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# G-POSITIVE AND G-REPOSITIVE SOLUTIONS TO SOME ADJOINTABLE OPERATOR EQUATIONS OVER HILBERT $C^*$-MODULES

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Abstract. Some necessary and sufficient conditions are given for the existence of a G-positive (G-repositive) solution of adjointable operator equations $AX = C, AXA^{(*)} = C$ and $AXB = C$ over Hilbert $C^*$-modules, respectively. Moreover, the expressions of these general G-positive (G-repositive) solutions are also derived. Some of the findings of this paper extend some known results in the literature.

## 1. Introduction

Hilbert $C^*$-module is a natural generalization of a Hilbert space and a $C^*$-algebra. The basic idea was to consider modules over $C^*$-algebras instead of linear spaces and to allow the inner product to take values in a $C^*$-algebra. The structure was first used by Kaplansky [11] in 1952. Compared to the Hilbert spaces, there exist some differences when we deal with operators in a general Hilbert $C^*$-module $\mathcal{H}$, for instances, a closed topologically complemented submodule of $\mathcal{H}$ may not be orthogonally complemented, meanwhile the fundamental Riesz representation theorem concerning the bounded linear functionals of $\mathcal{H}$ may also be not

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true. For more details of $C^*$-algebra and Hilbert $C^*$-modules, we refer the readers to [13, 21].

Let $\mathfrak{A}$ be a $C^*$-algebra. An inner-product $\mathfrak{A}$-module is a linear space $E$ which is a right $\mathfrak{A}$-module (with a scalar multiplication satisfying $\lambda(xa) = x(\lambda a) = (\lambda x)a$ for $x \in E, a \in \mathfrak{A}, \lambda \in \mathbb{C}$), together with a map $E \times E \to \mathfrak{A}$, $(x, y) \to \langle x, y\rangle$ such that
(1) $\langle x, \alpha y + \beta z\rangle = \alpha \langle x, y\rangle + \beta \langle x, z\rangle$;
(2) $\langle x, ya\rangle = \langle x, y\rangle a$;
(3) $\langle x, y\rangle = \langle y, x\rangle^*$;
(4) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$
for any $x, y, z \in E$, $\alpha, \beta \in \mathbb{C}$ and $a \in \mathfrak{A}$. An inner-product $\mathfrak{A}$-module $E$ is called a (right) Hilbert $\mathfrak{A}$-module if it is complete with respect to the induced norm $||x|| = |\langle x, x\rangle|^{1/2}$.

Assume that $\mathcal{H}$ and $\mathcal{K}$ are two Hilbert $\mathfrak{A}$-modules, and $\mathfrak{B}(\mathcal{H}, \mathcal{K})$ is the set of all maps $T$: $\mathcal{H} \to \mathcal{K}$ for which there is a map $T^*$: $\mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$, for any $x \in \mathcal{H}$ and $y \in \mathcal{K}$. We know that any element $T$ of $\mathfrak{B}(\mathcal{H}, \mathcal{K})$ is a bounded linear operator. We call $\mathfrak{B}(\mathcal{H}, \mathcal{K})$ the set of adjointable operators from $\mathcal{H}$ into $\mathcal{K}$. In case $\mathcal{H} = \mathcal{K}$, $\mathfrak{B}(\mathcal{H}, \mathcal{H})$ which we abbreviate to $\mathfrak{B}(\mathcal{H})$, is a $C^*$-algebra. An operator $A \in \mathfrak{B}(\mathcal{H})$ is called Hermitian (or self-adjoint) if $A^* = A$, and positive if $\langle Ax, x\rangle \geq 0$ for all $x \in H$, we write $A \geq 0$ if $A$ is positive. The Moore-Penrose inverse of $A \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$ is defined as the operator $A^\dagger \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ satisfying the Penrose equations

$$AA^\dagger A = A, A^\dagger AA^\dagger = A^\dagger, (A^\dagger A)^* = A^\dagger A, (AA^\dagger)^* = AA^\dagger.$$

An operator $A \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$ has the (unique) Moore-Penrose inverse if and only if $A$ has closed range, or equivalently it is regular. The notations $\mathcal{R}(A)$ and $\mathcal{N}(A)$ stand for the range of $A$ and the null space of $A$, respectively. For simplicity, we use $L_A$ and $R_A$ to stand for $I - A^\dagger A$ and $I - AA^\dagger$, respectively.

As a generalization of Hermitian operator, the quasi-hermitian operator over the Hilbert space was given in Dieudonné [6] as: $S$ is called a quasi-hermitian operator if there is a positive bounded operator $V$ such that $VS = S^*V$. Recently, Sun and Ma [15] studied a new kind of generalized Hermitian operator: $S$ is called a generalized Hermitian operator if there is an invertible operator $V$ such that $VS = S^*V$. In fact, many authors have shown some results concerning generalized Hermitian operators, although they have not used the terminology (generalized Hermitian operators). Such as, Radjavi and Williams [14] proved that a

bounded linear operator on a finite-dimensional Hilbert space is generalized Hermitian iff it is the product of two Hermitian operators; Jiang [10] showed that for a normal operator $S$ is generalized Hermitian iff it is the product of two Hermitian operators. Also he considered the case of hyponormal operators.

We are now interested in the following operators:

**Definition 1.1.** *(1) Suppose $\mathcal{H}, \mathcal{K}$ are two Hilbert $\mathfrak{A}$-modules, $V \in \mathfrak{B}(\mathcal{H}), W \in \mathfrak{B}(\mathcal{K})$ are Hermitian invertible, $\mathfrak{B}(\mathcal{H}, \mathcal{K})$ is the set of all maps $A : \mathcal{H} \to \mathcal{K}$ for which there is a map $A^{(*)} : \mathcal{K} \to \mathcal{H}$ such that*

$$\langle Ax, y\rangle = \left\langle Vx, A^{(*)}W^{-1}y \right\rangle, \text{ for all } x \in \mathcal{H} \text{ and } y \in \mathcal{K}.$$

*(2) An operator $A \in \mathfrak{B}(\mathcal{H})$ is called G-Hermitian (G-skew-Hermitian) if*

$$A = A^{(*)} = V^{-1}A^*V \ \left(A = -A^{(*)} = -V^{-1}A^*V\right).$$

*(3) An operator $A \in \mathfrak{B}(\mathcal{H})$ is called G-positive if and only if $A$ is G-Hermitian and $\langle Ax, Vx\rangle \geq 0$ for all $x \in \mathcal{H}$.*

Obviously, the class of G-positive operators (G-repositive operators), which of course contains the positive operators (repositive operators), is in fact much wider. Let $\mathfrak{B}(\mathcal{H})_G$ be the set of all G-Hermitian elements of $\mathfrak{B}(\mathcal{H})$, $\mathfrak{B}(\mathcal{H})^{(+)}$ be the set of all G-positive elements of $\mathfrak{B}(\mathcal{H})$. Then for $A, B \in \mathfrak{B}(\mathcal{H})$,
(1) $(AB)^{(*)} = B^{(*)}A^{(*)}$,
(2) $A \in \mathfrak{B}(\mathcal{H})_G$ if and only if $VA = A^*V$, for some invertible Hermitian operator $V$,
(3) $A$ is regular, then $(A^\dagger)^{(*)} = (A^{(*)})^\dagger, (L_A)^{(*)} = R_{A^{(*)}}$ and $(R_A)^{(*)} = L_{A^{(*)}}$,
(4) $A \in \mathfrak{B}(\mathcal{H})^{(+)}$ and is regular, then $A^\dagger \in \mathfrak{B}(\mathcal{H})^{(+)}$.

