ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 40 (2014), No. 1, pp. 55–81

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 40 (2014), No. 1, pp. 55–81
Online ISSN: 1735-8515

MAXIMAL PREHOMOGENEOUS SUBSPACES ON CLASSICAL GROUPS

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(Communicated by Abbas Salemi)

Abstract. Suppose G is a split connected reductive orthogonal or symplectic group over an infinite field F, $P = MN$ is a maximal parabolic subgroup of G, $\mathfrak{n}$ is the Lie algebra of the unipotent radical N. Under the adjoint action of its stabilizer in M, every maximal prehomogeneous subspaces of $\mathfrak{n}$ is determined.

Keywords: Prehomogeneous space, adjoint action, orthogonal, symplectic, orbit.
MSC(2010): Primary: 15A99; Secondary: 22F30.

1. Introduction

In representation theory on reductive algebraic groups, the intertwining operators (see [3,9] for definition) play a central role in studying induced representations. In $p-$adic case, by Langlands classification on tempered representations (cf [10]) and multiplicity of intertwining operators (cf [9]), it is necessary and sufficient to study only representations induced from maximal parabolic subgroups. The main purpose of this paper is to study a problem raised from the study of these intertwining operators.

More precisely, let K be a nonarchimedean field of characteristic zero, G be a subgroup consisting of $K-$rational points of a split connected reductive orthogonal or symplectic classical group of split rank l defined over K. Let $P = MN \cong \mathrm{GL}_n(K) \times G_m$ be a maximal parabolic subgroup of G. Where $l = m + n$; G_m is a connected reductive classical

Article electronically published on February 25, 2014.
Received: 09 December 2011, Accepted: 09 December 2012.
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group of same type as G of split rank m; M, N are the Levi subgroup and unipotent subgroup of P, respectively. In order to understand the reducibility and tempered spectrum of a representation of G induced from an irreducible unitary representation of M, or equivalently, to determine the pole of the standard intertwining operator attached to it at $s = 0$, we need to study the behavior of the adjoint action of M on $\mathfrak{n}$, the Lie algebra of N (cf [4, 5, 9, 16–18, 22, 23]). To be more accurate, we need to integrate the matrix coefficients of such a representation on certain orbits related to the conjugacy classes of N under $\mathrm{Int}\,(M)$. For this reason, the prehomogeneity of $\mathfrak{n}$ under $\mathrm{Ad}(M)$ is of great interest to us, and it is known that when $\mathfrak{n}$ is prehomogeneous, there is only a single possible pole at $\mathrm{Re}\,(s) = 0$, the residue of which can be expressed as finite number of orbital integrals ([18, 22, 23]).

Except for some special cases ([11, 21–23]), $\mathfrak{n}$ is generally not prehomogeneous under $\mathrm{Ad}(M)$. However, relating to the classification theory of induced representations and orbital integrals, the structure of prehomogeneous sub vector spaces (see Definition 2.1) of $\mathfrak{n}$ under the adjoint action of certain subgroups of M still shows its great importance. This paper aims to solve such a problem in a general way, it gives a necessary and sufficient condition for any subspace $\mathcal{V}$ of $\mathfrak{n}$ which contains the center $\mathfrak{n}_2$ of $\mathfrak{n}$ to be prehomogeneous.

Now suppose F is an infinite field, every group mentioned above is defined over F. An element in $\mathfrak{n}$ can be expressed as $L(X, Y)$, where $X \in M_{n\times 2m}(F)$ or $M_{n\times(2m+1)}(F)$, $Y \in \mathrm{GL}_n(F)$ is skew-symmetric (symmetric) with respect to the second diagonal if G is orthogonal (semplectic, resp). Suppose X' is determined by Lemma 3.3, then the main results of this paper, Theorem 4.5, 4.6 and 5.4 can be summarized as follows:

Theorem 1.1. *Let $\mathfrak{n}_2$ be the center of $\mathfrak{n}$. Suppose $\mathcal{V}$ is a subspace of $\mathfrak{n}$ containing $\mathfrak{n}_2$ as a proper subspace, $M_{\mathcal{V}} = \{m \in M|\ Ad(m) \circ v \in \mathcal{V},\ \forall\, v \in \mathcal{V}\}$ is the stabilizer of $\mathcal{V}$. Then*

I) If G is symplectic or G is orthogonal with n even, then $\mathcal{V}$ is a maximal prehomogeneous subspace of $\mathfrak{n}$ under $M_{\mathcal{V}}$ if and only if for all $L(X, Y) \in \mathcal{V}$, $XX' = 0$.

II) If G is orthogonal and n is odd, then $\mathcal{V}$ is maximal prehomogeneous under $M_{\mathcal{V}}$ if and only if for all $L(X, Y) \in \mathcal{V}$, $rank(XX') \leq 1$.

It should be mentioned that although we only give prehomogeneity for some special objects (containing the center), the subspaces in this paper can be substituted by others (change $\mathfrak{n}_2$ to any of its subspace). More generally, making some plain modification in the proof of these theorems, it can be seen that if $\mathcal{V}$ is an arbitrary subspace, then it is maximal prehomogeneous under the action of its stabilizer if and only if for all $L(X,Y) \in \mathcal{V}$,

$$\operatorname{rank}(XX' + Y) = \operatorname{rank}(XX') + \operatorname{rank}(Y) \leq n, \tag{1.1}$$

(see the remark at the end of section 5). This should give a classification of all prehomogeneous subspaces in $\mathfrak{n}$.

