

BIFLATNESS OF CERTAIN SEMIGROUP ALGEBRAS

M. ESSMAILI* AND A. MEDGHALCHI

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ABSTRACT. In the present paper, we consider biflatness of certain classes of semigroup algebras. Indeed, we give a necessary condition for a band semigroup algebra to be biflat and show that this condition is not sufficient. Also, for a certain class of inverse semigroups S , we show that the biflatness of $\ell^1(S)''$ is equivalent to the biprojectivity of $\ell^1(S)$.

1. Introduction

The concepts of biprojectivity and biflatness of Banach algebras were introduced by Khelemskiĭ in [10]. These notions have been studied for various classes of Banach algebras such as C^* -algebras, the group algebra $L^1(G)$ of a locally compact group G and Segal algebras. Recently these notions have been investigated for some classes of semigroup algebras; see [1, 7] and [12].

In [1], Y. Choi studied the biflatness of ℓ^1 -semilattice algebras and showed that for any semilattice S , the semigroup algebra $\ell^1(S)$ is biflat if and only if S is uniformly locally finite. Choi also characterized biflatness of Clifford semigroup algebras. Afterwards in [12], P. Ramsden extended these results to characterize the biflatness and biprojectivity of $\ell^1(S)$, for an inverse semigroup S . Briefly, the following results are obtained:

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*Corresponding author

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Theorem 1.1. ([12, Theorem 3.7]). *Let S be an inverse semigroup. Then*

- (i) $\ell^1(S)$ is biflat if and only if (S, \leq) is uniformly locally finite and each maximal subgroup of S is amenable.
- (ii) $\ell^1(S)$ is biprojective if and only if (S, \leq) is uniformly locally finite and each maximal subgroup of S is finite.

Also in [7], N. Grønbaek and F. Habibian considered the biflatness of semigroup algebras with a different approach. Precisely, they investigated the biflatness of ℓ^1 -graded Banach algebras over a semilattice and as an application characterized the biflatness of commutative semigroup algebras [7, Theorem 5.3].

Recently in [2], the authors studied simplicial cohomology of band semigroup algebras and showed that for every band semigroup S , the semigroup algebras of $\ell^1(S)$ is simplicially trivial, that is, the simplicial cohomology $\mathcal{H}^n(\ell^1(S), \ell^\infty(S))$ vanishes for each $n \geq 1$. Since every biflat Banach algebra is simplicially trivial it is natural to see that when band semigroup algebras are biflat. The aim of this paper is to investigate biflatness of certain semigroup algebras. Indeed, we give a partially answer to the above question. The paper is organized as follows:

In section 2, we recall some standard notations and define some basic concepts that we shall need. In section 3, we give a necessary condition for the biflatness of band semigroup algebras, by using the fact that each band semigroup is a semilattice of rectangular band semigroups (Theorem 3.6). We also investigate the biflatness of the second dual of semigroup algebras and show that for a certain class of inverse semigroups S , the biflatness of $\ell^1(S)''$ is equivalent to the biprojectivity of $\ell^1(S)$.

2. preliminaries

For a Banach algebra \mathcal{A} the projective tensor product $\mathcal{A} \hat{\otimes} \mathcal{A}$ is a Banach \mathcal{A} -bimodule with the following actions:

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in \mathcal{A}).$$

We define the multiplication map $\pi : \mathcal{A} \hat{\otimes} \mathcal{A} \rightarrow \mathcal{A}$ by

$$\pi(a \otimes b) = ab \quad (a, b \in \mathcal{A}).$$

Definition 2.1. *Let \mathcal{A} be a Banach algebra and $C > 0$. Then*

- (i) \mathcal{A} is biprojective if there is a bounded \mathcal{A} -bimodule homomorphism $\rho : \mathcal{A} \rightarrow \mathcal{A} \hat{\otimes} \mathcal{A}$ such that $\pi \circ \rho = I_{\mathcal{A}}$, where $I_{\mathcal{A}}$ is the identity map on \mathcal{A} . We say that \mathcal{A} is C -biprojective if $\|\rho\| \leq C$.
- (ii) \mathcal{A} is biflat if there is a bounded \mathcal{A} -bimodule homomorphism $\rho : \mathcal{A} \rightarrow (\mathcal{A} \hat{\otimes} \mathcal{A})''$ such that $\pi'' \circ \rho = \kappa_{\mathcal{A}}$, where $\kappa_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}''$ is the natural embedding of \mathcal{A} into its second dual. We say that \mathcal{A} is C -biflat if $\|\rho\| \leq C$.