Researches on the nonnegative definite and Re-nonnegative definite solutions to matrix equations, as well as the positive and Re-positive solutions of operator equations have been actively ongoing for many years. For instance, the nonnegative definite solutions to matrix equation

$$AXB = C \tag{1.1}$$

were studied by Khatri, Mitra [12] in 1976 and Zhang [24] in 2004, respectively. In 1998, Wang and Yang [20] derived the Re-nonnegative definite solution to matrix equation (1.1) by matrix decomposition. Recently, Arias and Gonzalez [1] studied the positive solutions to operator

equation (1.1). Under the assumption that the underlying space is finite-dimensional or the range of $B$ is contained in the range of $A^*$, Xu et al. [23] proposed some solvability conditions for operator equation (1.1) to have a positive solution and a Re-positive solution, as well as some expressions of these solutions in the general setting of Hilbert $C^*$-modules. Some other results can be found in [22]-[7].

In fact, the positive (Re-positive) operator can be seen as a special case of G-positive (G-repositive) operator, therefore investigating the G-positive (G-repositive) solutions to some operator equations over Hilbert $C^*$-modules are very meaningful. So far, to our knowledge, there has been little information on either the G-positive or the G-repositive to adjointable operator equations over Hilbert $C^*$-modules. Then in this paper, we mainly consider the G-positive and G-repositive solutions to some classical adjointable operator equations over the Hilbert $C^*$-modules. This paper is organized as follows: In Section 2, we present some criteria for an adjointable operator to be G-positive (G-repositive). In Section 3, we show some necessary and sufficient conditions for the existence of a G-positive solution to some adjointable operator equations over Hilbert $C^*$-modules. The expressions of these general G-positive solutions are also derived. In Section 4, we show some necessary and sufficient conditions for the existence of a G-repositive solutions to some adjointable operator equations over Hilbert $C^*$-modules, as well as the general expressions of these solutions. We conclude the paper by proposing in Section 5 some further research topics.

## 2. **Preliminaries**

For any operator $A \in \mathfrak{B}(\mathcal{H})$, there exist two orthogonal projections $P_1, P_2 = I - P_1$, such that $A$ can be expressed as

$$A = A_1 + A_2 + A_3 + A_4,$$

where

$$A_1 = P_1 A P_1^{(*)}, A_2 = P_1 A P_2^{(*)}, A_3 = P_2 A P_1^{(*)}, A_4 = P_2 A P_2^{(*)}.$$

The mapping $\Psi : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H} \oplus \mathcal{H})$ is an algebra $*-$*monomrophism* where $\mathfrak{B}(\mathcal{H} \oplus \mathcal{H})$ equipped with the usual matrix operations. Then we shall write $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ with respect to $P_1, P_2$, and have the following results:

**Theorem 2.1.** *Suppose that $A \in \mathfrak{B}(\mathcal{H})_G$ and has closed range. Then $A \in \mathfrak{B}(\mathcal{H})^{(+)}$ if and only if the following conditions are satisfied:*
*(1)* $A_1 \in \mathfrak{B}(\mathcal{H})^{(+)}$,
*(2)* $A_1 A_1^{\dagger} A_2 = A_2$,
*(3)* $A_4 - A_3 A_1^{\dagger} A_2 \in \mathfrak{B}(\mathcal{H})^{(+)}$.

*Proof.* Suppose that $A \in \mathfrak{B}(\mathcal{H})^{(+)}$, then there exists $H = V^{-1}\left(A^* V\right)^{\frac{1}{2}}$ such that $A = HVH^{(*)}$ and $A_1 = P_1 A P_1^{(*)} = P_1 HVH^{(*)} P_1^{(*)} \in \mathfrak{B}(\mathcal{H})^{(+)}$. In view of

$$\begin{aligned} A_1 A_1^{\dagger} &= P_1 HV\left(P_1 H\right)^{(*)} \left(P_1 HV\left(P_1 H\right)^{(*)}\right)^{\dagger} \\ &= P_1 HV\left(P_1 H\right)^{(*)} \left(\left(P_1 H\right)^{(*)}\right)^{\dagger} V^{-1}\left(P_1 H\right)^{\dagger} \\ &= P_1 HV\left(\left(P_1 H\right)^{\dagger} P_1 H\right)^{(*)} V^{-1}\left(P_1 H\right)^{\dagger} \\ &= P_1 H\left(P_1 H\right)^{\dagger}, \end{aligned}$$

we have

$$A_1 A_1^{\dagger} A_2 = P_1 H\left(P_1 H\right)^{\dagger} P_1 HVH^{(*)} P_2^{(*)} = A_2.$$

Finally,

$$\begin{aligned} & A_4 - A_2^{(*)} A_1^{\dagger} A_2 \\ =& P_2 A P_2^{(*)} - P_2 A P_1^{(*)} A_1^{\dagger} P_1 A P_2^{(*)} \\ =& P_2 HVH^{(*)} P_2^{(*)} - P_2 HVH^{(*)} P_1^{(*)} \left(P_1 HVH^{(*)} P_1^{(*)}\right)^{\dagger} P_1 HVH^{(*)} P_2^{(*)} \\ =& P_2 HVH^{(*)} P_2^{(*)} - P_2 HVH^{(*)} P_1^{(*)} \left(H^{(*)} P_1^{(*)}\right)^{\dagger} V^{-1}\left(P_1 H\right)^{\dagger} P_1 HVH^{(*)} P_2^{(*)} \\ =& P_2 HV\left(I - H^{(*)} P_1^{(*)} \left(H^{(*)} P_1^{(*)}\right)^{\dagger} \left(P_1 HV\right)^{\dagger} P_1 HV\right) H^{(*)} P_2^{(*)} \\ =& P_2 H\left(I - \left(P_1 H\right)^{\dagger} P_1 H\right) V \left(P_2 H\left(I - \left(P_1 H\right)^{\dagger} P_1 H\right)\right)^{(*)} \in \mathfrak{B}(\mathcal{H})^{(+)}. \end{aligned}$$

On the contrary, suppose that (1), (2) and (3) are satisfied, then for any $h = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H}$, let

$$V_1 = \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix},\ x = \begin{pmatrix} -A_1^{\dagger} A_2 x_2 \\ x_2 \end{pmatrix},\ y = \begin{pmatrix} x_1 + A_1^{\dagger} A_2 x_2 \\ 0 \end{pmatrix},$$

then

$$\begin{aligned}
&\langle Ax, V_1 y\rangle \\
=&y^* V_1 Ax \\
=&\begin{pmatrix} x_1^* + x_2^* \left(A_1^\dagger A_2\right)^* & 0 \end{pmatrix} \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{pmatrix} -A_1^\dagger A_2 x_2 \\ x_2 \end{pmatrix} \\
=&\begin{pmatrix} x_1^* + x_2^* \left(A_2^\dagger A_1\right)^* & 0 \end{pmatrix} \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{pmatrix} -A_1^\dagger A_2 x_2 \\ x_2 \end{pmatrix} \\
=&\begin{pmatrix} x_1^* V + x_2^* \left(A_2^\dagger A_1\right)^* V & 0 \end{pmatrix} \begin{pmatrix} 0 \\ A_4 x_2 - A_3 A_1^\dagger A_2 x_2 \end{pmatrix} = 0.
\end{aligned}$$

