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**A lower estimate of harmonic functions**

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## A LOWER ESTIMATE OF HARMONIC FUNCTIONS

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**ABSTRACT.** We shall give a lower estimate of harmonic functions of order greater than one in a half space, which generalize the result obtained by B. Ya. Levin in a half plane.

**Keywords:** Lower estimate, Harmonic function, Half space.

**MSC(2010):** Primary: 31B05; Secondary: 31B10.

### 1. Introduction

Let  $\mathbf{R}$  and  $\mathbf{R}_+$  be the sets of all real numbers and of all positive real numbers, respectively. Let  $\mathbf{R}^n$  ( $n \geq 2$ ) denote the  $n$ -dimensional Euclidean space with points  $x = (x', x_n)$ , where  $x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$  and  $x_n \in \mathbf{R}$ . The boundary and closure of an open set  $D$  of  $\mathbf{R}^n$  are denoted by  $\partial D$  and  $\bar{D}$ , respectively. The upper half space is the set  $H = \{(x', x_n) \in \mathbf{R}^n : x_n > 0\}$ , whose boundary is  $\partial H$ .

For a set  $E$ ,  $E \subset \mathbf{R}_+ \cup \{0\}$ , we denote  $\{x \in H : |x| \in E\}$  and  $\{x \in \partial H : |x| \in E\}$  by  $HE$  and  $\partial HE$ , respectively. We identify  $\mathbf{R}^n$  with  $\mathbf{R}^{n-1} \times \mathbf{R}$  and  $\mathbf{R}^{n-1}$  with  $\mathbf{R}^{n-1} \times \{0\}$ , writing typical points  $x, y \in \mathbf{R}^n$  as  $x = (x', x_n)$ ,  $y = (y', y_n)$ , where  $y' = (y_1, y_2, \dots, y_{n-1}) \in \mathbf{R}^{n-1}$  and putting

$$x \cdot y = \sum_{j=1}^n x_j y_j = x' \cdot y' + x_n y_n, \quad |x| = \sqrt{x \cdot x}, \quad |x'| = \sqrt{x' \cdot x'},$$

$$|x'| = |x| \cos \theta \text{ and } x_n = |x| \sin \theta \quad (0 < \theta \leq \pi/2).$$

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Let  $B_r$  denote the open ball with center at the origin and radius  $r$  ( $> 0$ ) in  $\mathbf{R}^n$ . We use the standard notations  $u^+ = \max\{u, 0\}$  and  $u^- = -\min\{u, 0\}$ . In the sense of Lebesgue measure  $dy' = dy_1 \cdots dy_{n-1}$  and  $dy = dy' dy_n$ . Let  $\sigma$  denote  $(n-1)$ -dimensional surface area measure and  $\partial/\partial n$  denote differentiation along the inward normal into  $H$ .

The estimate we deal with has a long history which can be traced back to Levin's estimate of harmonic functions from below (see, for example, Levin [6, p. 209]).

**Theorem 1.1.** *Let  $A_1$  be a constant,  $u(z)$  harmonic in the upper half space  $\mathbf{C}_+$  and continuous on  $\partial\mathbf{C}_+$ . Suppose that*

$$u(z) \leq A_1 R^\rho, \quad z \in \mathbf{C}_+, \quad R = |z| > 1, \quad \rho > 1$$

and

$$|u(z)| \leq A_1, \quad |z| \leq 1, \quad \text{Im}z \geq 0.$$

Then

$$u(Re^{i\varphi}) \geq -A_2 A_1 (1 + R^\rho) \sin^{-1} \varphi, \quad Re^{i\varphi} \in \mathbf{C}_+,$$

where  $A_2$  is a constant independent of  $A_1$ ,  $R$ ,  $\varphi$  and the function  $u(z)$ .

Further versions and refinements of Theorem 1.1 may be found in the monograph Nikol'skiĭ [7, Ch. 1] and in the paper Krasichkov-Ternovskii [3].