It is clear that every biprojective Banach algebra is biflat. Also we recall that a Banach algebra \mathcal{A} is amenable if and only if it is biflat and has a bounded approximate identity.

Definition 2.2. Let S be a semigroup and $\ell^1(S) = \{f : S \rightarrow \mathbb{C} : \|f\|_1 = \sum_{s \in S} |f(s)| < \infty\}$. We define the convolution of two elements $f, g \in \ell^1(S)$ by

$$(f * g)(s) = \sum_{uv=s} f(u)g(v),$$

where $\sum_{uv=s} f(u)g(v) = 0$, when there are no elements $u, v \in S$ with $uv = s$. Then $(\ell^1(S), *, \|\cdot\|_1)$ becomes a Banach algebra.

We now bring some definitions and basic properties of semigroups. The standard reference for the algebraic theory of semigroups is [9].

Definition 2.3. Let S be a semigroup and let $E(S) = \{p \in S : p^2 = p\}$. We say that S is a semilattice if S is commutative and $E(S) = S$. The canonical partial order on $E(S)$ is given by

$$(2.1) \quad s \leq t \iff s = st = ts \quad (s, t \in E(S)).$$

An idempotent is called a minimal idempotent if it is a minimal element in $(E(S), \leq)$.

Definition 2.4. Let S be a semigroup. Then

- (i) S is a band semigroup if $S = E(S)$.
- (ii) S is a regular semigroup if for each $s \in S$ there exists $t \in S$ with $sts = s$.
- (iii) S is an inverse semigroup if for each $s \in S$ there exists a unique element $s^* \in S$ with $ss^*s = s$ and $s^*s^*s^* = s^*$.

Let S be an inverse semigroup. By [9, Proposition 5.2.1], there is a partial order on S defined by

$$(2.2) \quad s \leq t \iff s = ss^*t \quad (s, t \in S).$$

It is easy to see that the partial order given in (2.2) coincide with the one given in (2.1) on $E(S)$.

Definition 2.5. *Let (P, \leq) be a partially ordered set. Then*

- (i) (P, \leq) is locally finite if $[x] = \{y \in P : y \leq x\}$ is finite, for each $x \in P$.
- (ii) (P, \leq) is locally C -finite if $\sup\{|[x]| : x \in P\} \leq C$, for some $C \geq 1$.
- (iii) (P, \leq) is uniformly locally finite if it is locally C -finite, for some $C \geq 1$.

Remark 2.6. *Let S be an inverse semigroup. By [12, Proposition 2.14], (S, \leq) is uniformly locally finite if and only if $(E(S), \leq)$ is uniformly locally finite.*

3. Biflatness of certain semigroup algebras

In this section, we give a necessary condition for biflatness of a band semigroup algebra and with an example show that this condition is not sufficient. We also consider biflatness of the second dual of semigroup algebras, for a certain class of inverse semigroups.

Recall that S is a right (left) zero semigroup if for each $s, t \in S$, $st = t$ ($st = s$).

Proposition 3.1. *Let S be a right (left) zero semigroup. Then $\ell^1(S)$ is 1-biprojective, so it is 1-biflat.*

Proof. Suppose that S is a right zero semigroup. For each $f, g \in \ell^1(S)$ we have

$$\begin{aligned} f * g &= \sum_{r \in S} \left(\sum_{uv=r} f(u)g(v) \right) \delta_r = \left(\sum_{s \in S} f(s) \right) g \\ &= \varphi_S(f)g, \end{aligned}$$

where φ_S is the augmentation character on $\ell^1(S)$. Take an arbitrary element $t_0 \in S$ and define $\rho : \ell^1(S) \rightarrow \ell^1(S) \hat{\otimes} \ell^1(S)$ by

$$\rho(f) = \delta_{t_0} \otimes f \quad (f \in \ell^1(S)).$$

Then for each $f, g \in \ell^1(S)$ we have

$$\begin{aligned} \rho(f * g) &= \delta_{t_0} \otimes (f * g) = \varphi_S(f) \delta_{t_0} \otimes g \\ &= f \cdot (\delta_{t_0} \otimes g) = f \cdot \rho(g) \end{aligned}$$

and

$$\begin{aligned} \rho(f * g) &= \delta_{t_0} \otimes (f * g) = (\delta_{t_0} \otimes f) \cdot g \\ &= \rho(f) \cdot g. \end{aligned}$$