Since

$$\begin{aligned}
&\langle Ah, V_1 h\rangle \\
=&\langle A(x+y), V_1(x+y)\rangle \\
=&\langle Ax, V_1 x\rangle + \langle Ax, V_1 y\rangle + \langle Ay, V_1 x\rangle + \langle Ay, V_1 y\rangle = \langle Ax, V_1 x\rangle + \langle Ay, V_1 y\rangle \\
=&\left\langle \left(A_4 - A_3 A_1^\dagger A_2\right) x_2, V_1 x_2 \right\rangle + \left\langle \left(A_4 - A_3 A_1^\dagger A_2\right) x_1, V_1 x_1 \right\rangle \geq 0,
\end{aligned}$$

then $A \in \mathfrak{B}(\mathcal{H})^{(+)}$. □

**Corollary 2.2.** *Suppose that $A \in \mathfrak{B}(\mathcal{H})_G$ has closed range. Then $A \in \mathfrak{B}(\mathcal{H})^{(+)}$ if and only if the following conditions are satisfied:*
*(1)* $A_4 \in \mathfrak{B}(\mathcal{H})^{(+)}$,
*(2)* $A_4 A_4^\dagger A_3 = A_3$,
*(3)* $A_1 - A_2 A_4^\dagger A_3 \in \mathfrak{B}(\mathcal{H})^{(+)}$.

The general real part of $A \in \mathfrak{B}(\mathcal{H})$ is defined by $\text{G-Re}\, A = \frac{1}{2}\left(A + A^{(*)}\right)$. Then $\text{G-Re}\, A$ is always G-Hermitian, $\text{G-Re}\, A = A$ if and only if $A \in \mathfrak{B}(\mathcal{H})_G$, moreover,

$$\text{G-}\,\text{Re}\,(A \pm B) = \text{G-}\,\text{Re}\, A + \text{G-}\,\text{Re}\, B,\ \ \text{G-}\,\text{Re}(X^{(*)} A X) = X^{(*)}\,(\text{G-}\,\text{Re}\, A)\, X.$$

An element $A \in \mathfrak{B}(\mathcal{H})$ is called G-repositive if and only if $\text{G-Re}\, A \in \mathfrak{B}(\mathcal{H})^{(+)}$. Similarly, we can prove the following results.

**Theorem 2.3.** *Suppose that $A \in \mathfrak{B}(\mathcal{H})$ and G-$\text{Re}\, A$ has closed range. Then $A + A^{(*)} \in \mathfrak{B}(\mathcal{H})^{(+)}$ if and only if the following conditions are satisfied:*
*(1)* $A_1 + A_1^{(*)} \in \mathfrak{B}(\mathcal{H})^{(+)}$,

*(2)* $\left(A_1+A_1^{(*)}\right)\left(A_1+A_1^{(*)}\right)^{\dagger}\left(A_2+A_2^{(*)}\right)=\left(A_2+A_2^{(*)}\right)$,

*(3)* $\left(A_4+A_4^{(*)}\right)-\left(A_3+A_3^{(*)}\right)\left(A_1+A_1^{(*)}\right)^{\dagger}\left(A_2+A_2^{(*)}\right)\in\mathfrak{B}(\mathcal{H})^{(+)}$.

**Corollary 2.4.** *Suppose that $A\in\mathfrak{B}(\mathcal{H})$ and G-*$\mathrm{Re}\,A$ *has closed range. Then $A+A^{(*)}\in\mathfrak{B}(\mathcal{H})^{(+)}$ if and only if the following conditions are satisfied:*

*(1)* $A_3+A_3^{(*)}\in\mathfrak{B}(\mathcal{H})^{(+)}$,

*(2)* $\left(A_3+A_3^{(*)}\right)\left(A_3+A_3^{(*)}\right)^{\dagger}\left(A_4+A_4^{(*)}\right)=\left(A_4+A_4^{(*)}\right)$,

*(3)* $\left(A_1+A_1^{(*)}\right)-\left(A_2+A_2^{(*)}\right)\left(A_4+A_4^{(*)}\right)^{\dagger}\left(A_3+A_3^{(*)}\right)\in\mathfrak{B}(\mathcal{H})^{(+)}$.

## 3. G-positive solution to some adjointable operator equations over Hilbert $C^*$-modules

In this Section, we mainly consider some necessary and sufficient conditions for some adjointable operator equations to have a G-positive solution, we also give expressions of these solutions.

**Theorem 3.1.** *Let $A,C\in\mathfrak{B}(\mathcal{H}_1,\mathcal{H}_2)$, $A$ and $CA^{(*)}$ have closed ranges. Then adjointable operator equation*

$$AX=C \tag{3.1}$$

*has a G-positive solution $X\in\mathfrak{B}(\mathcal{H}_2)^{(+)}$ if and only if $CA^{(*)}\in\mathfrak{B}(\mathcal{H}_1)^{(+)}$ and $\mathcal{R}(C)\subseteq\mathcal{R}\left(CA^{(*)}\right)$. In this case, the general a G-positive solution of equation (3.1) can be expressed as*

$$X=C^{(*)}\left(CA^{(*)}\right)^{\dagger}C+(I-A^{\dagger}A)U(I-A^{\dagger}A)^{(*)}, \tag{3.2}$$

*where $U\in\mathfrak{B}(\mathcal{H}_2)^{(+)}$ is arbitrary.*

*Proof.* Suppose that $CA^{(*)}$ is G-positive, then $\left(CA^{(*)}\right)^{\dagger}$ and $C^{(*)}\left(CA^{(*)}\right)^{\dagger}C$ are G-positive. Noting that

$$AC^{(*)}\left(CA^{(*)}\right)^{\dagger}C=CA^{(*)}\left(CA^{(*)}\right)^{\dagger}C=C,$$

then $C^{(*)}\left(CA^{(*)}\right)^{\dagger}C$ is a G-positive solution to equation (3.1). For the other direction. Let $X_0$ be a G-positive solution to equation (3.1). Then

$CA^{(*)} = AX_0A^{(*)}$ is G-positive, and

$$\begin{aligned}\mathcal{R}\left(CA^{(*)}\right) &= \mathcal{R}\left(AX_0A^{(*)}\right)\\ &= \mathcal{R}\left(AX_1V\left(AX_1\right)^{(*)}\right)\\ &= \mathcal{R}\left(AX_1\right) \supseteq \mathcal{R}\left(AX_0\right)\\ &= \mathcal{R}(C),\end{aligned}$$

where $X_0 = X_1VX_1^{(*)}$ for some Hermitian invertible operator $V \in \mathfrak{B}(\mathcal{H}_2)$.

