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LIMIT DISTRIBUTION OF DEGREES IN SCALED ATTACHMENT RANDOM RECURSIVE TREES

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Abstract. We study the limiting distribution of the degree of a given node in a scaled attachment random recursive tree, a generalized random recursive tree, which is introduced by Devroye et. al (2011). In a scaled attachment random recursive tree, every node i is attached to the node labeled $\lfloor iX_i \rfloor$ where $X_0, \ldots, X_n$ is a sequence of i.i.d. random variables, with support in [0, 1) and distribution function F. By imposing a condition on F, we show that the degree of a given node is asymptotically normal.

1. Introduction

The degree of a node is well motivated and of importance for the structure of the tree. It has been studied in a wide variety of tree models (see, e.g., [2, 7, 11]). This paper is devoted to establish a limit theorem for the degree of a node in scaled attachment random recursive trees.

A *uniform random recursive tree* of size n is a random tree that starts from a node labeled 0 (the root) and where at stage i ($i = 2, \ldots, n+1$) a new node labeled $i-1$ is attached uniformly at random to one of the existing nodes until the total number of nodes is equal to $n+1$. See [5] for a survey of early results for recursive trees.

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There are several generalizations of uniform random recursive trees (see, e.g., [1, 4, 10]). A *scaled attachment random recursive tree* is a recent generalization of uniform random recursive trees which has been defined in [3]. We are given a random variable X, with support in [0, 1) and distribution function F. In a scaled attachment random recursive tree, a node i is attached to the node labeled $\lfloor iX_i \rfloor$ where X_0, X_1, … ,X_n is a sequence of i.i.d. random variables with distribution function F. In particular, if X is uniform on [0, 1) we get a uniform random recursive tree.

In [6] the outdegree of a specified node has been studied for recursive trees, where exact and asymptotic formulas for the probability distribution are given. All factorial moments and limiting distribution results of node degree for random increasing trees (such as recursive trees, plane-oriented recursive trees and binary increasing trees), have been presented in [7]. Thus the exact and asymptotic formula presented in [7] extend the few known results on this subject.

Here we obtain an asymptotic normal distribution for the degree of an arbitrary node in scaled attachment random recursive trees. We base our argument of asymptotic normality of degrees on the following theorem quoted from [8], p. 41.

Theorem 1.1. (Lyapunov) *Let Y_1, Y_2, Y_3, … be a sequence of independent random variables. Let $\mathbf{E}[Y_n] = \mu_n$, $\mathbf{Var}[Y_n] = \sigma_n^2$, and $\mathbf{E}[|Y_n - \mu_n|^3] = \beta_n$ exist for each n ($\sigma_n \neq 0$ for at least one value of n). Furthermore, let*

$$A_n = \Big(\sum_{j=1}^{n} \beta_j\Big)^{\frac{1}{3}} \text{ and } B_n = \Big(\sum_{j=1}^{n} \sigma_j^2\Big)^{\frac{1}{2}}.$$

If $\lim_{n\to\infty} \frac{A_n}{B_n} = 0$, then $\sum_{j=1}^{n} \frac{Y_j - \mu_j}{B_n}$ converges in distribution to a standard normal random variable.

2. Main result

Let the random variable $D_{n,i}$ count the node degree (i.e., the outdegree) of a specified node i (with $0 \le i \le n$) in a scaled attachment random recursive tree of size n. Let I_A denote the indicator of the event

A. We have

$$\begin{aligned} D_{n,i} &= \sum_{j=i+1}^{n} I_{\{\text{node } i \text{ is the parent of node } j\}} \\ &= \sum_{j=i+1}^{n} I_{\{\lfloor jX_j \rfloor = i\}}, \end{aligned}$$

where the random variables $I_{\{\lfloor jX_j \rfloor = i\}}$ are independent. Also,

$$\begin{aligned} \mathbf{P}(I_{\{\lfloor jX_j \rfloor = i\}} = 1) &= \mathbf{P}(i \le jX_j < i+1) \\ &= \mathbf{P}\Big(\frac{i}{j} \le X_j < \frac{i+1}{j}\Big) \\ &= F\Big(\frac{i+1}{j} -\Big) - F\Big(\frac{i}{j} -\Big), \end{aligned}$$

where $F(x-)$ denotes the left hand limit of F at x.

