BIFURCATION OF LIMIT CYCLES FROM A QUADRATIC REVERSIBLE CENTER WITH THE UNBOUNDED ELLIPTIC SEPARATRIX

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ABSTRACT. The paper is concerned with the bifurcation of limit cycles in general quadratic perturbations of a quadratic reversible and non-Hamiltonian system, whose period annulus is bounded by an elliptic separatrix related to a singularity at infinity in the Poincaré disk. Attention goes to the number of limit cycles produced by the period annulus under perturbations. By using the appropriate Picard-Fuchs equations and studying the geometric properties of two planar curves, we prove that the maximal number of limit cycles bifurcating from the period annulus under small quadratic perturbations is two.

1. Introduction

In [23], a classification is given for quadratic integrable systems with at least one center. Following [12], such systems can be put into five classes in the complex form:

$$(1) \quad \dot{z} = -iz - z^2 + 2|z|^2 + (b+ic)\bar{z}^2, \quad \text{Hamiltonian}(Q_3^H),$$

(2) $\dot{z} = -iz + az^2 + 2|z|^2 + b\bar{z}^2,$ reversible (Q_3^R),

(3) $\dot{z} = -iz + 4z^2 + 2|z|^2 + (b + ic)\bar{z}^2,$ $|b + ic| = 2$, codimension four (Q_4),

(4) $\dot{z} = -iz + z^2 + (b + ic)\bar{z}^2,$ generalized Lotka-Volterra (Q_3^{LV}),

(5) $\dot{z} = -iz + \bar{z}^2,$ Hamiltonian triangle,
        where a, b and c stand for arbitrary real constants.

A quadratic integrable system is called generic if it belongs to one of the first four class and does not belong to other integrable class. Otherwise, it is called degenerate.

Generally, limit cycles arising from a quadratic center are those from either the center itself, or the period annulus surrounding it, or the seperatrix cycle which bounds the period annulus in the Poincaré disk. As usual, we use the notion of *cyclicity* (see Definition 2.2) for the total number of limit cycles which can emerge from a configuration of trajectories (center, period annulus, seperatrix cycle or singular loop) under perturbations. The problem of the cyclicity of the center point itself was completely solved by Bautin in the early 1950's (see [1]). The bifurcation of limit cycles from saddle-loop in perturbations of quadratic Hamiltonian systems has been studied in [11]. Moreover, if the loop contains only one saddle and certain genericity conditions hold, it was proved in [18] that the cyclicity of a singular loop can be transferred to the cyclicity of the period annuli. However, if the loop contains at least two saddles, this transfer in general is not true. For more details, we refer to [17] and references therein.

The present paper is devoted to studying the number of limit cycles produced by the period annulus of a generic quadratic reversible center with an unbounded elliptic separatrix under small perturbations. As is well known, the maximal number of limit cycles bifurcating from one period annulus under perturbations is less than or equal to the exact upper bound of zeros of the associated Abelian integral. Therefore the study of the number of limit cycles from the period annulus under perturbations can be transferred to counting the number of isolated zeros of the associated Abelian integral.

For quadratic Hamiltonian systems with quadratic perturbations, the exact upper bound of the number of zeros of the associated Abelian integrals is 2, see [4,7,10,16,24]. It is well known that the orbital topological

properties of quadratic reversible systems under quadratic perturbations are very rich and hence this case is very interesting. For some results concerning the quadratic perturbations from the reversible systems, we refer to [2, 3, 5, 6, 13, 14, 19–22] and references therein. However, known results are very limited.

Recently, Gautier et al [9] proved that there are 18 classes of reversible centers of genus one, whose phase portraits contain only elliptic curves. In the present paper, we take the following generic reversible system

$$\begin{aligned} \dot{x} &= y + 4x^2, \\ \dot{y} &= -x(1 - 6y) \end{aligned}$$

(1.1)

from [9] and investigate the bifurcation of limit cycles from its period annulus under small quadratic perturbations.

By taking the change $(x, y) \to (y, -x)$, system (1.1) is transformed into

$$\begin{aligned} \dot{x} &= y(1 + 6x), \\ \dot{y} &= -x + 4y^2, \end{aligned}$$

(1.2)

which has a first integral of the form

$$H^*(x, y) = (1 + 6x)^{-\frac{4}{3}}\left(\frac{1}{2}y^2 - \frac{1}{12}(1 + 6x) + \frac{1}{48}\right) = h$$

with the integrating factor $(1 + 6x)^{-7/3}$.

From [12], it is easy to know that the Abelian integral related to system (1.2) under small quadratic perturbations is the following

$$I(h) = \alpha I_0^*(h) + \beta I_1^*(h) + \gamma I_{-1}^*(h),$$

where the elliptic integrals $I_k^*(h) = \oint_{H^*(x,y)=h}(1 + 6x)^{k-7/3}ydx, k = 0, 1, -1$. α, β and γ are any real constants.

After making the change $(x, y, t) \to ((X - 1)/6, Y, \tau/6)$ and writing (x, y, t) instead of (X, Y, τ), we get the equivalent system of system (1.2)

$$\begin{aligned} \dot{x} &= xy, \\ \dot{y} &= \frac{2}{3}y^2 - \frac{1}{36}x + \frac{1}{36} \end{aligned}$$

(1.3)

with a first integral of the form

$$H(x, y) = x^{-\frac{4}{3}}\left(\frac{1}{2}y^2 - \frac{1}{12}x + \frac{1}{48}\right) = h$$

and the corresponding integrating factor $x^{-7/3}$. At the same time, the associated Abelian integral can be expressed as follows

$$I(h) = \alpha I_0(h) + \beta I_1(h) + \gamma I_{-1}(h),$$

where $I_k(h) = \oint_{H(x,y)=h} x^{k-7/3} y dx, k = 0, 1, -1$. α, β and γ are the same as above.

As usual in this theory, we derive the closed Picard-Fuchs equations, then get the related differential equation of $P(h) = I_1(h)/I_0(h), Q(h) = I_{-1}(h)/I_0(h)$ and $R(h) = I_{-1/3}(h)/I_0(h)$, which is in fact not easy to deal with. In addition to this, since the outer boundary of the period annulus of system (1.3) is an elliptic separatrix Γ_0 (see section 2) related to a singularity at infinity, we must use some other method to compute the asymptotic expansion of $I_k(h)$ with $k = 0, 1, -1$ and $-1/3$ near the boundary. In the present paper, we first define the functions $\nu(h) = I''_{-1}(h)/I''_0(h)$ and $\omega(s) = \nu(h)(s = h^3)$, then make an intensive study of the properties of two curves $\Sigma = \{(P,Q)(h) : h \in (-1/16, 0)\}$ and $C_\omega = \{(s, \omega(s)) : s \in (-1/4096, 0)\}$. Using the related Picard-Fuchs equations, Ricatti equations and other techniques, we prove the main result of this paper.

Theorem 1.1. *Under small quadratic perturbations, at most two limit cycles can bifurcate from the period annulus of system* (1.3)*, and this bound is sharp.*

Recall that the maximal number of limit cycles arising from the period annulus of system (1.3) under small quadratic perturbations is also called the cyclicity of the period annulus under the perturbations. Theorem 1.1 implies the following result.

Theorem 1.2. *The cyclicity of the period annulus of system* (1.3) *under small quadratic perturbations is equal to two.*

Theorem 1.2 is just Conjecture 1 for (1.1) in [9]. Hence this paper can be regarded as a part of verifying the conjecture.