Next we only need to show that every G-positive of equation (3.1) can be expressed as (3.2). Suppose $X_1$ is an arbitrary G-positive solution of equation (3.1), and denote $X_0 = C^{(*)}\left(CA^{(*)}\right)^{\dagger}C$. Then $X_1 - X_0$ is a G-Hermitian solution to equation $AX = 0$, so there exist a G-Hermitian matrix $T$ such that $X_1 - X_0 = (I - A^{\dagger}A)T(I - A^{\dagger}A)^{(*)}$. Choose $Y = C\left(CA^{(*)}\right)^{\dagger}CC^{\dagger}A$, then $YX_0 = X_0$, and $YA^{\dagger}A = Y$, so $(I-Y)X_0 = 0$ and $(I-Y)(I-A^{\dagger}A) = (I-A^{\dagger}A)$. Therefore,

$$\begin{aligned}(I-Y)X_1(I-Y)^{(*)} &= (I-Y)X_1(I-Y)^{(*)} - (I-Y)X_0(I-Y)^{(*)}\\ &= (I-Y)\left(X_1 - X_0\right)(I-Y)^{(*)}\\ &= (I-Y)(I-A^{\dagger}A)T(I-A^{\dagger}A)^{(*)}(I-Y)^{(*)}\\ &= X_1 - X_0.\end{aligned}$$

Since $(I-Y)X_1(I-Y)^{(*)} \in \mathfrak{B}(\mathcal{H})^{(+)}$, then

$$X_1 = C^{(*)}\left(CA^{(*)}\right)^{\dagger}C + (I-A^{\dagger}A)T(I-A^{\dagger}A)^{(*)},$$

implying that every G-positive solution of equation (3.1) can be written as (3.2) with some proper choice of $U$. □

**Remark 3.2.** *Similarly, on the assumption that*

$$\begin{bmatrix} A \\ C^{(*)} \end{bmatrix} \text{ and } \begin{bmatrix} BA^{(*)} & BC \\ (AD)^{(*)} & D^{(*)}C \end{bmatrix}$$

*have closed ranges, then we can derive some necessary and sufficient conditions for adjointable operator equations $AX = B$ and $XC = D$ to have a common G-positive solution ,as well as an expression for the common G-positive solutions.*

**Theorem 3.3.** *Let $\mathcal{H},\mathcal{K}$ be Hilbert $C^*$-modules. Assume that $A \in \mathfrak{B}(\mathcal{H},\mathcal{K})$, $C \in \mathfrak{B}(\mathcal{K})$ such that $A$ has closed range, $C$ is G-Hermitian and $\mathcal{R}(C) \subseteq \mathcal{R}(A)$. Then equation*

$$AXA^{(*)} = C \tag{3.3}$$

*has a G-positive solution if and only if $C$ is G-positive. If, in addition, $C$ has closed range, the general G-positive solution of equation (3.3) can be expressed as*

$$X = X_2 + X_2X_2^{\dagger}EL_A^{(*)} + L_AE^{(*)}\left(X_2X_2^{\dagger}\right)^{(*)} + L_AE^{(*)}X_2^{\dagger}EL_A^{(*)} + L_AFL_A^{(*)}, \tag{3.4}$$

*with $X_2 = A^{\dagger}C\left(A^{\dagger}\right)^{(*)}$ is a particular G-positive solution, $E \in \mathfrak{B}(\mathcal{H})$ and $F \in \mathfrak{B}(\mathcal{H})^{(+)}$ are arbitrary.*

*Proof.* When $X$ is given as (3.4), and in view of the assumption $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $C \in \mathfrak{B}(\mathcal{K})^{(+)}$, then $X \in \mathfrak{B}(\mathcal{H})^{(+)}$ and satisfies equation (3.3). Next we show that (3.4) provides the general G-positive solution. Assume that $C$ has closed range and $X$ is a G-positive solution to equation (3.3). Denote $P_1 = A^{\dagger}A, P_2 = \left(I - A^{\dagger}A\right)$, then

$$X = X_1 + X_2 + X_3 + X_4 = P_1XP_1^{(*)} + P_1XP_2^{(*)} + P_2XP_1^{(*)} + P_2XP_2^{(*)}.$$

Combining the mapping $\Psi : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H} \oplus \mathcal{H})$, $X$ can be expressed as

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

Since

$$X_1 = P_1XP_1^{(*)} = A^{\dagger}AXA^{(*)}\left(A^{\dagger}\right)^{(*)} = A^{\dagger}C\left(A^{\dagger}\right)^{(*)},$$

and note that $C \in \mathfrak{B}(\mathcal{H})^{(+)}$, then $X_1 \in \mathfrak{B}(\mathcal{H})^{(+)}$. Moreover,

$$\begin{aligned} X_1A^{(*)}C^{\dagger}AX_1 &= A^{\dagger}C\left(A^{\dagger}\right)^{(*)}A^{(*)}C^{\dagger}AA^{\dagger}C\left(A^{\dagger}\right)^{(*)} \\ &= A^{\dagger}CVAA^{\dagger}C^{\dagger}AA^{\dagger}CVA = A^{\dagger}CC^{\dagger}C\left(A^{\dagger}\right)^{(*)} = X_1, \end{aligned}$$

thus $X_1$ is regular and $X_1^{\dagger}$ exists. It follows Theorem 2.1 that $X_1X_1^{\dagger}X_2 = X_2$ and $X_4 - X_2^{(*)}X_1^{\dagger}X_2 \in \mathfrak{B}(\mathcal{H})^{(+)}$ are satisfied. Then we have $X_2 =$

$X_1X_1^\dagger Y$, where $Y \in \mathfrak{B}(H)$ is arbitrary, and

$$\begin{aligned} X_4 - X_3^{(*)}X_1^\dagger X_2 &= X_4 - L_A E^{(*)} \left(X_2 X_2^\dagger\right)^* X_2^\dagger X_2 X_2^\dagger E L_A^{(*)} \\ &= X_4 - L_A E^{(*)} X_2^\dagger E L_A^{(*)} \in \mathfrak{B}(\mathcal{H})^{(+)}. \end{aligned}$$

Then any arbitrary G-positive solution to equation (3.3) can be expressed as (3.4) with some proper choices of $E$ and $F$. □

We can now prove the following result.

**Theorem 3.4.** *Let $\mathcal{H}_i (i = 1, 2, \cdots, 4)$ be Hilbert $C^*$-modules, $A \in \mathfrak{B}(\mathcal{H}_1, \mathcal{H}_2)$, $B \in \mathfrak{B}(\mathcal{H}_3, \mathcal{H}_1)$, $C \in \mathfrak{B}(\mathcal{H}_3, \mathcal{H}_2)$. Denote*

$$\begin{aligned} A_1 &= A^{(*)}V^{-1}A, B_1 \\ &= BVB^{(*)}, C_1 \\ &= A^{(*)}CB^{(*)}, M \\ &= A_1 + B_1, T \\ &= B_1M^\dagger C_1 M^\dagger A_1. \end{aligned}$$

*Suppose that $A, B, T$ and $M$ have closed ranges. Then the consistent equation (1.1) has a G-positive solution if and only if*

$$\mathcal{R}(A_1M^\dagger C_1^{(*)}) \subseteq \mathcal{R}(T), \ \mathcal{R}(B_1M^\dagger C_1) \subseteq \mathcal{R}(T). \tag{3.5}$$

*In this case, the general G-positive solution to equation (1.1) can be expressed as*

$$\begin{aligned} X = M^\dagger \left(C_1 + C_1^{(*)} + Y + Z\right) \left(M^\dagger\right)^{(*)} + X_2 + X_2X_2^\dagger E L_A^{(*)} \\ + L_A E^{(*)} \left(X_2X_2^\dagger\right)^{(*)} + L_A E^{(*)} X_2^\dagger E L_A^{(*)} + L_A F L_A^{(*)}, \end{aligned} \tag{3.6}$$