Let $\mu_j := \mathbf{E}[I_{\{\lfloor jX_j \rfloor = i\}}]$, $\sigma_j^2 := \mathbf{Var}[I_{\{\lfloor jX_j \rfloor = i\}}]$, and $\beta_j := \mathbf{E}\big[\big|I_{\{\lfloor jX_j \rfloor = i\}} - \mu_j\big|^3\big]$. Then

$$\begin{aligned} \mu_j &= F\Big(\frac{i+1}{j} -\Big) - F\Big(\frac{i}{j} -\Big), \\ \sigma_j^2 &= \Big[F\Big(\frac{i+1}{j} -\Big) - F\Big(\frac{i}{j} -\Big)\Big] - \Big[F\Big(\frac{i+1}{j} -\Big) - F\Big(\frac{i}{j} -\Big)\Big]^2 \\ &:= \Big[F\Big(\frac{i+1}{j} -\Big) - F\Big(\frac{i}{j} -\Big)\Big] - G_{1j} \qquad (2.1) \\ &< F\Big(\frac{i+1}{j} -\Big) - F\Big(\frac{i}{j} -\Big) \qquad (2.2) \end{aligned}$$

and

$$\begin{aligned}
\beta_j \quad = \quad & \Big[1 - F\Big(\frac{i+1}{j} -\Big) + F\Big(\frac{i}{j} -\Big)\Big]^3 \Big[F\Big(\frac{i+1}{j} -\Big) - F\Big(\frac{i}{j} -\Big)\Big] \\
& + \Big[F\Big(\frac{i+1}{j} -\Big) - F\Big(\frac{i}{j} -\Big)\Big]^3 \Big[1 - F\Big(\frac{i+1}{j} -\Big) + F\Big(\frac{i}{j} -\Big)\Big] \\
= \quad & \Big[F\Big(\frac{i+1}{j} -\Big) - F\Big(\frac{i}{j} -\Big)\Big] - 3\Big[F\Big(\frac{i+1}{j} -\Big) - F\Big(\frac{i}{j} -\Big)\Big]^2 \\
& + 4\Big[F\Big(\frac{i+1}{j} -\Big) - F\Big(\frac{i}{j} -\Big)\Big]^3 - 2\Big[F\Big(\frac{i+1}{j} -\Big) - F\Big(\frac{i}{j} -\Big)\Big]^4 \\
(2.3) := \quad & \Big[F\Big(\frac{i+1}{j} -\Big) - F\Big(\frac{i}{j} -\Big)\Big] - G_{2j} \\
(2.4) < \quad & F\Big(\frac{i+1}{j} -\Big) - F\Big(\frac{i}{j} -\Big) \text{ (since } G_{2j} > 0).
\end{aligned}$$

Define $A_n := (\sum_{j=i+1}^{n} \beta_j)^{\frac{1}{3}}$ and $B_n := (\sum_{j=i+1}^{n} \sigma_j^2)^{\frac{1}{2}}$. Thus by verifying $\lim_{n\to\infty} \frac{A_n}{B_n} = 0$, we may apply Theorem 1 with $Y_j := I_{\{\lfloor jX_j \rfloor = i\}}$, $j \geq i+1$.