Two points should be stressed. First of all, we study the Abelian integral $I(h)$ related to one-parameter perturbations of system (1.1), and thus the cyclicity of the period annulus under one-parameter quadratic perturbations is two. However, the result will be the same for multi-parameter perturbations. In fact, let X_λ be a family of analytic plane vector fields depending analytically on a parameter $\lambda \in (\mathbb{R}^k, 0)$, also called a k-parameter deformation of X_0. It follows from Theorem 1 in [8] that for any k-parameter deformation X_λ about which the compact

invariant set of X_0 has finite cyclicity, there exists a germ of analytic curve $\varepsilon \to \lambda(\varepsilon)$, $\varepsilon \in (\mathbb{R}, 0), \lambda(0) = 0$ such that the cyclicity of the compact invariant set of X_0 with respect to the deformation X_λ is the same as that with respect to $X_{\lambda(\varepsilon)}$, which is one-parameter deformation of X_0. Second, the Abelian integral in the present paper includes all possible higher-order Melnikov functions, as explained in [9]. This allows to study cyclicity in general.

2. **Outline of the proof of Theorem 1.1**

System (1.3) has an invariant line $x = 0$. A unique finite singular point $(1, 0)$ is a center. The singularities at infinity in x- direction are a pair of degenerate ones, while the singularities at infinity in y- direction are a pair of saddles.

The period annulus is formed by the closed orbits

$$\Gamma_h : \{(x, y) : x^{-\frac{4}{3}} \left(\frac{1}{2}y^2 - \frac{1}{12}x + \frac{1}{48}\right) = h\}, h \in (-\frac{1}{16}, 0).$$

Γ_h shrinks to the center $(1, 0)$ as $h \to -1/16$, while Γ_h expands to an unbounded elliptic separatrix Γ_0 as $h \to 0$, see Figure 1.
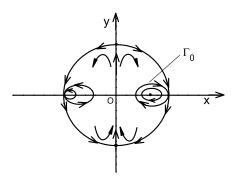

FIGURE 1. The phase portrait of system (1.3) in the Poincaré disk

Theorem 3 of [12] directly gives the following result.

Lemma 2.1. *The exact upper bound for the number of limit cycles produced by the period annulus of system* (1.3) *under quadratic perturbations is equal to the maximal number of zeros (counting multiplicities) of the following Abelian integral*

$$(2.1) \qquad I(h) = \alpha I_0(h) + \beta I_1(h) + \gamma I_{-1}(h)$$

in $h \in (-1/16, 0)$, *where* $I_k(h) = \oint_{\Gamma_h} x^{k-7/3} y \, dx$ *with* $k = 0, 1$ *and* -1. α, β *and* γ *are any real constants.*

For convenience, we give some definitions.

Definition 2.2. *(see [8]) Let* X_λ *be a family of analytic real plane vector fields depending analytically on a parameter* $\lambda \in (\mathbb{R}^n, 0)$, *and let* $K \subset \mathbb{R}^2$ *be a compact invariant set of* X_{λ_0}. *We say that the pair* (K, X_{λ_0}) *has cyclicity* $N = Cycl(K, X_{\lambda_0})$ *with respect to the deformation* X_λ, *provided that* N *is the smallest integer having the properties: there exists* $\varepsilon_0 > 0$ *and a neighborhood* V_K *of* K, *such that for every* λ, *such that* $\|\lambda - \lambda_0\| < \varepsilon_0$, *the vector field* X_λ *has no more than* N *limit cycles contained in* V_K.

Definition 2.3. *For* $h \in (-1/16, 0)$, *define*

$$\varphi(h) = -2(-h)^{\frac{1}{2}}((-h)^{\frac{1}{2}} I(h))', \ \ \widetilde{\varphi}(h) = 192h^3 (\frac{\varphi(h)}{h})',$$

$$P(h) = \frac{I_1(h)}{I_0(h)}, \ \ Q(h) = \frac{I_{-1}(h)}{I_0(h)}, \ \ R(h) = \frac{I_{-\frac{1}{3}}(h)}{I_0(h)},$$

$$\upsilon(h) = \frac{I''_{-1}(h)}{I''_0(h)},$$

where $I_{-1/3}(h) = \oint_{\Gamma_h} x^{-8/3} y \, dx$, $I_0(h), I_1(h)$ *and* $I_{-1}(h)$ *are the same as before, and* $I_0(h) \neq 0$, $I''_0(h) \neq 0$, *see section 3 for details.*
   Let $s = h^3$, *then for* $s \in (-1/4096, 0)$, *define* $\omega(s) = \upsilon(h)$.

Definition 2.4. *We call* $\Sigma = \{(P, Q)(h) : h \in (-1/16, 0)\}$ *the centroid curve, and* $C_\omega = \{(s, \omega(s)) : s \in (-1/4096, 0)\}$ *the auxiliary curve, where* $\omega(s)$ *is defined as before.*

Remark 2.5. *The definition of the centroid curve here is different from the one in [10]. For convenience, we also call it the centroid curve.*

For proving Theorem 1.1, we need the following four theorems.

Theorem 2.6. *The functions* $P(h), Q(h)$ *and the curve* $\sum$ *satisfy the following properties.*

(1) *For $h \in (-1/16, 0)$, we have $P'(h) > 0$;*
(2) *The centroid curve $\sum$ has the same concavity near the two endpoints, which means $d^2Q/dP^2 < 0$ for $0 \leq h + 1/16 \ll 1$ and $0 < -h \ll 1$.*

Theorem 2.7. *For $s \in (-1/4096, 0)$, the auxiliary curve C_ω is globally convex.*

Theorem 2.8. *For $h \in (-1/16, 0)$, the function $\tilde{\varphi}(h)$ has at most two zeros, taking into account the multiplicities, where $\tilde{\varphi}(h)$ is defined as above.*

Theorem 2.9. *For $h \in (-1/16, 0)$, the Abelian integral $I(h)$ defined by (2.1) has at most two zeros, taking into account the multiplicities, and $I(h)$ can have exactly two zeros for some constants α, β and γ.*

Theorem 1.1 follows from Lemma 2.1 and Theorem 2.9 immediately.

The rest of this paper is organized as follows. In section 3, we will derive the closed Picard-Fuchs equations satisfied by $I_k(h) = \oint_{\Gamma_h} x^{k-7/3} y \, dx$ with $k = 0, 1, -1, -1/3$, and Riccati equations by $v(h) = I''_{-1}(h)/I''_0(h) = \omega(s)$. In section 4, by using the differential equations on h, P, Q and R, and computing the asymptotic expansion of the elliptic integrals $I_k(h)$ with $k = 0, 1, -1$ and $-1/3$ near $h = 0$, we study the properties of the centroid curve $\sum$ and prove Theorem 2.6. After taking a qualitative analysis of the Riccati equation satisfied by $\omega(s)$, the global convexity of the auxiliary curve C_ω is gotten in section 5. This finishes the proof of Theorem 2.7. At the same time, we also get the monotonicity of C_ω. Theorem 2.8 is proven in section 6. In the last section, we prove Theorem 2.9, then Theorem 1.1 will be verified.