*where $X_2 = M^\dagger \left(C_1 + C_1^{(*)} + Y + Z\right) M^\dagger$, $Y$ and $Z$ are arbitrary G-positive solutions to equations*

$$YM^\dagger B_1 = C_1M^\dagger A_1, \ A_1M^\dagger Z = B_1M^\dagger C_1, \tag{3.7}$$

*such that $C_1 + C_1^{(*)} + Y + Z$ is G-positive, $E \in \mathfrak{B}(\mathcal{H}_1)$ and $F \in \mathfrak{B}(\mathcal{H}_1)^{(+)}$ are arbitrary.*

*Proof.* Suppose that equation (1.1) is consistent, then there is an $X$ such that

$$T = B_1M^\dagger C_1M^\dagger A_1 = B_1M^\dagger A_1 X B_1M^\dagger A_1.$$

Moreover, since

$$\begin{aligned} A_1 &= A^{(*)}V^{-1}A \\ &= {}^{-1}A^*A \\ &= V^{-1}A^*VV^{-1}A \\ &= V^{-1}A^*V\left(V^{-1}A^*\right)^{(*)} \in \mathfrak{B}(\mathcal{H})^{(+)}, \end{aligned}$$

and

$$\begin{aligned} B_1 &= BVB^{(*)} \\ &= BB^{(*)}V \\ &= BVV^{-1}B^*V \\ &= (BV)^{(*)}V^{-1}B^*V \in \mathfrak{B}(\mathcal{H})^{(+)}, \end{aligned}$$

thus

$$B_1M^{\dagger}A_1 = B_1\left(A_1+B_1\right)^{\dagger}A_1 = A_1\left(A_1+B_1\right)^{\dagger}B_1$$

is G-positive. When $X$ is G-positive, there exist $X_1$ such that

$$\begin{aligned} T &= B_1M^{\dagger}A_1XB_1M^{\dagger}A_1 \\ &= B_1M^{\dagger}A_1X_1V\left(B_1M^{\dagger}A_1X_1\right)^{(*)} \in \mathfrak{B}(\mathcal{H}_1)^{(+)}. \end{aligned}$$

Further, we have

$$\begin{aligned} \mathcal{R}\left(B_1M^{\dagger}C_1\right) &= \mathcal{R}\left(B_1M^{\dagger}A_1XB_1\right) \subseteq \mathcal{R}\left(B_1M^{\dagger}A_1X_1V\right) \\ &= \mathcal{R}\left(B_1M^{\dagger}A_1X_1V\left(B_1M^{\dagger}A_1X_1\right)^{(*)}\right) = \mathcal{R}\left(T\right) \end{aligned}$$

and $\mathcal{R}(A_1M^{\dagger}C_1^{(*)}) \subseteq \mathcal{R}\left(T\right)$.

For the sufficiency, we first show that when these conditions are true, G-positive solutions $Y$ and $Z$ to equations (3.7) can be so determined that

$$C_1 + C_1^{(*)} + Y + Z$$

is G-positive. Since $T$ is G-positive and

$$\mathcal{R}(A_1M^{\dagger}C_1^{(*)}) \subseteq \mathcal{R}\left(T\right),\ \mathcal{R}(B_1M^{\dagger}C_1) \subseteq \mathcal{R}\left(T\right),$$

then it follows from Theorem 3.1 that the general G-positive solutions to equations in (3.7) can be written as

$$Y = FT^{\dagger}F^{(*)} + \left(I - EE^{\dagger}\right)^{(*)} W_1 \left(I - EE^{\dagger}\right),$$
$$Z = L^{(*)}T^{\dagger}L + \left(I - K^{\dagger}K\right) W_2 \left(I - K^{\dagger}K\right)^{(*)},$$

where

$$F = C_1M^{\dagger}A_1, E = M^{\dagger}B_1, K = A_1M^{\dagger}, L = B_1M^{\dagger}C_1.$$

Hence

$$\begin{aligned} & C_1 + C_1^{(*)} + Y + Z \\ = & \left(F + L^{(*)}\right) T^{\dagger} \left(F^{(*)} + L\right) + \left(I - EE^{\dagger}\right)^{(*)} W_1 \left(I - EE^{\dagger}\right) \\ & + \left(I - K^{\dagger}K\right) W_2 \left(I - K^{\dagger}K\right)^{(*)} + C_1 - FT^{\dagger}L + C_1^{(*)} - L^{(*)}T^{\dagger}F^{(*)}. \end{aligned}$$

Recall (3.5), then

$$E^{(*)}(C_1 - FT^{\dagger}L) = B_1 \left(A_1 + B_1\right)^{\dagger} C_1 - TT^{\dagger}B_1 \left(A_1 + B_1\right)^{\dagger} C_1 = 0$$

and

$$\left(C_1 - FT^{\dagger}L\right) K^{(*)} = C_1 \left(A_1 + B_1\right)^{\dagger} A_1 - FT^{\dagger}T = 0.$$

Hence

$$\begin{aligned} C_1 + C_1^{(*)} + Y + Z = & \left(F + L^{(*)}\right) T^{\dagger} \left(F^{(*)} + L\right) \\ & + \left[ \begin{array}{cc} \left(I - EE^{\dagger}\right)^{(*)} \left(I - K^{\dagger}K\right) \end{array} \right] \\ & \left[ \begin{array}{cc} W_1 & C_1 - FT^{\dagger}L \\ \left(C_1 - FT^{\dagger}L\right)^{(*)} & W_2 \end{array} \right] \left[ \begin{array}{c} I - EE^{\dagger} \\ \left(I - K^{\dagger}K\right)^{(*)} \end{array} \right] \end{aligned}$$

is G-positive if $W_1$ and $W_2$ are so chosen that

$$\left[ \begin{array}{cc} W_1 & C_1 - FT^{\dagger}L \\ \left(C_1 - FT^{\dagger}L\right)^{(*)} & W_2 \end{array} \right]$$

is G-positive. One such choice is $W_1 = I$,
$W_2 = \left(C_1 - FT^{\dagger}L\right)^{(*)} \left(C_1 - FT^{\dagger}L\right)$, in view of Theorem 2.1.

Let $X_0$ be a G-positive solution of (1.1). Put $Y = A_1X_0A_1, Z = B_1X_0B_1$ and observe that $Y, Z$ are G-positive solutions to equations (3.7), such that

$$N = C_1 + C_1^{(*)} + Y + Z = \left(A_1 + B_1\right) X_0 \left(A_1 + B_1\right)$$

is also G-positive. Then by Theorem 3.2, $X_0$ can be expressed as (3.6). □

## 4. G-repositive solution to some adjointable operator equations over Hilbert $C^*$-modules

In order to get the G-repositive solution to some adjointable operator equations over the Hilbert $C^*$-modules, we need the following Lemma.