Moreover,

$$\begin{aligned} \pi \circ \rho(f) &= \pi(\delta_{t_0} \otimes f) = \delta_{t_0} * f \\ &= \varphi_S(\delta_{t_0})f = f. \end{aligned}$$

Thus $\ell^1(S)$ is 1-biprojective. The proof for a left zero semigroup is similar. \square

Proposition 3.2. *Let S be a rectangular band semigroup (i.e., S is an idempotent semigroup such that for each $x, y \in S$, $xyx = x$). Then $\ell^1(S)$ is 1-biprojective, so it is 1-biflat.*

Proof. Let S be a rectangular band semigroup. By [9, Theorem 1.1.3], S is isomorphic to $L \times R$ where L and R are left and right zero semigroups, respectively. So

$$\ell^1(S) \cong \ell^1(L \times R) \cong \ell^1(L) \hat{\otimes} \ell^1(R).$$

From Proposition 3.1, it follows that $\ell^1(L)$ and $\ell^1(R)$ are 1-biprojective. The result now follows from [12, Proposition 2.4]. \square

Theorem 3.3. *Let S be a rectangular band semigroup. Then $\ell^1(S)$ is amenable if and only if S is singleton.*

Proof. Let $\ell^1(S)$ be amenable. Then $E(S) = S$ is finite and by [3, Corollary 10.6], $K(S)$, the minimal ideal of S , is an amenable group. Also since S is an idempotent semigroup, $K(S)$ is singleton. Since S is finite, it has a minimal idempotent $p \in S$. By regularity of S and [3, Proposition 3.6], Sp is a minimal left ideal of S and

$$K(S) = \bigcup_{q \in S} (Sp)q = \{p\}.$$

So for each $q \in S$, we have $(Sp)q = \{p\}$. Thus for each $q \in S$, $qpq = p$ and since S is rectangular band semigroup we have $q = p$. Hence S is singleton. \square

Definition 3.4. *Let \mathcal{A} be a Banach algebra, Λ be a semilattice and $\{\mathcal{A}_\alpha : \alpha \in \Lambda\}$ be a collection of closed subalgebras of \mathcal{A} . Suppose that \mathcal{A} is ℓ^1 -direct sum of \mathcal{A}_α 's as Banach space such that*

$$\mathcal{A}_\alpha \mathcal{A}_\beta \subseteq \mathcal{A}_{\alpha\beta} \quad (\alpha, \beta \in \Lambda).$$

Then \mathcal{A} is called ℓ^1 -graded of \mathcal{A}_α 's over the semilattice Λ and is denoted by $\mathcal{A} = \bigoplus_{\alpha \in \Lambda}^{\ell^1} \mathcal{A}_\alpha$.

Our next aim is the characterization of amenability of band semigroup algebras. To do so we apply the following fact to band semigroups.

Let S be a semigroup and S^1 denote the unitization of S . We define the equivalence relation τ on S by

$$a\tau b \iff S^1aS^1 = S^1bS^1 \quad (a, b \in S).$$

If S is a regular semigroup, then by the argument in [9, Section 2.4],

$$a\tau b \iff SaS = SbS \quad (a, b \in S).$$

Now let S be a band semigroup. Then by [9, Theorem 4.4.1], S is a semilattice of rectangular band semigroups. Indeed, $S = \bigcup_{\alpha \in Y} S_\alpha$ where $Y = S/\tau$ and for each $\alpha = [s] \in Y$, $S_\alpha = [s]$.

In the following theorem, we characterize the amenability of band semigroup algebras.

Theorem 3.5. *Let S be a band semigroup. Then the following are equivalent:*

- (i) $\ell^1(S)$ is amenable.
- (ii) S is finite and each τ -class is singleton.
- (iii) S is a finite semilattice.