3. The Picard-Fuchs equations and Riccati equations

Lemma 3.1. *Suppose $J(h) = [I_0(h), I_1(h), I_{-1}(h), I_{-1/3}(h)]^T$, then*

$$(3.1) \qquad\qquad J(h) = M(h)J'(h),$$

where

$$M(h) = \begin{bmatrix} h & 0 & 0 & \frac{1}{16} \\ 0 & -2h & 0 & -\frac{1}{8} \\ \frac{6}{7}h & 0 & \frac{4}{7}h & \frac{5}{56} \\ -\frac{1}{160h} & 0 & \frac{1}{320h} & \frac{4}{5}h \end{bmatrix}.$$

Proof. Recall that the first integral of system (1.3) is

$$(3.2) \qquad H(x,\ y) = x^{-\frac{4}{3}} \left(\frac{1}{2} y^2 - \frac{1}{12} x + \frac{1}{48} \right) = h,\ h \in \left(-\frac{1}{16},\ 0 \right).$$

Differentiating (3.2) with respect to h and x respectively, we get

$$\frac{\partial y}{\partial h} = \frac{x^{\frac{4}{3}}}{y}, \quad \frac{\partial y}{\partial x} = \frac{\frac{4}{3} h x^{\frac{1}{3}} + \frac{1}{12}}{y}.$$

It is easy to know that

$$(3.3) \qquad I_k'(h) = \oint_{\Gamma_h} x^{k - \frac{7}{3}} \frac{\partial y}{\partial h} dx = \oint_{\Gamma_h} \frac{x^{k-1}}{y} dx > 0.$$

From (3.2) and (3.3), we have

$$I_k(h) = \oint_{\Gamma_h} \frac{x^{k - \frac{7}{3}} y^2}{y} dx = \oint_{\Gamma_h} \frac{x^{k - \frac{7}{3}} (2 h x^{\frac{4}{3}} + \frac{1}{6} x - \frac{1}{24})}{y} dx$$

$$(3.4) \qquad = 2 h I_k'(h) + \frac{1}{6} I_{k-\frac{1}{3}}'(h) - \frac{1}{24} I_{k-\frac{4}{3}}'(h).$$

On the other hand, when $k \neq 4/3$, by integrating by parts, we get

$$I_k(h) = \oint_{\Gamma_h} x^{k - \frac{7}{3}} y \, dx = \frac{1}{k - \frac{4}{3}} \oint_{\Gamma_h} y \, dx^{k - \frac{4}{3}}$$

$$= -\frac{1}{k - \frac{4}{3}} \oint_{\Gamma_h} x^{k - \frac{4}{3}} \frac{\frac{4}{3} h x^{\frac{1}{3}} + \frac{1}{12}}{y} dx$$

$$(3.5) \qquad = -\frac{4}{3k - 4} h I_k'(h) - \frac{1}{4(3k - 4)} I_{k-\frac{1}{3}}'(h).$$

Removing $I_{k-1/3}'(h)$ and $I_k(h)$ from (3.4) and (3.5) respectively, we obtain

$$(3.6) \qquad I_k(h) = -\frac{2}{6k - 5} h I_k'(h) - \frac{1}{8(6k - 5)} I_{k-\frac{4}{3}}'(h).$$

$$(3.7) \quad 24(6k - 4) h I_k'(h) + 2(6k - 5) I_{k-\frac{1}{3}}'(h) - (3k - 4) I_{k-\frac{4}{3}}'(h) = 0,\ k \neq \frac{4}{3}.$$

Taking $k = 0, k = -1$ and $k = -1/3$ in (3.5), $k = 1$ in (3.6), and $k = 2/3, k = 1/3$ and $k = 0$ in (3.7) respectively, the following equalities

hold.

$$I_0(h) = hI_0'(h) + \frac{1}{16}I_{-\frac{1}{3}}'(h),$$

$$I_{-1}(h) = \frac{4}{7}hI_{-1}'(h) + \frac{1}{28}I_{-\frac{4}{3}}'(h),$$

$$I_{-\frac{1}{3}}(h) = \frac{4}{5}hI_{-\frac{1}{3}}'(h) + \frac{1}{20}I_{-\frac{2}{3}}'(h),$$

$$I_1(h) = -2hI_1'(h) - \frac{1}{8}I_{-\frac{1}{3}}'(h),$$

$$I_{-\frac{2}{3}}'(h) = I_{\frac{1}{3}}'(h),$$

$$I_{\frac{1}{3}}'(h) = \frac{1}{16h}(I_{-1}'(h) - 2I_0'(h)),$$

(3.8) $$\frac{1}{28}I_{-\frac{4}{3}}'(h) = \frac{6}{7}hI_0'(h) + \frac{5}{56}I_{-\frac{1}{3}}'(h).$$

Let

$$J(h) = (I_0(h), I_1(h), I_{-1}(h), I_{-\frac{1}{3}}(h))^T,$$

then (3.1) follows from (3.8). This finishes the proof of Lemma 3.1. $\square$

Moreover, differentiating (3.1) with respect to h, we get

(3.9) $$G(h)J''(h) = M_1(h)J'(h),$$

where

$$G(h) = 4h(1 + 4096h^3),$$

$$M_1(h) = \begin{bmatrix} 2 & 0 & -1 & -256h^2 \\ 2 & -24576h^3 - 6 & -1 & -256h^2 \\ -24576h^3 - 4 & 0 & 2 + 12288h^3 & -256h^2 \\ -32h & 0 & 16h & 4096h^3 \end{bmatrix}.$$

Remark 3.2. *For $h \in (-1/16, 0)$, the following results are easily obtained.*

(1) *The definitions yield that*

$$I_k(h) = \oint_{\Gamma_h} x^{k-7/3}y\,dx > 0$$

with $k = 0, 1, -1$ and $-1/3$.

(2) *From Lemma 3.1 and equation* (3.9), *we have*

$$G(h)I_0''(h) = -320h^2 I_{-\frac{1}{3}}(h) < 0,$$

which implies that

$$I_0''(h) < 0.$$

Lemma 3.3. *The function* $\nu(h) = I_{-1}''(h)/I_0''(h)$ *satisfies the following Riccati equation*

$$12h(1 + 4096h^3)\nu' = -5\nu^2 + 8(9216h^3 - 1)\nu + (4 - 110592h^3).$$

Proof. Differentiating the first, third and fourth equations of (3.1) with respect to h respectively, we get

$$I_{-\frac{1}{3}}''(h) = -16hI_0''(h),$$

$$I_{-1}'(h) - 2I_0'(h) = \frac{4}{3}hI_{-1}''(h) + 2hI_0''(h) + \frac{5}{24}I_{-\frac{1}{3}}''(h),$$

(3.10) $$I_{-\frac{1}{3}}'(h) = 4hI_{-\frac{1}{3}}''(h) - \frac{1}{64h^2}(I_{-1}'(h) - 2I_0'(h))$$

$$+ \frac{1}{64h}(I_{-1}''(h) - 2I_0''(h)),$$

which implies that

$$I_{-\frac{1}{3}}'(h) = \frac{(-2 - 12288h^3)I_0''(h) - I_{-1}''(h)}{192h}.$$

Moreover, differentiating (3.10) with respect to h and removing $I_{-1/3}''(h)$ and $I_{-1/3}'''(h)$, we have

(3.11)

$$G_1(h) \begin{bmatrix} I_0'''(h) \\ I_{-1}'''(h) \end{bmatrix} = \begin{bmatrix} -86016h^3 + 10 & 5 \\ -110592h^3 + 4 & -12288h^3 + 2 \end{bmatrix} \begin{bmatrix} I_0''(h) \\ I_{-1}''(h) \end{bmatrix},$$

where

$$G_1(h) = 12h(1 + 4096h^3).$$

It follows from the definition of $\nu(h)$ and (3.11) that

$$G_1(h)\nu' = -5\nu^2 + 8(9216h^3 - 1)\nu + (-110592h^3 + 4),$$

which is the desired result. $\square$

Recalling $s = h^3, \omega(s) = \nu(h)$, we easily get the following lemma.