**Lemma 4.1.** *Let $\mathcal{H}, \mathcal{K}$ be Hilbert $C^*$-modules. Assume that $A \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$ and has closed range, $C \in \mathfrak{B}(\mathcal{K})$, then adjointable operator equation*

$$AX^{(*)} + XA^{(*)} = C, \tag{4.1}$$

*is consistent if and only if $C$ is G-Hermitian and $R_A C R_A^{(*)} = 0$. In this case, the general solution of equation (4.1) can be expressed as*

$$X = \frac{1}{2}\left(I + R_A\right)\left(C\left(A^{\dagger}\right)^{(*)} - Y\left(A^{\dagger}A\right)^{(*)}\right) + Y - \frac{1}{2}AY^{(*)}\left(A^{\dagger}\right)^{(*)}, \tag{4.2}$$

*where $V \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$ is arbitrary.*

*Proof.* On the assumption that $C$ is G-Hermitian and $R_A C R_A^{(*)} = 0$, it is easy to verify that $X = \frac{(I+R_A)C\left(A^{\dagger}\right)^{(*)}}{2}$ satisfies equation (4.1). For the other direction, suppose that $X_0$ satisfies the equation (4.1), then we have

$$C^{(*)} = \left(AX_0^{(*)} + X_0A^{(*)}\right)^{(*)} = C$$

and

$$R_A C R_A^{(*)} = R_A\left(AX^{(*)} + XA^{(*)}\right)R_A^{(*)} = 0.$$

Next we will prove that every solution of equation (4.1) can be expressed as (4.2). Suppose that $X_0$ is an arbitrary solution of the equation (4.1), and setting $Y = X_0$, we have

$$X_0 = \frac{1}{2}\left(I + R_A\right)\left(C\left(A^{\dagger}\right)^{(*)} - X_0\left(A^{\dagger}A\right)^{(*)}\right) + X_0 - \frac{1}{2}AX_0^{(*)}\left(A^{\dagger}\right)^{(*)}.$$

Then (4.2) is the general solution of equation (4.1). □

We can now prove the following result.

**Theorem 4.2.** *Let $A, C \in \mathfrak{B}(\mathcal{H}_1, \mathcal{H}_2)$, $A$ and $CA^{(*)}$ have closed ranges. Then adjointable operator equation (3.1) has a G-repositive solution $X \in \mathfrak{B}(\mathcal{H}_1)$ if and only if $CA^{(*)}$ is G-repositive and $\mathcal{R}(C) \subseteq \mathcal{R}\left(CA^{(*)}\right)$. In this case, the general G-repositive solution can be expressed as*

$$\begin{aligned} X &= X_0 + \frac{1}{2}L_A\left(H - \left(X_0 + X_0^{(*)}\right)\right)\left(I + \left(A^\dagger A\right)^{(*)}\right) \\ &\quad + \frac{1}{2}L_A Y L_A^* - \frac{1}{2}L_A Y^{(*)} L_A^{(*)} \end{aligned} \tag{4.3}$$

*with*

$$H = X_2 + X_2 X_2^\dagger E L_A^{(*)} + L_A E^{(*)}\left(X_2 X_2^\dagger\right)^{(*)} + L_A E^{(*)} X_2^\dagger E L_A^{(*)} + L_A F L_A^{(*)},$$

*and*

$$X_0 = A^\dagger C - \left(A^\dagger C\right)^{(*)} + A^\dagger A\left(A^\dagger C\right)^{(*)}, X_2 = A^\dagger\left(CA^{(*)} + AC^{(*)}\right)\left(A^\dagger\right)^{(*)},$$

*$E, Y \in \mathfrak{B}(\mathcal{H}_1)$ and $F \in \mathfrak{B}(\mathcal{H}_1)^{(+)}$ are arbitrary.*

*Proof.* Suppose equation (3.1) has a G-repositive solution $X$, then $AA^\dagger C = AA^\dagger AX = C$. Further, $CA^{(*)} + AC^{(*)} = A\left(X + X^{(*)}\right)A^{(*)} \in \mathfrak{B}(\mathcal{H}_1)^{(+)}$, implying that $CA^{(*)}$ is G-repositive. Conversely, suppose $AA^\dagger C = C$ and $CA^{(*)}$ is G-repositive. Denote $X_0 = A^\dagger C - \left(A^\dagger C\right)^{(*)} + A^\dagger A\left(A^\dagger C\right)^{(*)}$, then $AX_0 = C$ and

$$X_0 + X_0^{(*)} = A^\dagger\left(CA^{(*)} + AC^{(*)}\right)\left(A^\dagger\right)^{(*)} \in \mathfrak{B}(\mathcal{H}_1)^{(+)}.$$

Next we show that every G-repositive solution to equation (3.1) can be written as (4.4). It is well known that the general solution of equation (3.1) can be expressed as

$$X = X_0 + L_A W, \tag{4.4}$$

where $W$ is free to vary over $\mathfrak{B}(\mathcal{K}_2, \mathcal{H})$. Denote

$$X + X^{(*)} = X_0 + X_0^{(*)} + L_A W + (L_A W)^{(*)} = H$$

then

$$X + X^{(*)} \in \mathfrak{B}(\mathcal{H}_1)^{(+)} \Leftrightarrow H \in \mathfrak{B}(\mathcal{H}_1)^{(+)}.$$

If you fix $H$, it is easy to get $H = H^{(*)}$, from Lemma 4.1, equation

$$L_A W + (L_A W)^{(*)} = H - \left(X_0 + X_0^{(*)}\right)$$

is consistent for $W$ if and only if

$$R_{L_A}CR_{L_A}^{(*)} = A^\dagger A\left(H - \left(X_0 + X_0^{(*)}\right)\right)\left(A^\dagger A\right)^{(*)} = 0,$$

which is equivalent to equation

$$A^\dagger AH\left(A^\dagger A\right)^{(*)} = A^\dagger A\left(X_0 + X_0^{(*)}\right)\left(A^\dagger A\right)^{(*)} \tag{4.5}$$

has a G-positive solution $H$. Recall that $CA^{(*)}$ is G-repositive, thus

$$\begin{aligned}&A^\dagger A\left(X_0 + X_0^{(*)}\right)\left(A^\dagger A\right)^{(*)}\\ =&A^\dagger A\left(A^\dagger\left(CA^{(*)} + AC^{(*)}\right)\left(A^\dagger\right)^{(*)}\right)\left(A^\dagger A\right)^{(*)} \in \mathfrak{B}(\mathcal{H})^{(+)}.\end{aligned}$$

It follows from Theorem 3.2 that equation (4.5) has a G-positive solution which can be expressed as

$$\begin{aligned}H &= X_2 + X_2X_2^\dagger EL_A^{(*)} + L_AE^{(*)}\left(X_2X_2^\dagger\right)^{(*)} + L_AE^{(*)}X_2^\dagger EL_A^{(*)}\\ &\quad + L_AFL_A^{(*)},\end{aligned} \tag{4.6}$$

where $X_2 = A^\dagger\left(CA^{(*)} + AC^{(*)}\right)\left(A^\dagger\right)^{(*)}$, $E \in \mathfrak{B}(\mathcal{H})$ and $F \in \mathfrak{B}(\mathcal{H})^{(+)}$ are arbitrary. Combining Lemma 4.1, we have

$$\begin{aligned}W &= \frac{1}{2}\left(L_AH - L_A\left(X_0 + X_0^{(*)}\right) - L_AY\right)\left(I + \left(A^\dagger A\right)^{(*)}\right)\\ &\quad + Y - \frac{1}{2}L_AY^{(*)}L_A^{(*)},\end{aligned} \tag{4.7}$$

where $Y$ is arbitrary. Further, the G-repositive solution of equation (3.1) can be expressed as (4.4) after taking (4.8) into (4.4). □

The following Lemma plays an important role to get the G-repositive solution of equation (3.3).