Proof. (i) \Rightarrow (ii) By the above argument, let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice of rectangular band semigroups and $\ell^1(S)$ is amenable. Then $E(S) = S$ is finite and so $Y = S/\tau$ is a finite semilattice. Hence $\ell^1(S)$ is ℓ^1 -graded of Banach algebras on finite semilattice of Y . Indeed, we have

$$\ell^1(S) \cong \bigoplus_{\alpha \in Y}^{\ell^1} \ell^1(S_\alpha).$$

By [5, Proposition 3.1], $\ell^1(S_\alpha)$ is amenable for all $\alpha \in Y$. Since S_α 's are rectangular band semigroups, by Theorem 3.3, S_α is singleton for each $\alpha \in Y$ and the proof is complete.

(ii) \Rightarrow (iii) In this case for each $\alpha \in Y$, S_α is singleton and so $S \cong Y$. Thus S is a finite semilattice.

(iii) \Rightarrow (i) This follows from [6, Theorem 2.7]. □

Theorem 3.6. *Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a band semigroup, which is a strong semilattice of rectangular band semigroups S_α on a semilattice Y and let $\ell^1(S)$ be biflat. Then Y is a uniformly locally finite semilattice.*

Proof. Let $\mathcal{A} = \ell^1(S)$, $\mathcal{A}_\alpha = \ell^1(S_\alpha)$ and $\varphi_\alpha : \mathcal{A}_\alpha \rightarrow \mathbb{C}$ be the augmentation character on \mathcal{A}_α , for each $\alpha \in Y$. First we show that $\ker \varphi_\alpha =$

$\overline{\ker\varphi_\alpha\mathcal{A}_\alpha + \mathcal{A}_\alpha\ker\varphi_\alpha}$, for each $\alpha \in Y$. Let $s_\alpha, t_\alpha \in S_\alpha$. Since S_α is a rectangular band semigroup we have

$$\delta_{s_\alpha} - \delta_{t_\alpha} = (\delta_{s_\alpha} + \delta_{t_\alpha})(\delta_{t_\alpha s_\alpha} - \delta_{t_\alpha}) - (\delta_{t_\alpha s_\alpha} - \delta_{s_\alpha t_\alpha}),$$

and

$$\delta_{t_\alpha s_\alpha} - \delta_{s_\alpha t_\alpha} = (\delta_{t_\alpha s_\alpha} - \delta_{s_\alpha})(\delta_{s_\alpha} + \delta_{t_\alpha}) - (\delta_{t_\alpha} - \delta_{s_\alpha}).$$

By above relations we have

$$\delta_{s_\alpha} - \delta_{t_\alpha} = \frac{1}{2}(\delta_{s_\alpha} + \delta_{t_\alpha})(\delta_{t_\alpha s_\alpha} - \delta_{t_\alpha}) - \frac{1}{2}(\delta_{t_\alpha s_\alpha} - \delta_{s_\alpha})(\delta_{s_\alpha} + \delta_{t_\alpha}).$$

Since $\delta_{t_\alpha s_\alpha} - \delta_{t_\alpha}, \delta_{t_\alpha s_\alpha} - \delta_{s_\alpha} \in \ker\varphi_\alpha$, it follows that $\delta_{s_\alpha} - \delta_{t_\alpha} \in \ker\varphi_\alpha\mathcal{A}_\alpha + \mathcal{A}_\alpha\ker\varphi_\alpha$. By noting that $\ker\varphi_\alpha$ is generated by $\{\delta_{s_\alpha} - \delta_{t_\alpha} : s_\alpha, t_\alpha \in S_\alpha\}$, we have $\ker\varphi_\alpha = \overline{\ker\varphi_\alpha\mathcal{A}_\alpha + \mathcal{A}_\alpha\ker\varphi_\alpha}$. Now define $\varphi = \bigoplus_{\alpha \in Y} \varphi_\alpha : \mathcal{A} \rightarrow \ell^1(Y)$ by

$$\varphi(f) = \varphi\left(\sum_{\alpha \in Y} f_\alpha\right) = \sum_{\alpha \in Y} \varphi_\alpha(f_\alpha)\delta_\alpha \quad (f = \sum_{\alpha \in Y} f_\alpha \in \mathcal{A}).$$