As a direct application, we obtain the following interesting result (Theorem 4.9, 5.5):

Proposition 1.2. *Suppose G is symplectic or G is orthogonal with n even, then every maximal prehomogeneous subgroup of N containing the center of N is abelian, and it is unique up to the conjugate action of* $\operatorname{Int}(M)$.

It should be pointed out that the general research on conjugacy classes in parabolic subgroups started as early as 1974 ([12]), while the authors in [1,2,6–8,11,12] have determined the prehomogeneity on all descending central series of $\mathfrak{n}$ for general parabolic subgroup P, and all cases that $\mathfrak{n}$ is prehomogeneous when G is an exceptional group have also been determined later. These results are mainly based on the root system of the unipotent radical, while our approach to this problem relies basically on symmetric and skew-symmetric forms. Since these quadratic forms are invariant under $\mathrm{Ad}(M)$ ([4,5,23]), they would certainly reveal some explicit significance in representation theory (cf [16,17,19]), and further research on this topic should yield more interesting application to the study of $L-$functions and representation classification ([16,18,23]).

Our main idea comes from the following observation: for a subspace $\mathcal{V}$ of $\mathfrak{n}$ to be prehomogebeous, then (1.1) must hold by the finiteness of representatives of orbits (Theorem 4.6 shows that $\mathcal{V}$ is not prehomogeneous if the inequality in (1.1) breaks); and this prehomogeneity is maximal if and only if the equality of (1.1) is also fulfilled. The techniques we've adopted are the following: to show a subspace is prehomegeneous, we solely find the representatives of orbits of which the union is a dense subset; to illustrate the converse, we construct certain orbital invariant variables and show that there are infinitely many of them.

Although we are more interested in application to representation theory over $p-$adic groups, the results of this paper are also valid for any infinite field, that is the ground field in this paper. On the other hand, for unipotent radicals of more general parabolic subgroups, the task of classifying all prehomogeneous subspaces should be no doubt much harder, and using only quadratic forms to do that seems not adequate. We hope we can combine our techniques with the ideas in [1,2,6–8,11,12] to do more on this topic in future.

Finally, since we've taken many notations, we attach at the end a list of major mathematical symbols appearing in this paper.

2. **Preliminaries**

Let F be an infinite field. We start with the following (see also [14]):

Definition 2.1. *Let H be a linear algebraic group defined over F, V a finite dimensional vector space, and ρ a rational representation of H on V. We call a triple (H, ρ, V) a prehomogeneous vector space if there exists a proper closed algebraic subset S (in Zariski topology) of V such that $V \setminus S$ has finite $H-$orbits (a single $H-$orbit if F is algebraically closed).*

Furthermore, suppose X is a subspace of V, let

$$H_X = \{h \in H | \ \rho(h) \circ v \in X, \ \forall v \in X\}$$

be the stabilizer of X in H. Then we will say that (H_X, ρ, X) is a prehomogeneous sub vector space if X has finitely many H_X-orbits up to a proper closed algebraic subset. Moreover, we'll call it a maximal prehomogeneous subspace if it is maximal in sense of containment.

We will sometimes omit the group H and its action ρ and simply say V is prehomogeneous if this meaning in clear from the context.

For a positive integer r, let

$$w_r = \begin{pmatrix} & & 1 \\ & \cdot & \\ \cdot & & \\ 1 & & \end{pmatrix} \in M_r(F),$$

$$J_{2l} = \begin{cases} w_{2l+1} & \text{if } \mathbf{G} = \mathrm{SO}_{2l+1}(F); \\ w_{2l} & \text{if } \mathbf{G} = \mathrm{SO}_{2l}(F); \\ \begin{pmatrix} & w_l \\ -w_l & \end{pmatrix} & \text{if } \mathbf{G} = \mathrm{S}p_{2l}(F). \end{cases}$$

Let $\mathbf{G}$ be a split connected reductive classical group over F, defined with respect to J_{2l}. i.e.,

$$\mathbf{G} = \{g \in \mathrm{G}L_{2l} | {}^t g J_{2l} g = J_{2l}\}^\circ,$$

with the superscript indicating the connected component.

Let $\mathbf{T}$ be the maximal split torus of diagonal elements in $\mathbf{G}$, we can take:

$$T = \begin{cases} \{\mathrm{diag}(x_1, \cdots, x_l, 1, x_l^{-1}, \cdots, x_1^{-1}) | x_i \in F^*, i = 1, 2, \cdots, l\}, \mathrm{SO}_{2l+1}(F); \\ \{\mathrm{diag}(x_1, \cdots, x_l, x_l^{-1}, \cdots, x_1^{-1}) | x_i \in F^*, i = 1, 2, \cdots l\}, \text{ otherwise.} \end{cases}$$

Fix an F-Borel subgroup $\mathbf{B}$ such that $\mathbf{B} = \mathbf{TU}$, where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$. Let Δ be the set of simple roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{U}$. Denote by $\mathbf{P} = \mathbf{MN}$ a maximal parabolic subgroup of $\mathbf{G}$ in the sense that $\mathbf{N} \subset \mathbf{U}$. Assume $\mathbf{T} \subset \mathbf{M}$ and let $\theta = \Delta \setminus \{\alpha\}$ such that $\mathbf{M} = \mathbf{M}_\theta$. We use G, P, M, N, B, T, U to denote the subgroups of F-rational points of the groups $\mathbf{G}, \mathbf{P}, \mathbf{M}, \mathbf{N}, \mathbf{B}, \mathbf{T}, \mathbf{U}$, respectively.