Lemma 3.4. *For $s \in (-1/4096, 0)$, $\omega(s)$ satisfies the following equations*

$$\dot{s} = G_*(s) = 36s(1 + 4096s),$$

(3.12) $$\dot{\omega} = -5\omega^2 + 8(9216s - 1)\omega + (4 - 110592s).$$

In what follows, we begin to prove theorems listed in section 2.

4. Proof of Theorem 2.6

Lemma 4.1. *When $h \in (-1/16, 0)$, we have $P'(h) > 0$ and $Q'(h) > 0$.*
Proof. By Definition 2.3, we have

$$P(h) = \frac{\oint_{\Gamma_h} x^{-\frac{4}{3}} y \, dx}{\oint_{\Gamma_h} x^{-\frac{7}{3}} y \, dx},$$

where

$$\Gamma_h : H(x, y) = x^{-\frac{4}{3}}\left(\frac{1}{2}y^2 - \frac{1}{12}x + \frac{1}{48}\right) = h, \ h \in \left(-\frac{1}{16}, 0\right).$$

Let

$$\phi(x) = \frac{1}{2}x^{-\frac{4}{3}}, \ \Phi(x) = -\frac{1}{12}x^{-\frac{1}{3}} + \frac{1}{48}x^{-\frac{4}{3}},$$

$$f_1(x) = x^{-\frac{7}{3}}, \ f_2(x) = x^{-\frac{4}{3}}.$$

Define $\tilde{x} = \tilde{x}(x)$ as [15] such that $\Phi(x) = \Phi(\tilde{x})$ for $1/4 < x < 1 < \tilde{x} < +\infty$.

Noting

$$\Phi'(x) = \frac{1}{36}x^{-\frac{7}{3}}(x - 1),$$

we easily get

(4.1) $$\frac{d\tilde{x}}{dx} = \frac{\Phi'(x)}{\Phi'(\tilde{x})} = \frac{x^{-\frac{7}{3}}(x - 1)}{\tilde{x}^{-\frac{7}{3}}(\tilde{x} - 1)} < 0$$

and

$$\begin{aligned}
\zeta(x) &= \frac{f_2(x)\sqrt{\phi(\tilde{x})}\Phi'(\tilde{x}) - f_2(\tilde{x})\sqrt{\phi(x)}\Phi'(x)}{f_1(x)\sqrt{\phi(\tilde{x})}\Phi'(\tilde{x}) - f_1(\tilde{x})\sqrt{\phi(x)}\Phi'(x)} \\
&= \frac{x\tilde{x}^{-\frac{2}{3}}(\tilde{x} - 1) - \tilde{x}x^{-\frac{2}{3}}(x - 1)}{\tilde{x}^{-\frac{2}{3}}(\tilde{x} - 1) - x^{-\frac{2}{3}}(x - 1)},
\end{aligned}$$

where $\zeta(x)$ is such a criterion function that $\zeta'(x) > 0(< 0)$ implies $P'(h) < 0(> 0)$, see Theorem 2 of [15].

Let $\psi(x) = x^{-2/3}(x - 1)$, then

$$\zeta(x) = \frac{x\psi(\tilde{x}) - \tilde{x}\psi(x)}{\psi(\tilde{x}) - \psi(x)},$$

and

(4.2) $$\zeta'(x) = \zeta_x + \zeta_{\tilde{x}}\frac{d\tilde{x}}{dx},$$

where

$$\begin{aligned}
\zeta_x &= \frac{\psi(\tilde{x})[(\psi(\tilde{x}) - \psi(x)) + \psi'(x)(x - \tilde{x})]}{(\psi(\tilde{x}) - \psi(x))^2} \\
&= \frac{\psi(\tilde{x})(\tilde{x} - x)(\psi'(\xi) - \psi'(x))}{(\psi'(\tilde{x}) - \psi(x))^2}, \quad \frac{1}{4} < x < \xi < \tilde{x},
\end{aligned}$$

similarly

$$\zeta_{\tilde{x}} = \frac{\psi(x)(\tilde{x} - x)(\psi'(\tilde{x}) - \psi'(\eta))}{(\psi'(\tilde{x}) - \psi(x))^2}, \quad \frac{1}{4} < x < \eta < \tilde{x}.$$

Now we estimate the sign of the function $\zeta'(x)$.

Recalling the fact that $\psi''(x) = -2/9x^{-8/3}(x + 5) < 0$, $\psi(x) < 0$, and $\psi(\tilde{x}) > 0$, we have

(4.3) $$\zeta_{\tilde{x}} > 0, \quad \zeta_x < 0.$$

(4.1), (4.2) and (4.3) yield

$$\zeta'(x) < 0,$$

which implies

$$P'(h) > 0.$$

Similarly we can get $Q'(h) > 0$. ☐

By straightforward calculation, we easily obtain Lemma 4.2.

Lemma 4.2. *When $h \to -1/16, (P, Q)(h) \to (1, 1), R(h) \to 1$.*

Remark 4.3.   (1)   *From Lemma 4.2, we may extend the domains of the functions $P(h), Q(h)$ and $R(h)$ from $h \in (-1/16, 0)$ to $h \in [-1/16, 0)$.*

(2) *From Lemmas 4.1 and 4.2, we can rewrite the centroid curve*
$\sum$ *as follows*

$$\sum = \{(P, Q(P)) : P \geq 1\},$$

where $Q(P) = Q(h(P))$, $h = h(P)$ *is the inverse function of*
$P = P(h)$.

In order to get the global properties of the centroid curve, we first
study the convexities of Σ near the endpoints.

Lemma 4.4. *Near the left endpoint* $(1, 1)$, $Q(P)$ *is concave, which
means that* $d^2Q/dP^2 < 0$ *for* $0 \leq h + 1/16 \ll 1$.
Proof. From (3.1), we have

(4.4) $$G(h)J'(h) = M_2(h)J(h),$$

where

$$G(h) = 4h(1 + 4096h^3),$$

and

$$M_2(h) = \begin{bmatrix} 16384h^3 - 10 & 0 & 7 & -1280h^2 \\ -14 & -8192h^3 - 2 & 7 & -1280h^2 \\ -24576h^3 - 20 & 0 & 28672h^3 + 14 & -1280h^2 \\ 224h & 0 & -112h & 20480h^3 \end{bmatrix}.$$

By the definitions of $P(h), Q(h), R(h)$ and (4.4), we obtain

$$\dot{h} = G(h),$$
$$\dot{P} = -14 + (-24576h^3 + 8)P + 7Q - 1280h^2R - 7PQ$$
$$+ 1280h^2PR,$$
$$\dot{Q} = -20 - 24576h^3 + (24 + 12288h^3)Q - 1280h^2R - 7Q^2$$
$$+ 1280h^2QR,$$
(4.5) $$\dot{R} = 224h - 112hQ + (4096h^3 + 10)R - 7QR + 1280h^2R^2.$$