It is straightforward to check that φ is an epimorphism and $\ker\varphi = \overline{\ker\varphi\mathcal{A} + \mathcal{A}\ker\varphi}$. Thus the short sequence

$$0 \rightarrow \ker\varphi \xrightarrow{\iota} \mathcal{A} \xrightarrow{\varphi} \ell^1(Y) \rightarrow 0$$

is exact, where $\iota : \ker\varphi \rightarrow \mathcal{A}$ is the inclusion map. Now from [7, Theorem 3.3], it follows that $\ell^1(Y)$ is biflat. Thus Y is a uniformly locally finite semilattice. \square

Remark 3.7. *The converse of Theorem 3.6 dose not hold in general. For example suppose that $S_0 = \{\mathbf{a}, \mathbf{b}\}$ and $S_1 = \{\mathbf{c}\}$, where S_0 is a left zero semigroup and*

$$\mathbf{ca} = \mathbf{ac} = \mathbf{a}, \quad \mathbf{bc} = \mathbf{cb} = \mathbf{b}.$$

Set $S = \bigcup_{\alpha \in \mathbb{Z}_2} S_\alpha$, where (\mathbb{Z}_2, \cdot) is the multiplicative semigroup. Then S is a band semigroup. We show that $\ell^1(S)$ is not biflat. Suppose to the contrary that $\ell^1(S)$ is biflat. Then by regarding δ_c as the unit element of $\ell^1(S)$, it follows that $\ell^1(S)$ is amenable. By using Theorem 3.5, it follows that S is a finite semilattice, which is a contradiction.

We now consider biflatness of the second dual of certain semigroup algebras. We show that for a certain class of inverse semigroups the biflatness of $\ell^1(S)''$ is equivalent to the biprojectivity of $\ell^1(S)$. We use the following relation on an inverse semigroup.

Definition 3.8. Let S be an inverse semigroup. We define a relation \mathcal{D} on S by $s\mathcal{D}t$ if and only if there exists $x \in S$ such that $s^*s = xx^*$ and $t^*t = x^*x$.

Let $\{\mathcal{A}_\alpha : \alpha \in I\}$ be a collection of Banach algebras. The ℓ^1 -direct sum of \mathcal{A}_α 's is denoted by $\ell^1 - \bigoplus \{\mathcal{A}_\alpha : \alpha \in I\}$ which is a Banach algebra with componentwise operations. We recall that $\ell^1 - \bigoplus \{\mathcal{A}_\alpha : \alpha \in I\}$ is biflat (biprojective) if and only if there exists $C > 0$ such that for each $\alpha \in I$, \mathcal{A}_α is C -biflat (C -biprojective).

Theorem 3.9. Let S be an inverse semigroup and $\{D_\lambda : \lambda \in \Lambda\}$ be the collection of all \mathcal{D} -classes of S . Suppose that Λ is finite and every \mathcal{D} -class has finitely many idempotents. Then the following are equivalent:

- (i) $\ell^1(S)''$ is biflat.
- (ii) $\ell^1(S)$ is biprojective.
- (iii) $\ell^1(S)''$ is biprojective.

Proof. First for each $\lambda \in \Lambda$, choose an idempotent $p_\lambda \in D_\lambda$ and let G_{p_λ} be the maximal subgroup of S at p_λ .

(i) \Rightarrow (ii) Suppose that $\ell^1(S)''$ is biflat. By [11, Theorem 2.2], we conclude that $\ell^1(S)$ is biflat, so (S, \leq) is uniformly locally finite [12, Theorem 3.7(i)]. On the other hand, by [12, Theorem 2.18], we have

$$\ell^1(S) \cong \ell^1 - \bigoplus \{\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda})) : \lambda \in \Lambda\},$$

as Banach algebras, where $\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ denotes the ℓ^1 -Munn algebra on $\ell^1(G_{p_\lambda})$. Since Λ is finite and every \mathcal{D} -class has finitely many idempotents we have

$$(3.1) \quad \begin{aligned} \ell^1(S)'' &\cong \ell^1 - \bigoplus \{\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))'' : \lambda \in \Lambda\} \\ &\cong \ell^1 - \bigoplus \{\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))'' : \lambda \in \Lambda\}. \end{aligned}$$

Since $\ell^1(S)''$ is biflat, $\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))''$ is biflat for each $\lambda \in \Lambda$. Using [12, Proposition 2.7], we conclude that $\ell^1(G_{p_\lambda})''$ is biflat and so G_{p_λ} is finite, for each $\lambda \in \Lambda$. The result now follows from [12, Theorem 3.7(ii)].