If $\sum_{j=i+1}^{\infty} \big[F\big(\frac{i+1}{j} -\big) - F\big(\frac{i}{j} -\big)\big] < \infty$, by (2.2) and (2.4), we have

$$0 < \lim_{n\to\infty} A_n = \Big(\sum_{j=i+1}^{\infty} \beta_j\Big)^{\frac{1}{3}} < \Big(\sum_{j=i+1}^{\infty} \Big[F\Big(\frac{i+1}{j} -\Big) - F\Big(\frac{i}{j} -\Big)\Big]\Big)^{\frac{1}{3}} < \infty$$

and

$$0 < \lim_{n\to\infty} B_n = \Big(\sum_{j=i+1}^{\infty} \sigma_j^2\Big)^{\frac{1}{2}} < \Big(\sum_{j=i+1}^{\infty} \Big[F\Big(\frac{i+1}{j} -\Big) - F\Big(\frac{i}{j} -\Big)\Big]\Big)^{\frac{1}{2}} < \infty.$$

So the condition $\lim_{n\to\infty} \frac{A_n}{B_n} = 0$ does not hold, unless $\sum_{j=i+1}^{\infty} \big[F\big(\frac{i+1}{j} -\big) - F\big(\frac{i}{j} -\big)\big] = \infty$.

Since for all n, $\sum_{j=i+1}^{n} \sigma_j^2 > 0$ and $\sum_{j=i+1}^{n} \beta_j > 0$, then, by (2.1) and (2.3),

$$(2.5) \qquad 0 < \frac{\sum_{j=i+1}^{n} G_{1j}}{\sum_{j=i+1}^{n} \big[F\big(\frac{i+1}{j} -\big) - F\big(\frac{i}{j} -\big)\big]} < 1, \text{ for all } n,$$

and

$$(2.6) \qquad 0 < \frac{\sum_{j=i+1}^{n} G_{2j}}{\sum_{j=i+1}^{n} \big[F\big(\frac{i+1}{j} -\big) - F\big(\frac{i}{j} -\big)\big]} < 1, \text{ for all } n.$$

Now if $\sum_{j=i+1}^{\infty}\left[F\left(\frac{i+1}{j}-\right)-F\left(\frac{i}{j}-\right)\right]=\infty$, by (2.1), (2.3), (2.5) and (2.6) we have

$$\lim_{n\to\infty}\frac{A_n}{B_n}=\lim_{n\to\infty}\frac{1}{\left(\sum_{j=i+1}^{n}\left[F\left(\frac{i+1}{j}-\right)-F\left(\frac{i}{j}-\right)\right]\right)^{\frac{1}{6}}}$$
$$\times\lim_{n\to\infty}\frac{\left(1-\sum_{j=i+1}^{n}G_{2j}/\sum_{j=i+1}^{n}\left[F\left(\frac{i+1}{j}-\right)-F\left(\frac{i}{j}-\right)\right]\right)^{\frac{1}{3}}}{\left(1-\sum_{j=i+1}^{n}G_{1j}/\sum_{j=i+1}^{n}\left[F\left(\frac{i+1}{j}-\right)-F\left(\frac{i}{j}-\right)\right]\right)^{\frac{1}{2}}}$$
$$=0.$$

This completes the proof of the following theorem.

Theorem 2.1. *Let $D_{n,i}$ denote the degree of the node labeled i in a scaled attachment random recursive tree of size n. If the attachment distribution F satisfies*

$$\sum_{j=i+1}^{\infty}\left[F\left(\frac{i+1}{j}-\right)-F\left(\frac{i}{j}-\right)\right]=\infty,$$

then

$$\frac{D_{n,i}-\sum_{j=i+1}^{n}\left[F\left(\frac{i+1}{j}-\right)-F\left(\frac{i}{j}-\right)\right]}{\sqrt{\sum_{j=i+1}^{n}\left(\left[F\left(\frac{i+1}{j}-\right)-F\left(\frac{i}{j}-\right)\right]-\left[F\left(\frac{i+1}{j}-\right)-F\left(\frac{i}{j}-\right)\right]^2\right)}}$$

converges in distribution to a standard normal random variable $\mathcal{N}(0,1)$ as $n\longrightarrow\infty$.

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