For any $g \in \mathbf{G}$, We will use $\mathrm{Int}(g)$ to denote the inner automorphism of $\mathbf{G}$ induced by g. i.e., for any $u \in \mathbf{G}$, $\mathrm{Int}(g) \circ u = gug^{-1}$. Let $\mathfrak{g} = \mathrm{Lie}(G)$, the Lie algebra of G, we will use $\mathrm{Ad}(g)$ to denote the adjoint action on $\mathfrak{g}$ induced from $\mathrm{Int}(g)$.

Let $\mathfrak{n} = \mathrm{Lie}(N)$, be the Lie algebra of N. Then $\mathfrak{n}$ can be graded by α as $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$. i.e., for any $t \in \{\text{center of } M\}$, and any $X_1 \in \mathfrak{n}_1$, $X_2 \in \mathfrak{n}_2$, we have:

$$\begin{cases} \mathrm{Ad}(t) \circ X_1 = \alpha(t) X_1; \\ \mathrm{Ad}(t) \circ X_2 = 2\alpha(t) X_2. \end{cases}$$

Notice that $\mathfrak{n}_2$ is the center of $\mathfrak{n}$. We will set $N_1 = \exp(\mathfrak{n}_1)$, $N_2 = \exp(\mathfrak{n}_2)$.

Let e_i $(1 \leq i \leq l) \in \mathrm{Hom}(T, F^*)$ such that $e_i(T) = x_i$. Let $\alpha_i = e_i - e_{i+1}, i = 1, 2, \cdots, l-1$, and

$$\alpha_l = \begin{cases} e_l, & \text{if } G = \mathrm{SO}_{2l+1}(F); \\ e_{l-1} + e_l, & \text{if } G = \mathrm{SO}_{2l}(F); \\ 2e_l, & \text{if } G = \mathrm{S}p_{2l}(F). \end{cases}$$

These α_i is a set of simple roots determined by T.

Suppose $\alpha = \alpha_n$, then up to a conjugation, we may assume $M = \mathrm{G}L_n(F) \times \mathrm{SO}_{2m+1}(F)$, $\mathrm{G}L_n(F) \times \mathrm{S}p_{2m}(F)$, or $\mathrm{G}L_n(F) \times \mathrm{SO}_{2m}(F)$, depending on wether G is of type B_l, C_l or D_l, respectively. For convenience of notation, let $G' = \mathrm{G}L_n(F)$,

$$G_m = \begin{cases} \mathrm{SO}_{2m+1}(F), & \text{if } G = \mathrm{SO}_{2l+1}(F); \\ \mathrm{SO}_{2m}(F), & \text{if } G = \mathrm{SO}_{2l}(F); \\ \mathrm{S}p_{2m}(F), & \text{if } G = \mathrm{S}p_{2l}(F). \end{cases}$$

And

$$\kappa = \begin{cases} 2m+1, & \text{if } G = \mathrm{SO}_{2l+1}(F); \\ 2m, & \text{otherwise.} \end{cases}$$

When $G = \mathrm{SO}_{2l+1}(F)$, we choose a representative s_{e_i} of $S_{e_i}, i = 1, 2 \cdots, l$, the Weyl group element which is the reflection about e_i as follows: $s_{e_i} = D_{i,2l-i+2}$, where for any pair of positive integers $\{i,j\}, D_{i,j} \in M_{(2l+1)\times(2l+1)}(F)$ is an elementary matrix obtained by interchanging the $i-$th and $j-$th rows of I_{2l+1} and then setting the $i-$th row to its negative. When $G = \mathrm{S}p_{2l}(F)$, we choose a representative s_{2e_i} of S_{2e_i} as follows:

$$s_{2e_i} = \begin{pmatrix} I_l - E_{i,i} & E_{i,l-i+1} \\ -E_{l-i+1,i} & I_l - E_{l-k+1,l-k+1} \end{pmatrix}.$$

Here for any pair of positive integers $\{i,j\}, E_{i,j} \in M_{l\times l}(F)$ is an elementary matrix such that its (i,j)'s entry is 1, all other entries are 0.

Also for any pair of positive integers $\{i,j\}$ with $i \neq j$, we choose a representative $s_{e_i-e_j}$ of $S_{e_i-e_j}$, the Weyl group element which is the reflection about $e_i - e_j$ as follows:

$$s_{e_i-e_j} = \begin{pmatrix} D_{i,j} & \\ & D_{i,j} \end{pmatrix}.$$

3. **The Unipotent radical** N

For any $Y \in M_n(F)$, we set $\varepsilon(Y) = w_n\, {}^tY w_n^{-1}$. Then $\varepsilon(\varepsilon(Y)) = Y$ since $w_n^{-1} = w_n$. We define an action ε of G' on $M_n(F)$ by: $\varepsilon(g) \circ A = gA\varepsilon(g), \ \forall\ g \in G', A \in M_n(F)$. Every element $m \in M$ may be written as $m = \mathrm{diag}(g, h, \varepsilon(g)^{-1})$ with $g \in G', h \in G_m$. Throughout this paper, when m is written in this way, it will always be referred to with the same meaning.