In this article, we will consider functions  $u(x)$  harmonic in  $H$  and continuous on  $\overline{H}$ . In what follows we shall denote by  $M$  various values which does not depend on  $K$ ,  $R$  ( $= |x|$ ),  $\theta$  and the function  $u(x)$ .

In this note we prove analogous estimates for  $u(x)$  in  $H$ .

**Theorem 1.2.** *Suppose that*

$$(1.1) \quad u(x) \leq KR^{\rho(R)}, \quad x \in H, \quad R = |x| > 1, \quad \rho(R) > 1$$

and

$$(1.2) \quad u(x) \geq -K, \quad |x| \leq 1, \quad x_n \geq 0.$$

Then

$$u(x) \geq -MK \left(1 + (2R)^{\rho(2R)}\right) \sin^{1-n} \theta,$$

where  $x \in H$  and  $\rho(R)$  is nondecreasing on  $[1, +\infty)$ .

**Remark 1.3.** *If  $n = 2$  and  $\rho(R) \equiv \rho$ , Theorem 1.2 is just a consequence of Theorem 1.1.*

**Theorem 1.4.** *If (1.1) and (1.2) hold, then*

$$u(x) \geq -MK \left( 1 + \left( \frac{N+1}{N} R \right)^{\rho \left( \frac{N+1}{N} R \right)} \right) \sin^{1-n} \theta,$$

where  $x \in H$ ,  $N(\geq 1)$  is a sufficiently large number and  $\rho(R)$  is as defined in Theorem 1.2.

## 2. Lemmas

Carleman's formula [2] connects the modulus and the zeros of a function analytic in  $\mathbf{C}_+$  (see, for example, [5, p. 224]). Nevanlinna's formula (see [6, p. 193]) refers to a harmonic function in a half disk. Armitage and Kuran obtained a generalized Nevanlinna-type formula in a half space and Poisson integral formula for half balls respectively, which play important roles in our discussions.

**Lemma 2.1.** ([1]). *If  $R > 1$ , then we have*

$$\int_{\{x \in H: |x|=R\}} u(x) \frac{nx_n}{R^{n+1}} d\sigma(x) + \int_{\partial H(1,R)} u(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' = c_1 + \frac{c_2}{R^n},$$

where

$$c_1 = \int_{\{x \in H: |x|=1\}} \left( (n-1)x_n u(x) + x_n \frac{\partial u(x)}{\partial n} \right) d\sigma(x)$$

and

$$c_2 = \int_{\{x \in H: |x|=1\}} \left( x_n u(x) - x_n \frac{\partial u(x)}{\partial n} \right) d\sigma(x).$$

**Lemma 2.2.** ([4]). *Let  $R > 1$ ,  $u(x)$  be a function in  $B_R^+ = B_R \cap H$  and continuous in  $\overline{B_R^+}$ . Then*

$$\begin{aligned} u(x) &= \int_{\{y \in H: |y|=R\}} \frac{R^2 - |x|^2}{\omega_n R} \left( \frac{1}{|y-x|^n} - \frac{1}{|y-x^*|^n} \right) u(y) d\sigma(y) \\ &\quad + \frac{2x_n}{\omega_n} \int_{\partial H[0,R)} \left( \frac{1}{|y'-x|^n} - \frac{R^n}{|x|^n} \frac{1}{|y'-\tilde{x}|^n} \right) u(y') dy', \end{aligned}$$

where  $x \in B_R^+$ ,  $\tilde{x} = R^2 x / |x|^2$ ,  $x^* = (x', -x_n)$  and  $\omega_n = \pi^{\frac{n}{2}} / \Gamma(1 + \frac{n}{2})$  is the volume of the unit  $n$ -ball in  $\mathbf{R}^n$ .