(ii) \Rightarrow (iii) Let $\ell^1(S)$ be biprojective. By [12, Theorem 3.7(ii)], (S, \leq) is uniformly locally finite and every maximal subgroup of S is finite. Thus $\ell^1(G_{p_\lambda})'' = \ell^1(G_{p_\lambda})$ is 1-biprojective, for each $\lambda \in \Lambda$. By using [12, Proposition 2.7], $\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))''$ is 1-biprojective, for each $\lambda \in \Lambda$. Now according to (3.1), it follows that $\ell^1(S)''$ is biprojective.

The implication (iii) \Rightarrow (i) is clear. \square

The following proposition is proved in [12, Proposition 2.19]. We give a short proof.

Proposition 3.10. ([12, Proposition 2.19]) *Let S be an inverse semigroup. Suppose that either*

- (i) S is locally finite and $\ell^1(S)$ has an identity, or
- (ii) S is uniformly locally finite and $\ell^1(S)$ has a bounded approximate identity.

Then $E(S)$ is finite.

Proof. i) Since S is an inverse semigroup and $\ell^1(S)$ has an identity, $\ell^1(E(S))$ has an identity too. Since $E(S)$ is a semilattice by [8, Theorem 7.3], $E(S)$ has a finite relative unit, that is, there exist $p_1, p_2, \dots, p_n \in E(S)$ such that for all $a \in E(S)$, there is $1 \leq i \leq n$ such that

$$ap_i = p_i a = a.$$

So $E(S) \subseteq \bigcup_{j=1}^n (p_j]$. Since (S, \leq) is locally finite, $E(S)$ is finite.

(ii) By [4, Lemma 13], $\ell^1(S)$ has a bounded approximate identity if and only if $\ell^1(E(S))$ has a bounded approximate identity. So we may suppose that S is a semilattice. Since S is uniformly locally finite, $\ell^1(S)$ is biflat and so is amenable. Thus $E(S)$ is finite. \square

Corollary 3.11. *Let S be an inverse semigroup such that $\ell^1(S)$ has a bounded approximate identity. Then the following are equivalent:*

- (i) $\ell^1(S)''$ is biflat.
- (ii) $\ell^1(S)$ is biprojective.
- (iii) $\ell^1(S)''$ is biprojective.

Proof. Each of the conditions (i), (ii) and (iii) implies that (S, \leq) is uniformly locally finite. By Proposition 3.10 (ii), it follows that $E(S)$ is finite. Since every \mathcal{D} -class has at least an idempotent, we conclude that S has finitely many \mathcal{D} -class. Using Theorem 3.9, the proof is complete. \square

Let G be a group and I be a non-empty set. Define

$$\mathcal{M}^0(G, I) = \{(g)_{ij} : g \in G, i, j \in I\} \cup \{0\},$$

where $(g)_{ij}$ is $I \times I$ -matrix with $g \in G$ in the $(i, j)^{th}$ place and 0 elsewhere and the zero matrix 0. Then $\mathcal{M}^0(G, I)$ with the multiplication given by

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \quad (g, h \in G, i, j, k, l \in I),$$

is an inverse semigroup with $(g)_{ij}^* = (g^{-1})_{ji}$, that is called the Brandt semigroup corresponding to G and I .

Corollary 3.12. *Let G be a group, I be a non-empty finite set and $S = \mathcal{M}^0(G, I)$ be the Brandt semigroup corresponding to G and I . Then $\ell^1(S)''$ is biflat if and only if G is finite.*

Proof. Let $\ell^1(S)''$ be biflat. Since I is finite, $E(S) = \{(e_G)_{ii} : i \in I\} \cup \{0\}$ is finite. From Theorem 3.9, it follows that $\ell^1(S)$ is biprojective. Using [12, Theorem 3.7(ii)], we conclude that each maximal subgroup of S is finite. Hence G is a finite group. □

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Morteza Esmaili

Faculty of Mathematical and Computer Science, Kharazmi University, P.O. Box
15618, Tehran, Iran

Email: m.esmaili@tmu.ac.ir

Alireza Medghalchi

Faculty of Mathematical and Computer Science, Kharazmi University, P.O. Box
15618, Tehran, Iran

Email: medghalchi@saba.tmu.ac.ir