Definition 3.1. *For any $A \in M_n(F)$, we say that A is $\varepsilon-$symmetric if $\varepsilon(A) = A$; skew$-\varepsilon-$symmetric if $\varepsilon(A) = -A$. Denote by $M_n^{\varepsilon}(F)$ the subspace of $M_n(F)$ consisting of $\varepsilon-$symmetric elements, $\bar{M}_n^{\varepsilon}(F)$ the subspace of $M_n(F)$ consisting of skew$-\varepsilon-$symmetric elements.*

Lemma 3.2. $M_n(F) = M_n^\varepsilon(F) \oplus \bar{M}_n^\varepsilon(F)$.

Proof. This is obvious. $\square$

Lemma 3.3. *Let $u \in N$ and suppose that*

$$u = \begin{pmatrix} I_n & X & Y \\ 0 & I_\kappa & X' \\ 0 & 0 & I_n \end{pmatrix}.$$

Then

$$X' = \begin{cases} -J_\kappa\, {}^tXw_n, & \text{if } G \text{ is orthogonal;} \\ J_\kappa\, {}^tXw_n, & \text{otherwise,} \end{cases}$$

and

$$XX' = \begin{cases} Y + \varepsilon(Y), & \text{if } G \text{ is orthogonal;} \\ Y - \varepsilon(Y), & \text{otherwise.} \end{cases}$$

In particular, if $u \in N_2$, then $X = 0$ and $Y \in \bar{M}_n^\varepsilon(F)$(or $M_n^\varepsilon(F)$) if G is orthogonal (or symplectic, resp). If $u \in N_1$, then $Y \in M_n^\varepsilon(F)$(or $\bar{M}_n^\varepsilon(F)$) if G is orthogonal (or symplectic, resp).

Proof. This is Lemma 3.3 in [23]. $\square$

If $u \in N$ is as above, we will denote it by $n(X,Y)$. Notice for any $m = \mathrm{diag}(g, h, \varepsilon(g^{-1})) \in M$, $\mathrm{Int}(m) \circ n(X,Y) = n(gXh^{-1}, gY\varepsilon(g))$. For any X as above, we use $C_i(X)$ to denote its i-th column, $i = 1, 2, \cdots, \kappa$. We say that $C_i(X)$ and $C_{\kappa-i+1}(X)$ are conjugate columns. In particular, when $G = \mathrm{SO}_{2l+1}(F)$, X_{m+1} is conjugate to itself.

From Lemma 3.3, it is not hard to see that:

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & X & Y \\ 0 & 0 & X' \\ 0 & 0 & 0 \end{pmatrix} \mid Y \in \begin{cases} \bar{M}_n^\varepsilon(F) \text{ if } G \text{ is orthogonal} \\ M_n^\varepsilon(F) \text{ if } G \text{ is symplectic} \end{cases} \right\}.$$

Suppose L is the Lie algebra element of $n(X,Y)$, then

(3.1)
$$L = \begin{pmatrix} 0 & X & Y' \\ 0 & 0 & X' \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & X & 0 \\ 0 & 0 & X' \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & Y' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = L_1 + L_2.$$

Where $Y' = Y - \frac{1}{2}XX'$ and $L_i \in \mathfrak{n}_i, i = 1, 2$. We will use $L(X, Y')$ to denote L.

Let $k = \min(n, m), l = [(\kappa+1)/2], q = \min(n, l)$. Let Ω be a subset of N satisfying $XX' = 0$. Note Ω is invariant under $\mathrm{Int}(M)$. Let

$$\mathcal{E} = \{L(X, Y') \in \mathfrak{n} | XX' = 0\}.$$

Then

$$\Omega = \{exp(L(X, Y'))|L(X, Y') \in \mathcal{E}\}.$$

Where exp is the usual exponential map.

Definition 3.4. *Suppose $\{X_1, X_2, \cdots, X_i\}$ is a subset of columns in X, where X is as in Lemma 3.3. If none of these columns is conjugate to another one, then they are said to be conjugate irrelevant.*

Definition 3.5. *For X as above, we define the relative rank of X, denoted by* $\mathrm{rrank}(X)$, *to be the maximal rank of $\{X_1, X_2, \cdots, X_i\}$ as $\{X_1, X_2, \cdots, X_i\}$ runs through all collections of conjugate irrelevant columns in X. If $\mathbf{O}$ $(\mathcal{O})$ is a subset of N $(\mathfrak{n}$, resp), we define*

$$\mathrm{rrank}(\mathbf{O}) = \max\{\mathrm{rrank}(X)|n(X, Y) \in \mathbf{O}\},$$

$$\mathrm{rrank}(\mathcal{O}) = \max\{\mathrm{rrank}(X)|L(X, Y') \in \mathcal{O}\}.$$

Notice for any X as above, $\mathrm{rrank}(X) \leq q$, and rrank will remain unchanged for any action $\mathrm{Int}(M)(\mathrm{Ad}(M))$ on $n(X, Y)$ $(L(X, Y')$, resp). For any non-negative integer r with $1 \leq r \leq q$, if we set $\Omega_r = \{n(X, Y) \in \Omega|\mathrm{rrank}(X) = r\}$. Then

$$\Omega = \bigcup_{r=1}^{q} \Omega_r.$$

Each Ω_r is invariant under $\mathrm{Int}(M)$ and it's obvious that Ω_q is a dense open subset of Ω.