### 3. Proof of Theorem 1

By applying Lemma 2.1 to  $u(x)$ , we have

$$\begin{aligned}
 (3.1) \quad & \int_{\{x \in H: |x|=R\}} u^+(x) \frac{nx_n}{R^{n+1}} d\sigma(x) + \int_{\partial H(1,R)} u^+(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' \\
 &= \int_{\{x \in H: |x|=R\}} u^-(x) \frac{nx_n}{R^{n+1}} d\sigma(x) \\
 & \quad + \int_{\partial H(1,R)} u^-(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' + c_1 + \frac{c_2}{R^n}.
 \end{aligned}$$

It immediately follows from (1.1) that

$$(3.2) \quad \int_{\{x \in H: |x|=R\}} u^+(x) \frac{nx_n}{R^{n+1}} d\sigma(x) \leq MKR^{\rho(R)-1}$$

and

$$(3.3) \quad \int_{\partial H(1,R)} u^+(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' \leq MKR^{\rho(R)-1}.$$

Hence from (3.1), (3.2) and (3.3) we have

$$(3.4) \quad \int_{\{x \in H: |x|=R\}} u^-(x) \frac{nx_n}{R^{n+1}} d\sigma(x) \leq MKR^{\rho(R)-1}$$

and

$$(3.5) \quad \int_{\partial H(1,R)} u^-(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' \leq MKR^{\rho(R)-1}.$$

And (3.5) gives

$$\begin{aligned}
 & \int_{\partial H(1,R)} \frac{u^-(x')}{|x'|^n} dx' \\
 & \leq \frac{2^n}{2^n - 1} \int_{\partial H(1,R)} u^-(x') \left( \frac{1}{|x'|^n} - \frac{1}{(2R)^n} \right) dx' \\
 (3.6) \quad & \leq MK(2R)^{\rho(2R)-1}.
 \end{aligned}$$

Since  $-u(x) \leq u^-(x)$ , by applying Lemma 2.2 to  $-u(x)$  we have

$$(3.7) \quad -u(x) \leq I_1(x) + I_2(x),$$

where

$$I_1(x) = \int_{\{y \in H: |y|=R\}} \frac{R^2 - |x|^2}{\omega_n R} \left( \frac{1}{|y-x|^n} - \frac{1}{|y-x^*|^n} \right) u^-(y) d\sigma(y)$$

and

$$I_2(x) = \frac{2x_n}{\omega_n} \int_{\partial H[0,R]} \left( \frac{1}{|y' - x|^n} - \frac{R^n}{|x|^n} \frac{1}{|y' - \tilde{x}|^n} \right) u^-(y') dy'.$$

We remark that

$$(3.8) \quad \frac{1}{|y - x|^n} - \frac{1}{|y - x^*|^n} \leq \frac{2nx_n y_n}{|y - x|^{n+2}}$$

and

$$(3.9) \quad |y - x|^n \geq x_n^n = |x|^n \sin^n \theta, \quad x \in H, \quad y_n = 0.$$

If we put  $|x| = r > 1/2$  and  $R = 2r$  in (3.7), then we finally have from (3.4), (3.8) and (3.9)

$$(3.10) \quad \begin{aligned} I_1(x) &\leq \int_{\{y \in H: |y|=R\}} \frac{R^2 - r^2}{\omega_n R} \frac{2nx_n y_n}{\omega_n |y - x|^{n+2}} u^-(y) d\sigma(y) \\ &\leq MKR^{\rho(R)} \end{aligned}$$

and

$$(3.11) \quad I_2(x) \leq I_{21}(x) + I_{22}(x),$$

where

$$I_{21}(x) = \frac{2}{\omega_n x_n^{n-1}} \int_{\partial H(1,R)} u^-(y') dy'$$

and

$$I_{22}(x) = \frac{2}{\omega_n x_n^{n-1}} \int_{\partial H[0,1]} u^-(y') dy'.$$

We obtain that

$$(3.12) \quad \begin{aligned} I_{21}(x) &\leq \frac{2R^n}{\omega_n x_n^{n-1}} \int_{\partial H(1,R)} \frac{u^-(y')}{|y'|^n} dy' \\ &\leq MK(2R)^{\rho(2R)} \sin^{1-n} \theta \end{aligned}$$

and

$$(3.13) \quad \begin{aligned} I_{22}(x) &\leq \frac{2K}{\omega_n x_n^{n-1}} \int_{\partial H[0,1]} dy' \\ &\leq MK \sin^{1-n} \theta \end{aligned}$$

from (3.6) and (1.2), respectively.