**Lemma 4.3.** *Let $\mathcal{H}, \mathcal{K}$ be Hilbert $C^*$-modules. Assume that $A \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$ and has closed range. Then the general G-skew-Hermitian solution of*

$$AXA^{(*)} = 0 \tag{4.8}$$

*can be expressed as*

$$X = L_AW - W^{(*)}L_A^{(*)}, \tag{4.9}$$

*where $W \in \mathfrak{B}(\mathcal{K})$ is arbitrary.*

*Proof.* Obviously, $X$ defined by (4.9) is G-skew-Hermitian and satisfies equation (4.8). Suppose $X_0$ is a G-skew-Hermitian solution to (4.8), setting $W = \frac{1}{2}X_0\left(I + A^\dagger A\right)$, then

$$X_0 = \frac{1}{2}L_A X_0\left(I + A^\dagger A\right) - \frac{1}{2}\left(X_0\left(I + A^\dagger A\right)\right)^{(*)} L_A^{(*)}.$$

Thus (4.9) is the general G-skew-Hermitian solution to equation (4.8). □

We can now prove the following result.

**Theorem 4.4.** *Let $\mathcal{H}, \mathcal{K}$ be Hilbert $C^*$-modules. Assume that $A \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$ and has closed range, $C \in \mathfrak{B}(\mathcal{K})$ is G-Hermitian and $\mathcal{R}(C) \subseteq \mathcal{R}(A)$. Then equation* (3.3) *has a G-repositive solution $X \in \mathfrak{B}(H)$ if and only if $C$ is G-repositive. If, in addition, $C$ has closed range, the general G-repositive solution of (3.3) can be expressed as*

$$\begin{aligned}(4.10) X \;=\; & X_2 + X_2X_2^\dagger EL_A^{(*)} + L_AE^{(*)}\left(X_2X_2^\dagger\right)^{(*)} + L_AE^{(*)}X_2^\dagger EL_A^{(*)} \\ & + L_AFL_A^{(*)} + L_AW - W^{(*)}L_A^{(*)}\end{aligned}$$

*with $X_2 = A^\dagger C\left(A^\dagger\right)^{(*)}$ is a particular G-repositive solution, $E, W \in \mathfrak{B}(\mathcal{H})$ and $F \in \mathfrak{B}(\mathcal{H})^{(+)}$ are arbitrary.*

*Proof.* Denote $H_p(X) = \frac{1}{2}\left(X + X^{(*)}\right)$. If $X$ is a G-repositive solution to equation (3.3), then $H_p(C) = AH_p(X)A^{(*)} \in \mathfrak{B}(\mathcal{H})^{(+)}$. For the other direction, if $C$ is G-repositive then $X_2 = A^\dagger C\left(A^\dagger\right)^{(*)}$ is a G-repositive solution to equation (3.3). Next we will show every G-repositive solution of (3.3) can be expressed as (4.11). If $X$ is an arbitrary G-repositive solution to equation (3.3), then $H_p(X)$ is a G-positive solution to

$$AH_p(X)A^{(*)} = H_p(C). \tag{4.11}$$

It follows from Theorem 3.2 that

$$H_p(X) = X_2 + X_2X_2^\dagger EL_A^{(*)} + L_AE^{(*)}\left(X_2X_2^\dagger\right)^{(*)} + L_AE^{(*)}X_2^\dagger EL_A^{(*)} + L_AFL_A^{(*)}$$

where $X_2 = A^\dagger H_p(C)\left(A^\dagger\right)^{(*)}$ is a particular G-positive solution to equation (4.11), $E \in \mathfrak{B}(\mathcal{H})$ and $F \in \mathfrak{B}(\mathcal{H})^{(+)}$ are arbitrary. Note that

$$\begin{aligned} H_p(X) &= H_p\left(A^\dagger C\left(A^\dagger\right)^{(*)} + X_2X_2^\dagger EL_A^{(*)} + L_AE^{(*)}\left(X_2X_2^\dagger\right)^{(*)}\right. \\ &\quad \left. + L_AE^{(*)}X_2^\dagger EL_A^{(*)} + L_AFL_A^{(*)}\right) \end{aligned}$$

implies that

$$\begin{aligned} X &= A^\dagger C\left(A^\dagger\right)^{(*)} + X_2X_2^\dagger EL_A^{(*)} + L_AE^{(*)}\left(X_2X_2^\dagger\right)^{(*)} + L_AE^{(*)}X_2^\dagger EL_A^{(*)} \\ &\quad + L_AFL_A^{(*)} + Z \end{aligned}$$

where $H_p(Z) = 0$ and $AZA^{(*)} = 0$. By the use of Lemma 4.3, we have

$$Z = L_AW - W^{(*)}L_A^{(*)},$$

where $W \in \mathfrak{B}(\mathcal{H})$ is arbitrary. Hence, (4.11) is the general G-repositive solution to equation (3.3). □

**Theorem 4.5.** *Let $\mathcal{H}_i (i = 1, 2, \cdots, 4)$ be Hilbert $C^*$-modules, $A \in \mathfrak{B}(\mathcal{H}_1, \mathcal{H}_2)$, $B \in \mathfrak{B}(\mathcal{H}_3, \mathcal{H}_1)$, $C \in \mathfrak{B}(\mathcal{H}_3, \mathcal{H}_2)$. Denote*

$$\begin{aligned} A_1 &= A^{(*)}V^{-1}A, B_1 \\ &= BVB^{(*)}, C_1 \\ &= A^{(*)}CB^{(*)}, M \\ &= A_1 + B_1, T \\ &= B_1M^\dagger C_1M^\dagger A_1, \end{aligned}$$

*and*

$$H(X) = X + X^{(*)}.$$

*Suppose that $A, B, T$ and $M$ have closed ranges. Then consistent equation (1.1) has a G-repositive solution if and only if $T$ is G-repositive. In this case, the general G-repositive solution of equation (1.1) can be expressed as*

(4.12)

$$\begin{aligned} X &= M^\dagger(C_1 + Y + Z + W)M^\dagger + X_2 + X_2X_2^\dagger EL_A^{(*)} \\ &\quad + L_AE^{(*)}\left(X_2X_2^\dagger\right)^{(*)} + L_AE^{(*)}X_2^\dagger EL_A^{(*)} + L_AFL_A^{(*)} + L_AW - W^{(*)}L_A^{(*)}, \end{aligned}$$

*where $Y$, $Z$ and $W$ are arbitrary solutions of*

$$YM^{\dagger}B_1 = C_1M^{\dagger}A_1,\ A_1M^{\dagger}Z = B_1M^{\dagger}C_1,\ AM^{\dagger}WM^{\dagger}B = T \tag{4.13}$$

*such that $C_1+Y+Z+W$ is G-repositive, $X_2 = M^{\dagger}\left(C_1 + Y + Z + W\right)M^{\dagger}$, $E, W \in \mathfrak{B}(\mathcal{H}_1)$ and $F \in \mathfrak{B}(\mathcal{H}_1)^{(+)}$ are arbitrary.*