Let

$$\mathcal{E}_r = \{L(X, Y)|L(X, Y) \in \mathcal{E},\ \mathrm{rrank}(X) = r\},\ \ r = 1, 2, \cdots, q.$$

Then

$$\Omega_r = \{exp(L(X, Y'))|L(X, Y') \in \mathcal{E}_r\}.$$

4. Maximal Prehomogeneous Spaces on B_l and C_l

From now on, we will determine all prehomogeneous subspaces of $\mathfrak{n}$ by studying its maximal prehomogeneous subspaces. If $\mathcal{V}$ is a subspace of $\mathfrak{n}$, we set

$$M_{\mathcal{V}} = \{m \in M|\ \mathrm{Ad}(m) \circ v \in \mathcal{V},\ \forall\ v \in \mathcal{V}\}$$

to be the stabilizer of $\mathcal{V}$ in M. Then $M_{\mathcal{V}}$ is a subgroup of M.

Throughout this paper, we will say a subspace $\mathcal{V}$ of $\mathfrak{n}$ is prehomogeneous without mentioning the group $M_{\mathcal{V}}$ and its action if this causes no ambiguity. Meanwhile, since the adjoint action $\mathrm{Ad}(M)$ on $\mathfrak{n}$ is completely determined by the action $\mathrm{Int}(M)$ on N, we will abuse the term "prehomogeneous" to say that a subgroup of N is prehomogeneous if its corresponding sub Lie algebra is prehomogeneous under its stabilizer. Moreover, for the purpose of convenience, we will also use "prehomogeneous" to describe an algebraic set, by means that it has finite number of orbits up to a closed subset (in Zariski topology).

Lemma 4.1. *Suppose G is type B_l or C_l and rrank$(X) = r$. Then there is an s which is a representative of a Weyl group element in G_m, such that the first r columns of Xs are linearly independent.*

Proof. Suppose $\mathrm{rank}(X_{i_1}, X_{i_2}, \cdots, X_{i_r}) = r$, one may assume $i_1 < i_2 < \cdots < i_r$. Then there is a positive integer p with $1 \le p \le r$, such that $i_{p-1} \le \frac{\kappa+1}{2} < i_p$.

For any j with $p \le j \le r$, set $n_j = i_j + n - m$. Let

$$s_1 = \begin{cases} \displaystyle\prod_{j=p}^{r} s_{e_{n_j}}, & \text{if } G = \mathrm{SO}_{2l+1}(F); \\ \displaystyle\prod_{j=p}^{r} s_{2e_{n_j}}, & \text{if } G = \mathrm{S}p_{2l}(F). \end{cases}$$

Multiplying X by s_1 from right, it will interchange the i_j-th column $(j = p, p+1, \cdots, r)$ with its conjugate. So if we set $Xs_1 = \{V_1, V_2, \cdots, V_\kappa\}$, then $\mathrm{rank}\{V_1, V_2, \cdots, V_\lambda\} = r$.

Suppose $\mathrm{rank}(V_{j_1}, V_{j_2}, \cdots, V_{j_r}) = r$ with $j_1 < j_2 < \cdots < j_r \le \lambda$. Set

$$s_2 = \prod_{i=1}^{r} s_{e_{n+i} - e_{n+j_i}} \in G_m,$$

$$W = \{W_1, W_2, \cdots, W_\kappa\} = (Xs_1)s_2 = Xs.$$

Then W has the desired property as stated in the lemma. □

Lemma 4.2. *Suppose G is type B_l or C_l. Then for any $n(X,Y) \in \Omega_r$, there is $m = diag(g, h, \varepsilon(g)^{-1}) \in M$, such that $Int(m) \circ n = n(E_r, gY\varepsilon(g))$, where*

$$E_r = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \in M_{n\times\kappa}(F).$$

In particular, rrank$(X) =$ *rank*(X).

Proof. Suppose $X \in \Omega_r$, by Lemma 4.1, we can assume that $\operatorname{rank}(X_1, X_2, \cdots, X_r) = r$. Then there exists $g \in G'$ such that

$$gX = \begin{pmatrix} I_r & B & A \\ 0 & B_1 & A_1 \end{pmatrix}.$$

Where $B \in M_{r\times(\kappa-2r)}(F), A \in M_{r\times r}(F), B_1 \in M_{(n-r)\times(\kappa-2r)}(F)$ and $A_1 \in M_{(n-r)\times r}(F)$.

Since $\operatorname{rrank}(gX) = \operatorname{rrank}(X) = r$, one must have: $B_1 = 0$. Therefore,

$$(gX)(gX)' = \begin{cases} -\begin{pmatrix} w_r^t A_1 w_{n-r} & C_1 \\ 0 & A_1 \end{pmatrix}, & \text{if } G \text{ is orthogonal;} \\ \begin{pmatrix} w_r^t A_1 w_{n-r} & C_2 \\ 0 & -A_1 \end{pmatrix}, & \text{if } G \text{ is symplectic.} \end{cases}$$

Where $C_1 = A + \varepsilon(A) - BB'$ and $C_2 = \varepsilon(A) - A + BB'$.