From (3.7), (3.10), (3.11), (3.12) and (3.13), we have for  $|x| > 1/2$

$$(3.14) \quad -u(x) \leq MK \left( 1 + (2R)^{\rho(2R)} \right) \sin^{1-n} \theta.$$

For  $|x| \leq 1/2$ , we have from (1.2)

$$(3.15) \quad -u(x) \leq K \leq K \left(1 + (2R)^{\rho(2R)}\right) \sin^{1-n} \theta.$$

Thus the conclusion immediately follows from (3.14) and (3.15).

#### 4. Proof of Theorem 2

By modifying (3.6), we have

$$\begin{aligned} & \int_{\partial H(1,R)} \frac{u^-(x')}{|x'|^n} dx' \\ & \leq \frac{(N+1)^n}{(N+1)^n - N^n} \int_{\partial H(1,R)} u^-(x') \left( \frac{1}{|x'|^n} - \frac{1}{\left(\frac{N+1}{N}R\right)^n} \right) dx' \\ & \leq MK \left(\frac{N+1}{N}R\right)^{\rho\left(\frac{N+1}{N}R\right)-1}. \end{aligned}$$

Then (3.12), (3.14) and (3.15) are replaced by the following estimates

$$(4.1) \quad I_{21}(x) \leq MK \left(\frac{N+1}{N}R\right)^{\rho\left(\frac{N+1}{N}R\right)-1} \sin^{1-n} \theta.$$

$$(4.2) \quad -u(x) \leq MK \left(1 + \left(\frac{N+1}{N}R\right)^{\rho\left(\frac{N+1}{N}R\right)}\right) \sin^{1-n} \theta.$$

$$(4.3) \quad -u(x) \leq K \leq MK \left(1 + \left(\frac{N+1}{N}R\right)^{\rho\left(\frac{N+1}{N}R\right)}\right) \sin^{1-n} \theta.$$

All (3.7), (3.10), (3.11), (4.1), (3.12), (4.2) and (4.3) give

$$u(x) \geq -MK \left(1 + \left(\frac{N+1}{N}R\right)^{\rho\left(\frac{N+1}{N}R\right)}\right) \sin^{1-n} \theta$$

from which the conclusion immediately follows.

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## REFERENCES

- [1] D. H. Armitage, A Nevanlinna theorem for superharmonic functions in half-spaces, with applications, *J. London Math. Soc.(2)* **23** (1981), no. 1, 137–157.
- [2] T. Carleman, Über die approximation analytischer Funktionen durch lineare aggregate von vorgegebenen potenzen, *Ark. Mat., Astr. Och Fysik.* **17** (1923) 1–30.
- [3] I. F. Krasichkov-Ternovskii, Estimates for the subharmonic difference of subharmonic functions. II, *Mat. Sb.* **32** (1977) 32–59.
- [4] Ü. Kuran, Harmonic majorizations in half-balls and half-spaces, *Proc. London Math. Soc. (3)* **21** (1970) 614–636.
- [5] B. Ya. Levin, Distribution of Zeros of Entire Functions, Revised ed., Translations of Mathematical Monographs, 5, Amer. Math. Soc., Providence, 1980.
- [6] B. Ya. Levin, Lectures on Entire Functions, Translations of Mathematical Monographs, 150, Amer. Math. Soc., Providence, 1980.
- [7] N. K. Nikol'skii, Selected problems of the weighted approximation and of spectral analysis, *Trudy Mat. Inst. Steklov. Inst. Steklov.* **21** (1974); *English transl. in Proc. Steklov Inst. Math.* **120** (1976).

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