*Proof.* Since $\mathcal{R}(B_1) \subseteq \mathcal{R}(M)$, then $M^{\dagger}B_1B_1^{\dagger}MM^{\dagger}B_1 = M^{\dagger}B_1$, which gives that $M^{\dagger}B_1$ is regular. Note that

$$\begin{aligned} C_1M^{\dagger}A_1\left(M^{\dagger}B_1\right)^{-}M^{\dagger}B_1 &= C_1M^{\dagger}A_1B_1^{\dagger}MM^{\dagger}B_1 \\ &= C_1M^{\dagger}A_1B_1^{\dagger}B_1 \\ &= C_1B_1^{\dagger}B_1M^{\dagger}A_1B_1^{\dagger}B_1 \\ &= C_1B_1^{\dagger}A_1M^{\dagger}B_1 \\ &= C_1M^{\dagger}A_1, \end{aligned}$$

then equation $YM^{\dagger}B_1 = C_1M^{\dagger}A_1$ is consistent. Similarly, we can get the other equations in (4.13) are consistent. Now suppose that equation (1.1) has a G-repositive solution $X$, then

$$\begin{aligned} H\left(T\right) &= H\left(B_1M^{\dagger}C_1M^{\dagger}A_1\right) = H\left(B_1M^{\dagger}A_1XB_1M^{\dagger}A_1\right) \\ &= B_1M^{\dagger}A_1H\left(X\right)\left(B_1M^{\dagger}A_1\right)^{(*)} \in \mathfrak{B}(\mathcal{H})^{(+)}. \end{aligned}$$

For the other direction, suppose $T$ is G-repositive. Since

$$\begin{aligned} &A_1M^{\dagger}\left(C_1 + Y + Z + W\right)M^{\dagger}B_1 \\ &= A_1M^{\dagger}C_1M^{\dagger}B_1 + A_1M^{\dagger}YM^{\dagger}B_1 + A_1M^{\dagger}ZM^{\dagger}B_1 + A_1M^{\dagger}WM^{\dagger}B_1 \\ &= A_1M^{\dagger}C_1M^{\dagger}B_1 + A_1M^{\dagger}C_1M^{\dagger}A_1 + B_1M^{\dagger}C_1M^{\dagger}B_1 + B_1M^{\dagger}C_1M^{\dagger}A_1 \\ &= \left(A_1 + B_1\right)M^{\dagger}C_1M^{\dagger}\left(A_1 + B_1\right) = C_1, \end{aligned}$$

thus

$$X_0 = M^{\dagger}\left(C_1 + Y + Z + W\right)\left(M^{\dagger}\right)^{(*)}$$

satisfies equation (1.1), where $Y, Z$ and $W$ are solutions of equations in (4.13). Next, we only need to prove that for some proper choices of $Y, Z$ and $W$, $C_1 + Y + Z + W$ is G-repositive, which is equivalent to $X_0$ is G-repositive. Setting

$$M^{\dagger}B_1 = E, F = C_1M^{\dagger}A_1, K = A_1M^{\dagger}, L = B_1M^{\dagger}C_1,$$

and choose some special solutions of equations in (4.13),

$$\begin{aligned} Y &= FE^{\dagger} - \left(FE^{\dagger}\right)^{(*)} + \left(E^{\dagger}\right)^{(*)} F^{(*)}EE^{\dagger} + R_E^{(*)}R_E, \\ Z &= K^{\dagger}L - \left(K^{\dagger}L\right)^{(*)} + K^{\dagger}K\left(K^{\dagger}L\right)^{(*)} + L_KQL_K^{(*)}, \\ W &= K^{\dagger}TE^{\dagger} - L_KS - SR_E, \end{aligned}$$

where

$$Q = \left(C_1^{(*)} - K^{\dagger}T^{(*)}E^{\dagger}\right)\left(C_1^{(*)} - K^{\dagger}T^{(*)}E^{\dagger}\right)^{(*)}, S = K^{\dagger}KC^{(*)} + C^{(*)}EE^{\dagger}.$$

Obviously,

$$\begin{aligned} H(Y) &= \left(E^{\dagger}\right)^{(*)} G(T)\, EE^{\dagger} + R_E^{(*)}R_E, \\ H(Z) &= K^{\dagger}G(T)\left(K^{\dagger}\right)^{(*)} + L_KG(Q)\, L_K^{(*)}, \\ H(W) &= K^{\dagger}TE^{\dagger} + \left(K^{\dagger}TE^{\dagger}\right)^{(*)} - G\left(C_1^{(*)}EE^{\dagger} + K^{\dagger}KC_1^{(*)} - 2K^{\dagger}T^{(*)}E^{\dagger}\right). \end{aligned}$$

Noting that

$$\begin{aligned} K^{\dagger}KK^{\dagger}T^{(*)}E^{\dagger} &= K^{\dagger}KL^{(*)}E^{\dagger} = K^{\dagger}T^{(*)}E^{\dagger}, \\ K^{\dagger}T^{(*)}E^{\dagger}EE^{\dagger} &= K^{\dagger}F^{(*)}EE^{\dagger} = K^{\dagger}T^{(*)}E^{\dagger}, \\ KC^{(*)}E &= KL^{(*)} = T^{(*)}, \end{aligned}$$

we have

$$\begin{aligned} &H(C_1 + Y + Z + W) \\ &= \left(\left(E^{\dagger}\right)^{(*)} + K^{\dagger}\right) H(T) \left(\left(E^{\dagger}\right)^{(*)} + K^{\dagger}\right)^{(*)} \\ &+ \begin{bmatrix} \left(I - EE^{\dagger}\right)^{(*)} & \left(I - K^{\dagger}K\right) \end{bmatrix} D \begin{bmatrix} I - EE^{\dagger} \\ \left(I - K^{\dagger}K\right)^{(*)} \end{bmatrix} \end{aligned}$$

where

$$D = \begin{bmatrix} I & C_1 - \left(E^{\dagger}\right)^{(*)} T^{\dagger}\left(K^{\dagger}\right)^{(*)} \\ \left(C_1 - \left(E^{\dagger}\right)^{(*)} T^{\dagger}\left(K^{\dagger}\right)^{(*)}\right)^{(*)} & H(Q) \end{bmatrix}.$$

It follows from Theorem 2.1, that $D$ is G-positive, then $X_0$ is G-repositive. Hence, with such a choice of $Y, Z$ and $W$, it can be seen the $X_0$ is a G-repositive solution to equation (3.6). Next we only need to prove that

every G-repositive solution of (1.1) can be written as form (4.12). Suppose $X$ is an arbitrary G-repositive solution of equation (1.1). It is easy to verify that $Y = AXA, Z = BXB, W = BXA$ are solutions of equations in (4.13), and

$$(A+B)\,X\,(A+B) = C + Y + Z + W,$$

is G-repositive. Then it follows by Theorem 4.4 that $X$ can be expressed as (4.12). □

## 5. **Conclusion**

In this paper, we have presented a criterion for an adjointable operator to be G-positive (G-repositive), and derived some necessary and sufficient conditions for the existence of a G-positive (G-repositive) solution to adjointable operator equations $AX = C$, $AXA^{(*)} = C$ and $AXB = C$ over Hilbert $C^*$-modules, respectively. Moreover, we have shown the expressions of the general G-positive and G-repositive solutions to these equations when the solvability conditions are satisfied.

Motivated by the results in the paper, and observing that equations (3.1), (3.3) and (1.1) are all special cases of the system of adjointable operator equations

$$A_1X = C_1, XB_2 = C_2,\ A_3XB_3 = C_3 \tag{5.1}$$

over Hilbert $C^*$-modules, it would be of interest to investigate the G-positive solutions to system (5.1). We will show the results in a forthcoming paper.

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