At the singularity $(-1/16, 1, 1, 1)$, the linear matrix of system (4.5) is

$$-12 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 24 & -1 & 0 & 0 \\ 12 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

thus near the point $(-1/16, 1, 1, 1)$, we suppose the functions $P(h), Q(h)$ and $R(h)$ have the following asymptotic expansions

$$P = 1 + p_1(h + \frac{1}{16}) + \frac{p_2}{2!}(h + \frac{1}{16})^2 + \cdots,$$

$$Q = 1 + q_1(h + \frac{1}{16}) + \frac{q_2}{2!}(h + \frac{1}{16})^2 + \cdots,$$

$$(4.6) \qquad R = 1 + r_1(h + \frac{1}{16}) + \frac{r_2}{2!}(h + \frac{1}{16})^2 + \cdots.$$

Substituting (4.6) into (4.5), we can get

$$p_1 = 12, p_2 = \frac{980}{3},$$

$$q_1 = 6, \ q_2 = \frac{58}{3},$$

$$(4.7) \qquad r_1 = 0, \ r_2 = \frac{80}{9}.$$

From (4.7), we get

$$\frac{d^2Q}{dP^2}\big|_{h=-\frac{1}{16}} = \frac{q_2p_1 - p_2q_1}{p_1^3} = -1 < 0,$$

which implies $Q(P)$ is concave near the left endpoint. This proves the conclusion of Lemma 4.4. □

Before studying the concavity of the centroid curve near the right endpoint, we prove the following lemma.

Lemma 4.5. *Near $h = 0$, $I_0(h), I_1(h), I_{-1}(h)$ and $I_{-1/3}(h)$ have the following expansion*

$$I_0(h) = \frac{7}{10}\lambda_1 + (\frac{1}{2}\lambda_1 k_1 + \frac{1}{2}\lambda_2 k_2 + \frac{1}{2}\lambda_3)h - 40\lambda_2 h^2$$
$$- \frac{448}{3}\lambda_1 h^3 + \cdots,$$

$$I_1(h) = -\frac{7}{5}\lambda_1 + \lambda_4 h^{-\frac{1}{2}} - 16\lambda_2 h^2 - \frac{256}{3}\lambda_1 h^3 + \cdots,$$

$$I_{-1}(h) = \lambda_1 + (\lambda_1 k_1 + \lambda_2 k_2 + \lambda_3)h + 80\lambda_2 h^2 - \frac{896}{15}\lambda_1 h^3 + \cdots,$$

$$(4.8) \quad I_{-\frac{1}{3}}(h) = \lambda_2 + \frac{56}{5}\lambda_1 h + \frac{640}{3}\lambda_2 h^3 + \cdots,$$

where $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_4 \neq 0$.

Proof. First extending the functions $I_k(h)(k = 0, 1, -1, -1/3)$ to complex plane, then using the theorem in [25, 26], we compute their expansions near $h = 0$.

Let $X(h) = (I_0(h), I_1(h), I_{-1}(h), I_{-1/3}(h))^T$. By (3.1), we have

$$(4.9) \qquad\qquad X' = \frac{B(h)}{h} X,$$

where

$$B(h) = \begin{bmatrix} \frac{-5+8192h^3}{2(1+4096h^3)} & 0 & \frac{7}{4(1+4096h^3)} & -\frac{320h^2}{1+4096h^3} \\ -\frac{7}{2(1+4096h^3)} & -\frac{1}{2} & \frac{7}{4(1+4096h^3)} & -\frac{320h^2}{1+4096h^3} \\ -\frac{5+6144h^3}{1+4096h^3} & 0 & \frac{7(1+2048h^3)}{1+4096h^3} & -\frac{320h^2}{1+4096h^3} \\ \frac{56h}{1+4096h^3} & 0 & -\frac{28h}{1+4096h^3} & \frac{5120h^3}{1+4096h^3} \end{bmatrix}.$$

It is easy to know that for matrix

$$B(0) = \begin{bmatrix} -\frac{5}{2} & 0 & \frac{7}{4} & 0 \\ -\frac{7}{2} & -\frac{1}{2} & \frac{7}{4} & 0 \\ -5 & 0 & \frac{7}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

there exists

$$T = \begin{bmatrix} \frac{7}{10} & 0 & \frac{1}{2} & 0 \\ -\frac{7}{5} & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

such that

$$T^{-1}B(0)T = J = \Lambda^* + Z,$$

where J is the Jordan matrix

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix},$$

and Λ^* is the diagonal matrix $\Lambda^* = diag(0, 0, 1, -1/2)$, and Z is the matrix $Z = 0$.

For any diagonal matrix $\Lambda = diag(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, we define the constant matrix

$$V = (v_{ij}) \in C^{4 \times 4},$$

where $v_{ij} = 0$, if $\lambda_i - \lambda_j \notin N$(the set of natural numbers), $i, j = 1, 2, 3, 4$, see [26]. Hence for the diagonal matrix Λ^*,

$$
V^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vartheta_{31} & \vartheta_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

where $\vartheta_{31}, \vartheta_{32}$ are constants.

Let

$$
\Phi(h) = \Lambda^* + h^{\Lambda^*} V^* h^{-\Lambda^*} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vartheta_{31}h & \vartheta_{32}h & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}
$$

and

$$
\begin{aligned}
\Psi(h) &= T^{-1} B(h) T = \left(1 + \sum_{k=1}^{+\infty} (-4096h^3)^k\right) \\
&\quad \cdot \left(\Lambda^* + \begin{pmatrix} 7168h^3 & -800h^2 & 0 & 0 \\ \frac{56}{5}h & 5120h^3 & 0 & 0 \\ -\frac{21504}{5}h^3 & 480h^2 & 4096h^3 & 0 \\ \frac{64512}{5}h^3 & -1440h^2 & 0 & -2048h^3 \end{pmatrix}\right).
\end{aligned}
$$

Consider the matrix function

$$
W(h) = (a(h), b(h), c(h), d(h))
$$

such that $W(0) = E$(the unit matrix), where the vectors are given as

$$
a(h) = \sum_{k=0}^{+\infty} a_k h^k, b(h) = \sum_{k=0}^{+\infty} b_k h^k, c(h) = \sum_{k=0}^{+\infty} c_k h^k, d(h) = \sum_{k=0}^{+\infty} d_k h^k.
$$

Using the theorem in [25, 26], we have

$$
h\frac{d}{dh}W(h) = \Psi(h)W(h) - W(h)\Phi(h),
$$

which is equivalent to

$$
\begin{aligned}
ha'(h) &= \Psi(h)a(h) - \vartheta_{31}hc(h), \\
hb'(h) &= \Psi(h)b(h) - \vartheta_{32}hc(h), \\
hc'(h) &= \Psi(h)c(h) - c(h), \\
hd'(h) &= \Psi(h)d(h) + \frac{1}{2}d(h).
\end{aligned}
$$

Substituting $a(h), b(h), c(h)$ and $d(h)$ into the above equalities and comparing the coefficients in terms of $h^k (k = 1, 2, 3)$ on both sides, we get

$$W_1(h) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{56}{5} & 0 & 0 & 0 \\ k_1 & k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, W_2(h) = \begin{bmatrix} 0 & -400 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 480 & 0 & 0 \\ 0 & -576 & 0 & 0 \end{bmatrix},$$

$$W_3(h) = \begin{bmatrix} -\frac{1792}{3} & 0 & 0 & 0 \\ 0 & \frac{640}{3} & 0 & 0 \\ \frac{2688}{5} & 0 & 0 & 0 \\ -\frac{4608}{5} & 0 & 0 & 0 \end{bmatrix},$$

where k_1 and k_2 are any constants. Moreover $\vartheta_{31} = 0, \vartheta_{32} = 0$, i.e., $V^* = 0$.