From the fact that $(gX)(gX)' = g(XX')\varepsilon(g) = 0$, we must have $A_1 = 0$ and $C_1 = C_2 = 0$. In either orthogonal or symplectic case,

$$h = \begin{pmatrix} I_r & B & A \\ 0 & I_{\kappa-2r} & B' \\ 0 & 0 & I_r \end{pmatrix} \in G_m.$$

Let $m = \operatorname{diag}(g, h, \varepsilon(g)^{-1})$, then it can be checked that $\operatorname{Int}(m) \circ n = n(E_r, gY\varepsilon(g))$. □

Theorem 4.3. *Suppose G is as above, then for each $r \leq q$, Ω_r is a prehomogeneous algebraic set under Int(M). In particular, Ω is a prehomogeneous algebraic set.*

Proof. For any $n(X, Y) \in \Omega_r$, $XX' = 0$. By Lemma 3.3, we have: $Y \in \bar{M}_n^\varepsilon(F)$ if $G = \mathrm{SO}_{2l+1}(F)$; or $Y \in M_n^\varepsilon(F)$ if $G = \mathrm{Sp}_{2l}(F)$. Therefore, if we set

$$n' = \begin{cases} 2\left[\dfrac{n}{2}\right], & \text{if } G = \mathrm{SO}_{2l+1}(F); \\ n, & \text{if } G = \mathrm{Sp}_{2l}(F). \end{cases}$$

Then $\operatorname{rank}(Y) \leq n'$. Let

$$\Omega_r' = \{n(X, A) | n(X, A) \in \Omega_r, \ \operatorname{rank}(A) = n'\}.$$

Then Ω_r' is a dense open subset of Ω_r. Let $n(X, A)$ be an arbitrary element in Ω_r', by Lemma 4.2, there is an $m_1 \in M$, such that $\operatorname{Int}(m_1) \circ n(X, A) = (E_r, Y)$, where $Y = gA\varepsilon(g)$ for some $g \in G'$. We'll prove by induction on n that Ω_r' is prehomogeneous under $\operatorname{Int}(M)$.

First suppose that $G = \mathrm{SO}_{2l+1}(F)$. If $n = 1$, then $Y = A = 0$. By Lemma 4.2, no matter wether $r = 0$ or 1 (notice $r \leq k = n$), there is only one generator $(E_r, 0)$ of Ω_r under $\mathrm{Int}(M)$. Hence, Ω_r is prehomogeneous.

Assume now the theorem is true for all $n < i$ and suppose $n = i$.

If $r = n$, then by an extension of Gaussian Elimination as in the proof of Lemma 3.7 in [23], there is a dense open subset W of $\bar{M}_n^{\varepsilon}(F)$, such that for any $Y \in W$, there is $g' \in G'$, satisfying $g'Y\varepsilon(g') = B_n$. Where

$$B_n = \begin{pmatrix} & & & & \cdot & \\ & & & \cdot & & \\ & & 1 & 0 & & \\ & & 0 & -1 & & \\ 1 & 0 & & & & \\ 0 & -1 & & & & \end{pmatrix} \in M_n(F).$$

Suppose $g'E_r = (X_1, 0)$, where $X_1 \in G'$. Then there is $g_1 \in G'$ such that $X_1 g_1 = I_r$. Let $h = \mathrm{diag}(g_1^{-1}, I_{\kappa-2n}, \varepsilon(g_1)) \in G_m$, then $g'E_r h^{-1} = E_r$. Thus, for any $n(X, A)$ belonging to a dense open subset $\{n(X, Y) \in \Omega_r | Y \in W\}$ of Ω_r, there is an $m = \mathrm{diag}(g', h, \varepsilon(g')^{-1}) \cdot m_1 \in M$, such that $\mathrm{Int}(m) \circ n(X, A) = (E_r, B_n)$. Hence, Ω_r is prehomogeneous under $\mathrm{Int}(M)$.

If $r < n$, assume

$$Y = \begin{pmatrix} Y_{1,1} & Y_{1,2} & \cdot & \cdot & Y_{1,n-1} & 0 \\ Y_{2,1} & Y_{2,2} & \cdot & \cdot & 0 & -Y_{1,n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ Y_{n-1,1} & 0 & \cdot & \cdot & -Y_{2,2} & -Y_{1,2} \\ 0 & -Y_{n-1,1} & \cdot & \cdot & -Y_{2,1} & -Y_{1,1} \end{pmatrix}.$$

We may further assume that $Y_{n-1,1} \neq 0$, this assumption will only amount to an open subset of Ω_r. Let

$$g_1 = \begin{pmatrix} 1 & 0 & \cdot & \cdot & 0 & 0 & \frac{-Y_{1,1}}{Y_{n-1,1}} \\ 0 & 1 & \cdot & \cdot & 0 & 0 & \frac{-Y_{2,1}}{Y_{n-1,1}} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 1 & 0 & \frac{-Y_{n-2,1}}{Y_{n-1,1}} \\ 0 & 0 & \cdot & \cdot & 0 & \frac{1}{Y_{n-1,1}} & 0 \\ 0 & 0 & \cdot & \cdot & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$Y_1 = \varepsilon(g_1) \circ Y = \begin{pmatrix} 0 & Y'_{1,2} & \cdot & \cdot & Y'_{1,n-1} & 0 \\ 0 & Y'_{2,2} & \cdot & \cdot & 0 & -Y'_{1,n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & \cdot & -Y'_{2,2} & -Y'_{1,2} \\ 0 & -1 & \cdot & \cdot & 0 & 0 \end{pmatrix}.$$

