f_G(2^{n-t-1}p^m) = 2^{t+2^k+n-t-2}p^{m-1} = 2^{n+2^k-2}p^{m-1}$. Then $n + 2^k - 2 \leq n$, that is $k = 1$. Thus $p = 5$ and $U \cong D_8$. Since two elements of order 4 of U are conjugate, we have that $f_{\bigcup_{g\in G} U^g}(4)$ is a 2-power, say 2^i. Also clearly

$$f_G(4) = f_{\bigcup_{g\in G} U^g}(4) + f_{P_2 \setminus \bigcup_{g\in G} U^g}(4).$$

But $5 \mid f_{P_2 \setminus \bigcup_{g\in G} U^g}(4)$ and $f_G(4) \mid 2^n 5^m$, so $5 \nmid f_G(4)$. Let $f_G(4) = 2^h$ for $h \geq 1$. Set $f_G(2) = 5^j$ for $j \geq 1$. By Frobenius's theorem we can obtain that $4 \mid 1 + 2^i + 5^j$, so $h = 1$, that is $f_G(4) = 2$. In view of Lemma 4.1, it is easy to see that P_2 is also a dihedral group. Obviously, for all $x \in P$, the element x is an automorphism of P_2. On the other hand, the order of the automorphism group of a dihedral 2-group is still a 2-group, so P acts trivially on P_2. It leads to $G = P \times P_2$, i.e., $G \cong Z_{5^m} \times D_{2^n}$. Moreover, $f_G(2) = 1 + 2^{n-1}$. So we get a Diophantine equation $1 + 2^{n-1} = 5^j$. By Lemma 3.4, the solution is $n = 3$ and $j = 1$. Thus $G \cong D_8 \times Z_{5^m}$.

If U is a semi-dihedral one, then $f_U(2) = 1 + 2^{n-t-2}$ and $f_U(4) = 2 + 2^{n-t-2}$. Since $f_U(4) > 2$, $f_U(4) = 2f_U(2)$, which is impossible.

Next assume that $f_U(4) = 0$. Then U is an elementary abelian 2-group of order 2^{n-t}. We may let $U > 1$ (otherwise $f_G(p^m) = 2^{n+2^k}p^{m-1} \mid 2^n p^m$, a contradiction). By Lemma 2.3, we have $2^n \mid f_G(p^m) + f_G(2p^m) = 2^{t+2^k}p^{m-1} + 2^{t+2^k}p^m$, so $n - t \leq 2^k + 1$. In addition, let P act on the set of elements of order 2 in G, so we have

$$f_G(2) \equiv f_U(2) = 2^{n-t} - 1 (\mathrm{mod}\, p). \tag{4.8}$$

We make the equation (4.8) into two cases to consider.

Case I. $f_G(2) = f_U(2) = 2^{n-t} - 1$. Then $2^{n-t} - 1 \mid p^m$, and hence $n - t = 2$ and $p = 3$.

Case II. $f_G(2) > f_U(2)$. Then $n - t > 2^k$, and thus $n - t = 2^k + 1$. Since $f_G(2) \mid p^m$, we have $p \mid f_U(2) = 2^{2^k+1} - 1$. So $p = 3$ and $n - t = 2$. $\square$

Using GAP software, we checked all small POS-groups G ($|G| \leq 2000$), G has a cylic Sylow p-subgroup or a generalized quaternion Sylow 2-group or a normal p-complement or a normal Sylow p-subgroup for every $p \in \pi(G)$. We put a conjecture to close this note.

Conjecture 4.8. *Let G be a POS-group and $p \in \pi(G)$. Then G satisfies one of following conditions:*

(a) *G has a cylic Sylow p-subgroup or a generalized quaternion Sylow 2-group;*

(b) *G has a normal p-complement;*

(c) *G has a normal Sylow p-subgroup.*

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References

[1] Ya. Berkovich, On p-groups of finite order, *Siberian Math. J.* **9** (1968) 963–978.
[2] A. K. Das, On finite groups having perfect order subsets, *Int. J. Algebra* **13** (2009), no. 3, 629–637.
[3] W. Feit, On large Zsigmondy primes, *Proc. Amer. Math. Soc.* **102** (1988), no. 1, 29–36.
[4] C. E. Finch and L. Jones, A curious connection between Fermat numbers and finite groups, *Amer. Math. Monthly* **109** (2002), no. 6, 517–524.
[5] C. E. Finch and L. Jones, Nonabelian groups with perfect order subsets, *JP J. Algebra Number Theory Appl.* **3** (2003), no. 1, 13–26.
[6] C. E. Finch and L. Jones, Corrigendum to: "Nonabelian groups with perfect order subsets", *JP J. Algebra Number Theory Appl.* **4** (2004), no. 2, 413–416.
[7] D. Gorenstein, Finite Groups, Chelsea Publishing Co., New York, 1980.
[8] GAP software, http://www.gap-system.org/.
[9] M. Hall, The Theory of Groups, The Macmillan Co., New York, 1959.
[10] G. Higman, Finite groups in which every element has prime power order, *J. London Math. Soc.* **32** (1957) 335–342.
[11] T. M. Keller, Solvable groups with a small number of prime divisors in the element orders, *J. Algebra* **170** (1994), no. 2, 625–648.
[12] G. Malle, A. Moreto and G. Navarro, Element orders and Sylow structure, *Math. Z.* **252** (2006), no. 1, 223–230.
[13] P. Ribenboim, The Book of Prime Number Records, Second Edition, Springer-Verlag, New York, 1989.

[14] R. Shen, A note on finite groups having perfect order subsets, *Int. J. Algebra* **13** (2010), no. 4, 643–646.
[15] J. S. Williams, Prime graph components of finite groups, *J. Algebra* **69** (1981), no. 2, 487–513.
[16] L. Weisner, On the number of elements of a group which have a power in a given conjugate set, *Bull. Amer. Math. Soc.* **31** (1925), no. 9–10, 492–496.
[17] K. Zsigmondy, Zur Theorie der Potenzreste, *Monatsh. Math. und Phys.* **3** (1892), no. 1, 265–284.
[18] H. Zassenhaus, Überendliche Fastkörpe, *Abh. Math. Sem. Univ. Hamburg* **11** (1936), no. 1, 187–220.

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