Hence we get

$$W(h) = E + W_1 h + W_2 h^2 + W_3 h^3 + \cdots,$$

thus the fundamental solution matrix of system (4.9) is

$$\widetilde{X}(h) = TWh^{\Lambda^*} h^{Z+V^*} = TWh^{\Lambda^*} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix},$$

where

$$x_{11} = \frac{7}{10} + \frac{1}{2}k_1 h - \frac{448}{3}h^3 + O(h^{\frac{7}{2}}), \quad x_{12} = \frac{1}{2}k_2 h - 40h^2 + O(h^{\frac{7}{2}}),$$

$$x_{13} = \frac{1}{2}h + O(h^{\frac{7}{2}}), \qquad\qquad\qquad x_{14} = O(h^{\frac{7}{2}}),$$

$$x_{21} = -\frac{7}{5} - \frac{256}{3}h^3 + O(h^{\frac{7}{2}}), \qquad x_{22} = -16h^2 + O(h^{\frac{7}{2}}),$$

$$x_{23} = O(h^{\frac{7}{2}}), \qquad\qquad\qquad\qquad x_{24} = h^{-\frac{1}{2}} + O(h^{\frac{7}{2}}),$$

$$x_{31} = 1 + k_1 h - \frac{896}{15}h^3 + O(h^{\frac{7}{2}}), \qquad x_{32} = k_2 h + 80h^2 + O(h^{\frac{7}{2}}),$$

$$x_{33} = h + O(h^{\frac{7}{2}}), \qquad\qquad\qquad x_{34} = O(h^{\frac{7}{2}}),$$

$$x_{41} = \frac{56}{5}h + O(h^{\frac{7}{2}}), \qquad\qquad x_{42} = 1 + \frac{640}{3}h^3 + O(h^{\frac{7}{2}}),$$

$$x_{43} = O(h^{\frac{7}{2}}), \qquad\qquad\qquad\qquad x_{44} = O(h^{\frac{7}{2}}).$$

This implies that $I_0(h), I_1(h), I_{-1}(h), I_{-1/3}(h)$ have the expansions (4.8) near $h = 0$.

The fact that $\lambda_1 \neq 0, \lambda_2 \neq 0$ follows from $I_k(0) \neq 0 (k = 0, 1, -1, -1/3)$. Moreover, we assert that $\lambda_4 \neq 0$. Otherwise, $P(0) = -(7/5\lambda_1)/(7/10\lambda_1) = -2$. On the other hand, $P(-1/16) = 1$ and $P'(h) > 0$, this is a contradiction. Hence $\lambda_4 \neq 0$. The lemma is proved. $\qquad\square$

Lemma 4.6. *Near the right endpoint, $Q(P)$ is concave, which means that $d^2Q/dP^2 < 0$ for $0 < -h \ll 1$.*
Proof. From Lemma 4.5, we know that when $h \to 0^-$

$$I_0(h) \to \frac{7}{10}\lambda_1, \ I_{-1}(h) \to \lambda_1, \ I_1(h) \to \infty.$$

Noting Lemmas 4.1 and 4.5, we have that when $h \to 0^-$

$$P(h) \to +\infty, \ P'(h) \to +\infty,$$
$$Q(h) \to \frac{10}{7}, \ Q'(h) \to \frac{50(\lambda_1 k_1 + \lambda_2 k_2 + \lambda_3)}{49\lambda_1},$$

which implies $dQ/dP \to 0$ as $h \to 0^-$.

Recalling $Q'(h)/P'(h) > 0$ for $h \in (-1/16, 0)$, the centroid curve $\sum$ has an asymptotical line $Q = 10/7$ as $h \to 0^-$, see Figure 2.

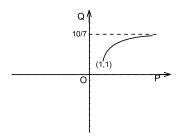Therefore $d^2Q/dP^2 < 0$ for $0 < -h \ll 1$. This finishes the proof of the lemma. $\qquad\square$

FIGURE 2. The behavoir of the centroid curve $\sum$

Proof of Theorem 2.6 Theorem 2.6 follows from Lemmas 4.1, 4.4 and 4.6.

5. **Proof of Theorem 2.7**

It is easy to know that system (3.12) have four singularities: saddles $C(0, 2/5)$ and $A(-1/4096, -31/5)$, an unstable node $B(0, -2)$ and a stable improper node $D(-1/4096, 1)$.

Using (3.9) and (4.4), we get

$$\omega(s) = \nu(h) = \frac{I''_{-1}(h)}{I''_0(h)} \quad = \quad \frac{672hI_0(h) - 336hI_{-1}(h) - 10I_{-\frac{1}{3}}(h)}{5I_{-\frac{1}{3}}(h)}$$

$$(5.1) \qquad\qquad\qquad = \quad \frac{672h - 336hQ - 10R}{5R}.$$

Lemma 4.2 implies that

$$P(-\frac{1}{16}) = Q(-\frac{1}{16}) = R(-\frac{1}{16}) = 1,$$

substituting the above equalities into (5.1), we have $\omega(-1/4096) = -31/5$.

On the other hand, Lemma 4.5 yields $\omega(0) = -2$. Hence the graph of $\omega(s)$ which we are interested in, denoted by C_ω, is just the trajectory of system (3.12) from node B to saddle A, see Figure 3.
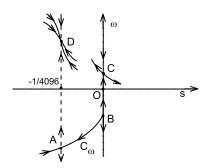

FIGURE 3. The behavior of the auxiliary curve C_ω

Proof of Theorem 2.7 Since the curve C_ω is located on the stable manifold at saddle A, $\omega(s)$ has the following asymptotic expansion at $s = -1/4096$

$$(5.2) \qquad \omega(s) = -\frac{31}{5} + \omega_1(s + \frac{1}{4096}) + \frac{\omega_2}{2!}(s + \frac{1}{4096})^2 + \cdots.$$

Substituting (5.2) into the following equation

$$\dot{\omega} - G_*(s)\frac{d\omega}{ds} = 0.$$

Solving it, we get

$$\omega_1 = \frac{39424}{5}, \ \omega_2 = \frac{2230452224}{135}.$$

Hence the curve C_ω is convex near the left endpoint $(-1/4096, -31/5)$. Moreover, C_ω is globally convex for $s \in (-1/4096, 0)$. Otherwise it has at least an inflection point. Noting the convexity of C_ω at the left endpoint A, we easily find the straight line $l_{a,b} : \omega(s) = as + b$ which has three intersection points with the curve C_ω, denoted by M_1 the first intersection point from A to B, and intersects the straight line: $s = -1/4096$ at A_1(see Figure 4) below A. As a consequence of saddle A, there must exist a tangent point M_0 with system (3.12) locating between A_1 and M_1. Hence on the line $l_{a,b}$, there are at least three points M_0, M_2 and M_3 having the same direction with system (3.12).
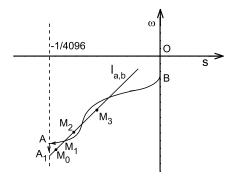

FIGURE 4. The intersection points of the curve C_ω with the straight line $l_{a,b}$

On the other hand

$$(\dot{\omega} - a\dot{s})|_{\omega(s)=as+b}$$
$$= (-73728a - 5a^2)s^2 + (-110592 - 44a + 73728b - 10ab)s$$
(5.3) $$-5b^2 - 8b + 4.$$

Obviously (5.4) has at most two roots. This leads to a contradiction. Therefore Theorem 2.7 is proved. □

From Theorem 2.7, we can easily get the following result.