Next, let

$$g_2 = \begin{pmatrix} 1 & 0 & \cdot & \cdot & 0 & 0 & Y'_{1,2} \\ 0 & 1 & \cdot & \cdot & 0 & 0 & Y'_{2,2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 1 & 0 & Y'_{n-2,2} \\ 0 & 0 & \cdot & \cdot & 0 & 1 & 0 \\ 0 & 0 & \cdot & \cdot & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$Y_2 = \varepsilon(g_2) \circ Y_1 = \begin{pmatrix} 0 & 0 & Y_{1,3}" & \cdot & \cdot & Y_{1,n-1}" & 0 \\ 0 & 0 & Y_{2,3}" & \cdot & \cdot & 0 & -Y_{1,n-1}" \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & -Y_{2,3}" & -Y_{1,3}" \\ 1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & -1 & 0 & \cdot & \cdot & 0 & 0 \end{pmatrix}.$$

Applying the induction hypothesis, one can see that there exists a $g \in G'$ such that $\varepsilon(g) \circ Y = B_n$. Moreover, by the above process of Gaussian Elimination, g has the property that

$$gE_r = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix},$$

with $X_1 \in \mathrm{G}L_r(F)$. Therefore, there exists $h_1 \in \mathrm{GL}_r(F)$ such that $X_1 h_1 = I_r$. Let

$$h = \mathrm{diag}(h_1^{-1}, I_{\kappa-2r}, \varepsilon(h_1)) \in G_m,$$

$$m_2 = \mathrm{diag}(g, h, \varepsilon(g)^{-1}) \in M.$$

Then $\mathrm{Int}(m_2) \circ n(X, Y) = n(E_r, B_n)$. Hence, we can choose a dense open subset Ω'_r of Ω_r properly so that it has only one orbit under $\mathrm{Int}(M)$. i.e., Ω_r is prehomogeneous.

Now we prove for the case $G = \mathrm{S}p_{2l}(F)$. For any $n(X, A) \in \Omega'_r$, one has $A \in M_n^{\varepsilon}(F)$ by Lemma 3.3. If $n = 1$, then Ω'_r is prehomogeneous when $r = 1$ by Theorem 4.2 in [23]. When $r = 0$, then every non-zero element in Ω'_r has the form $n(0, u)$ for some $u \in F^*$. Let S be a complete set of representatives of $F^*/(F^*)^2$. Choose $\varepsilon_i \in S$ such that $u = \varepsilon_i t^2$ for some $t \in F^*$, let $m = \mathrm{diag}(t^{-1}, I_{2l-2}, t) \in M$, then $\mathrm{Int}(m) \circ n(0, u) = n(0, \varepsilon_i)$. Therefore, there are only finitely many number of orbits of Ω'_0 under $\mathrm{Int}(M)$.

Assume now the theorem is true for all $n < i$ and suppose $n = i$.

If $r = n$, by Corollary 3.6 in [23], there is $g \in G'$ such that

$$\delta = \varepsilon(g) \circ Y = \begin{pmatrix} & & & & \varepsilon_1 \\ & & & \varepsilon_2 & \\ & & \cdot & & \\ & \cdot & & & \\ \varepsilon_n & & & & \end{pmatrix}, \tag{4.1}$$

with $\varepsilon_i \in S,\ i = 1, 2, \cdots, n$. Then we set $X_1 = gE_r$.

If $r < n$, then by a similar proof as above, we can find $g \in G'$ such that $\varepsilon(g) \circ Y = \delta$ with δ carrying the same form as in equation (4.1). Moreover, such g has a property that

$$gE_r = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix},$$

with $X_1 \in \mathrm{GL}_r(F)$.

In both cases, let $h = \mathrm{diag}(X_1, I_{\kappa-2r}, \varepsilon(X_1)^{-1}) \in G_m$, and $m_2 = \mathrm{diag}(g, h, \varepsilon(g)^{-1}) \in M$. Then $\mathrm{Int}(m_2) \circ n(E_r, Y) = n(E_r, \delta)$. Since there are only finitely many possibilities of δ, there are only finitely many generators of Ω'_r under $\mathrm{Int}(M)$. i.e., Ω'_r is prehomogeneous. □

Corollary 4.4. *If $G = \mathrm{SO}_{2l+1}(F)$, then there is only one open orbit of $\mathcal{E}$ under $Ad(M)$; if $G = \mathrm{S}p_{2l}(F)$, then the number of open orbits is $C_{n+|S|-1}^{|S|-1}$. Where S is a complete set of representatives of $F^*/(F^*)^2$, $C_{n+|S|-1}^{|S|-1}$ is a combination number.*

Proof. The first statement is obvious from the proof of Theorem 4.3.