Corollary 5.1. *When $s \in (-1/4096, 0)$, we have $\omega'(s) > 0$.*

6. **Proof of Theorem 2.8**

Lemma 6.1. *For any constants α, β and γ, we have*

$$(6.1) \qquad \varphi(h) := I(h) + 2hI'(h) = mhI_0'(h) + khI_{-1}'(h) + nI_{-\frac{1}{3}}'(h),$$

where $m = 3\alpha + 6\beta/7, k = 18\gamma/7, n = \alpha/16 - \beta/8 + 5\gamma/56$.
Proof. Recall that

$$(6.2) \qquad I(h) = \alpha I_0(h) + \beta I_1(h) + \gamma I_{-1}(h).$$

Noting Picard-Fuchs equations (3.1) and removing $I_k(h)(k = 0, 1, -1)$ from (6.2), we get

$$
\begin{aligned}
I(h) &= (\alpha h + \frac{6}{7}\gamma h)I_0'(h) - 2\beta h I_1'(h) + \frac{4}{7}\gamma h I_{-1}'(h) \\
&\quad + (\frac{1}{16}\alpha - \frac{1}{8}\beta + \frac{5}{56}\gamma)I_{-\frac{1}{3}}'(h).
\end{aligned}
$$
(6.3)

On the other hand, differentiating (6.2) with respect to h, we have

$$(6.4) \qquad I'(h) = \alpha I_0'(h) + \beta I_1'(h) + \gamma I_{-1}'(h).$$

Thus (6.1) follows from (6.3) and (6.4). □

Lemma 6.2. *When $h \in (-1/16, 0)$, the following equalities hold.*
(1) *If $k = 0$ in (6.1), then*

$$
\begin{aligned}
\tilde{\varphi}(h) : &= 192h^3(\frac{\varphi(h)}{h})' \\
&= nI_{-1}''(h) + [192(m + 48n)h^3 + 2n]I_0''(h).
\end{aligned}
$$
(2) *If $k \neq 0$, without loss of generality, suppose $k = 1$ in (6.1), then*

$$
\begin{aligned}
\tilde{\varphi}(h) : &= 192h^3(\frac{\varphi(h)}{h})' \\
&= (b_1 h^3 + b_2)I_{-1}''(h) + (a_1 h^3 + a_2)I_0''(h),
\end{aligned}
$$
(6.5)

where $a_1 = 192(m + 48n), \ a_2 = 2n, \ b_1 = 192$ and $b_2 = n$.

Proof. By using (6.1) and straightforward calculation, we have

$$\left(\frac{\varphi(h)}{h}\right)' = mI_0''(h) + kI_{-1}''(h) + \frac{n}{h^2}\left(hI_{-\frac{1}{3}}''(h) - I_{-\frac{1}{3}}'(h)\right).$$

After removing $I_{-1/3}''(h)$ and $I_{-1/3}'(h)$ from the above equality, the results are obtained. $\qquad\square$

Proof of Theorem 2.8 Since $I_0''(h) \neq 0$, the zeros of the functions $\tilde{\varphi}(h)$ and $\bar{\varphi}(h) = \tilde{\varphi}(h)/I_0''(h)$ are the same.

If $n = 0$, then $\bar{\varphi}(h)$ has at most one zero. Without loss of generality, we suppose $n \neq 0$.

1. $k = 0$.

Let $s = h^3, \omega(s) = \nu(h)$ for $s \in (-1/4096, 0)$, then

$$\begin{aligned}
\bar{\varphi}(h) &= n\nu(h) + [192(m + 48n)h^3 + 2n] \\
&= n(\omega(s) - \vartheta(s)),
\end{aligned}$$

where $\vartheta(s) = c_1 s + c_2$, c_1 and c_2 are constants depending on m and n. Obviously the zero of $\bar{\varphi}(h)$ is the intersection point of C_ω and C_ϑ, where $C_\vartheta = \{(s, \vartheta(s)) : s \in [-1/4096, 0]\}$. Noting that C_ω is globally convex, while the curve C_ϑ is a line segment, we conclude that they have at most two intersection points. That is, $\bar{\varphi}(h)$ has at most two zeros for $h \in (-1/16, 0)$.

2. $k = 1$.

In this case, $\bar{\varphi}(h)$ can be expressed as follows

$$\begin{aligned}
\bar{\varphi}(h) &= (a_1 h^3 + a_2) + (b_1 h^3 + b_2)\nu(h) \\
&= (a_1 s + a_2) + (b_1 s + b_2)\omega(s).
\end{aligned} \tag{6.6}$$

Suppose that $a_1 b_2 - a_2 b_1 = 0$. Then there exists some constant λ, such that

$$\bar{\varphi}(h) = (b_1 s + b_2)(\omega(s) + \lambda).$$

Since $\omega(s)$ is increasing, $\bar{\varphi}(h)$ has at most two zeros.

If $a_1 b_2 - a_2 b_1 \neq 0$, and there exists s_1 such that $b_1 s_1 + b_2 = 0$, then $\bar{\varphi}(h_1) = a_1 s_1 + a_2 \neq 0$, where $h_1 = (s_1)^{1/3}$. Without loss of generality, suppose $b_1 s + b_2 \neq 0$ for $s \in [-1/4096, 0]$, then expression (6.6) becomes

$$\bar{\varphi}(h) = (b_1 s + b_2)(\omega(s) - \chi(s)),$$

where

$$\chi(s) = -\frac{a_1 s + a_2}{b_1 s + b_2},$$

a_1, a_2, b_1 and b_2 are the same as that of (6.5). Hence the zero of $\bar{\varphi}(h)$ is the intersection point of C_ω and C_χ, where $C_\chi = \{(s, \chi(s)) : s \in [-1/4096, 0]\}$.

Since $b_1 = 192 \neq 0$, then C_χ is a hyperbola. If the hyperbola is decreasing, shown as Figure 5(a), then it follows from $\omega'(s) > 0$ that C_ω and C_χ has at most two intersection points. Suppose the hyperbola is increasing, shown as Fig.5(b). In this case, C_ω only intersect with one branch of the hyperbola. If this happens for the right-lower branch, then we get the same conclusion, because the right-lower one is concave and C_ω is globally convex.

Now we consider the case that C_ω intersects with the left-upper branch of the hyperbola, denoted by C_1, see Figure 5(b).
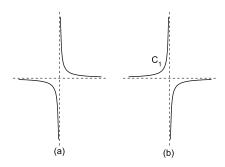

FIGURE 5. The behavior of the curve C_χ

Since

$$(6.7) \qquad (\dot{\omega} - \chi'(s)\dot{s})|_{\omega=\chi(s)} = \frac{-4608s}{(192s + n)^2}(\alpha_1 s^2 + \beta_1 s + \gamma_1),$$

where α_1, β_1 and γ_1 are constants depending on m and n. Noting (6.7) and $\omega(0) = \chi(0) = -2$, we get that for $s \in [-1/4096, 0)$, C_ω and C_1 have at most three intersection points. Moreover, two is the maximal number of the intersection points of C_ω and C_1 for $s \in (-1/4096, 0)$. Otherwise, C_ω and C_1 have exactly three intersection points. By using the same arguments as proof of Theorem 2.7, we can find at least three contact points with system (3.12) on C_1 for $s \in (-1/4096, 0)$, and it contradicts (6.7). This finishes the proof in the second case.

Therefore the function $\bar{\varphi}(h)$ has at most two zeros for $h \in (-1/16, 0)$, taking into account the multiplicities, this implies the result of Theorem 2.8.

7. Proof of Theorem 2.9

In what follows, we will prove that some assertions hold.

Assertion 1: For $h \in (-1/16, 0)$, the number of zeros of $I(h)$ is no more than three.

Without loss of generality, suppose there are four zeros. Noting $I(-1/16) = 0$, by mean value theorem, $((-h)^{1/2} I(h))'$ has at least four zeros for $h \in (-1/16, 0)$. Then $\varphi(h) = -2(-h)^{1/2}((-h)^{1/2} I(h))'$ also has at least four zero for $h \in (-1/16, 0)$. From Lemma 6.2, $\tilde{\varphi}(h)$ has at least three zeros. This contradicts Theorem 2.8.

Assertion 2: $I(h)$ has at most two zeros for $h \in (-1/16, 0)$.

Theorem 2.6 shows that the centroid curve Σ has the same concavity near the two endpoints. This implies that its inflection points (if exists) must appear in pairs. In the following, we will prove that the centroid curve $\sum$ is globally concave without zero curvature.

First, the centroid curve $\sum$ is globally located in the right side of the tangent line of $\sum$ at the left endpoint. Otherwise, we may find a point near the left endpoint on the curve $\sum$, at which the tangent line of $\sum$ cuts two more points of $\sum$. That is, there exist some α, β and γ such that the associated Abelian integral $I(h)$ has at least four zeros for $h \in (-1/16, 0)$, taking into account their multiplicities, which contradicts assertion 1. Secondly, $\sum$ is monotonously increasing and globally located below its asymptotical line $Q = 10/7$. If there exist two inflection points, then we can find a straight line $l_{\alpha_0, \beta_0, \gamma_0}$ such that it has four intersection points with $\sum$, which also implies $I(h) = \alpha_0 I_0(h) + \beta_0 I_1(h) + \gamma_0 I_{-1}(h)$ has four zeros for $h \in (-1/16, 0)$ and leads to the same contradiction. Hence we conclude that the centroid curve Σ is globally concave. Moreover, it also has no zero-curvature point. This is because the zero-curvature point here means that $I(h)$ has one zero point of multiplicity at least four.

Assertion 3: For some constants α, β and γ, $I(h)$ has exactly two zeros in $h \in (-1/16, 0)$. The assertion follows from the fact that the centroid curve $\sum$ is strictly concave.

Assertions 1-3 yield the result of Theorem 2.9.

Acknowledgments

References

[1] N. N. Bautin, On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type, *Amer. Math. Soc. Transl.* **100** (1954) 1–19.

[2] C. Chicone and M. Jacobs, Bifurcation of limit cycles from quadratic isochrones, *J. Differential Equations* **91** (1991), no. 2, 268–326.

[3] G. Chen, C. Li, C. Liu and J. Llibre, The cyclicity of period annuli of some classes of reversible quadratic systems, *Discrete Contin. Dyn. Syst.* **16** (2006), no. 1, 157–177.

[4] F. Chen, C. Li, C. Liu and J. Llibre, A unified proof on the weak Hilbert 16th problem for $n = 2$, *J. Differential Equations* **221** (2006), no. 2, 309–342.

[5] B. Coll, C. Li and R. Prohens, Quadratic perturbations of a class of quadratic reversible systems with two centers, *Discrete Contin. Dyn. Syst.* **24** (2009), no. 3, 699–729.

[6] F. Dumortier, C. Li and Z. Zhang, Unfolding of a quadratic integrable system with two centers and two unbounded heteroclinic loops, *J. Differential Equations* **139** (1997), no. 1, 146–193.

[7] L. Gavrilov, The infinitesimal 16th Hilbert problem in the quadratic case, *Invent. Math.* **143** (2001), no. 3, 449–497.

[8] L. Gavrilov, Cyclicity of period annuli and principalization of Bautin ideals, *Ergodic Theory Dynam. Systems* **28** (2008), no. 5, 1497–1507.

[9] S. Gautier, L. Gavrilov and I. D. Iliev, Perturbations of quadratic centers of genus one, *Discrete Contin. Dyn. Syst.* **25** (2009), no. 2, 511–535.

[10] E. Horozov and I. D. Iliev, On the number of limit cycles in perturbations of quadratic Hamiltonian system, *Proc. Lond. Math. Soc. (3)* **69** (1994), no. 1, 198–224.

[11] E. Horozov and I. D. Iliev, On saddle-loop bifurcation of limit cycles in perturbations of quadratic Hamiltonian system, *J. Differential Equations* **113** (1994), no. 1, 84–105.

[12] I. D. Iliev, Perturbations of quadratic centers, *Bull. Sci. Math.* **122** (1998), no. 2, 107–161.

[13] I. D. Iliev, C. Li and J. Yu, Bifurcations of limit cycles from quadratic non-Hamiltonian systems with two centres and two unbounded heteroclinic loops, *Nonlinearity* **18** (2005), no. 1, 305–330.

[14] I. D. Iliev, C. Li and J. Yu, Bifurcation of limit cycles in a reversible quadratic system with a center, a saddle and two nodes, *Commun. Pure Appl. Anal.* **9** (2010), no. 3, 583–610.

[15] C. Li and Z. Zhang, A criterion for determing the monotonicity of ratio of two Ablian integrals, *J. Differential Equations* **124** (1996), no. 2, 407–424.

[16] C. Li and Z. H. Zhang, Remarks on 16th weak Hilbert problem for $n = 2$, *Nonlinearity* **15** (2002), no. 6, 1975–1992.

[17] C. Li and R. Roussarie, The cyclicity of the elliptic segment loops of the reversible quadratic Hamiltonian systems under quadratic perturbations, *J. Differential Equations* **205** (2004), no. 2, 488–520.

[18] P. Mardesic, Chebyshev Systems and the Versal Unfolding of the Cusps of Order n, Travauxen en Cours, Paris, 1998.

[19] H. Liang and Y. Zhao, Quadratic perturbations of a class of quadratic reversible systems with one center, *Discrete Contin. Dyn. Syst.* **27** (2010), no. 1, 325–335.

[20] L. Peng, Unfolding of a quadratic integrable system with a homoclinic loop, *Acta Math. Sin. (Engl. Ser.)* **18** (2002), no. 4, 737–754.

[21] G. Swirszcz, Cyclicity of infinite contour around certain reversible quadratic center, *J. Differential Equations* **154** (1999), no. 2, 239–266.

[22] J. Yu and C. Li, Bifurcation of a class of planar non-Hamiltonian integrable systems with one center and one homoclinic loop, *J. Math. Anal. Appl.* **269** (2002), no. 1, 227–243.

[23] H. Zoladék, Quadratic systems with center and their perturbations, *J. Differential Equations* **109** (1994), no. 2, 223–273.

[24] Z. Zhang and C. Li, On the number of limit cycles of a class of quadratic Hamiltonian systems under quadratic perturbations, *Adv. Math. (China)* **26** (1997), no. 5, 445–460.

[25] Y. Zhao, Abelian integrals for cubic Hamiltonian vector fields, Ph.D Thesis, Peking University, 1998.

[26] B. ГоπγσeB, Analytic theory of differential equations, Translated by J. K. Lu and M. Y. Qi, Higher Education Press, Beijing, 1985.

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