Suppose $G = \mathrm{S}p_{2l}(F)$. By Theorem 4.3, the number of orbits equals to the number of $\varepsilon-$conjugate classes of all those δ, where δ is defined in equation (4.1). And this number equals to the number of all $\bar{\varepsilon}-$conjugate classes of δw_n, where the action $\bar{\varepsilon}$ of G' on $M_n(F)$ is defined as the usual congruence:

$$\bar{\varepsilon}(g) \circ Y = gY({}^t g), \quad g \in G', \; Y \in M_n(F).$$

Since $\delta w_n = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$, it is easily seen that those elements which have the same number and multiplicity of each $\varepsilon_i (i = 1, 2, \cdots, |S|)$, are $\bar{\varepsilon}-$conjugate to each other.

Suppose the multiplicity of each ε_i appearing in δw_n is k_i, then

$$\sum_{i=1}^{|S|} k_i = n.$$

For each i, there is a k_i-dimensional subspace V_i of F^n, such that for any $v \in V_i$, the quadratic form ${}^t v(\delta w_n) v \in \varepsilon_i (F^*)^2$. Notice the dimension of such space is invariant under the adjoint action. For if $m = \mathrm{diag}(g, h, \varepsilon(g)) \in M$, then $\mathrm{Ad}(m) \circ L(X, Y) = L(gXh^{-1}, gY\varepsilon(g))$, and the subspace ${}^t g^{-1} V_i$ has the desired property in the quadratic form determined by $gY\varepsilon(g)w_n$, which is of dimension k_i.

Thus, the numbers $k_i (i = 1, 2, \cdots, |S|)$, are invariant under $\mathrm{Ad}(M)$. This implies that the generator of each orbit of $\mathfrak{n}_2$ is completely determined by the combination of ε_i's. It is then easily calculated that there are $C_{n+|S|-1}^{|S|-1}$ such combinations in total. □

Theorem 4.5. *If G is symplectic and $\mathcal{V}$ is a subspace of $\mathfrak{n}$ containing $\mathfrak{n}_2$. Then $\mathcal{V}$ is a prehomogebeous space under $Ad(M_{\mathcal{V}})$ if and only if for any $L(X, Y') \in \mathcal{V}$, $rank(XX') = 0$. In particular, $\mathfrak{n}$ is prehomogeneous if and only if $n = 1$ or l.*

Proof. We will first prove the "if" statement.

Let $\mathcal{V}_i = \mathcal{E}_i \cap \mathcal{V}, i = 0, 1, \cdots, q$, then $\mathcal{V}$ is a disjoint union of all $\mathcal{V}_i$. Let r be the largest integer so that $\mathcal{V}_r$ is not empty, then $\mathcal{V}_r$ is a dense open subset of $\mathcal{V}$. We will prove $\mathcal{V}_r$ has finite number of open orbits under $\mathrm{Ad}(M_{\mathcal{V}})$.

Pick up an arbitrary element $L(X_0, Y_0)$ in $\mathcal{V}_r \cap \Omega'_r$, by Theorem 4.3, there exists $m \in M$ so that $\mathrm{Ad}(m) \circ L(X_0, Y_0) = L(E_r, \delta)$. Since $\mathcal{V}$ contains $\mathfrak{n}_2$ as a subspace, $L(X_0, Y_0) \in \mathcal{V}_r$ for any $Y_0 \in M_n^{\varepsilon}(F)$. Therefore, we may fix m and make Y_0 vary inside a certain finite set of candidates so that δ runs through all representatives of $\varepsilon-$conjugacy classes in $M_n^{\varepsilon}(F)$.

Let $\bar{\mathcal{V}} = \mathrm{Ad}(m) \circ \mathcal{V}$. Then $\bar{\mathcal{V}} \subset \mathcal{E}$ is a vector space and for any $L(\bar{X}, \bar{Y}) \in \bar{\mathcal{V}}$, $\mathrm{rrank}(\bar{X}) \leq r$. Let $\bar{\mathcal{V}}_r = \mathrm{Ad}(m) \circ \mathcal{V}_r$, then $\bar{\mathcal{V}}_r$ is also a dense open subset of $\bar{\mathcal{V}}$. For any $L(X, Y') \in \bar{\mathcal{V}}_r$, there is $m' \in M$ so that $\mathrm{Ad}(m') \circ L(X, Y') = L(E_r, \delta)$ by Theorem 4.3. Since both $L(X, Y')$ and

$L(E_r, \delta)$ belong to $\bar{\mathcal{V}}$, we know $m' \in M_{\bar{\mathcal{V}}}$. Therefore, $\bar{\mathcal{V}}$ is prehomogeneous under its stabilizer. So is $\mathcal{V}$.

Now we'll prove the "only if" statement.

If $\mathcal{O}$ is a subspace of $\mathfrak{n}$ containing $\mathcal{V}$ as a proper subspace, then there is $L(\tilde{X}, \tilde{Y}) \in \mathcal{O}$ such that $\tilde{X}\tilde{X}' = E \neq 0$, where $\tilde{X}' = J_\kappa\, {}^t\tilde{X} w_n$. Since $\mathcal{O}$ is a vector space containing $\mathfrak{n}_2$, $L(u\tilde{X}, \tilde{Y}) \in \mathcal{O}$ for any $u \in F$ and $\tilde{Y} \in M_n^\varepsilon(F)$. Let

$$\tilde{M}_n^\varepsilon(F) = \{A | A \in M_n^\varepsilon(F), \mathrm{rank}(A) = n\},$$

$$\mathcal{O}' = \{L(X,Y) | L(X,Y) \in \mathcal{O}, Y \in \tilde{M}_n^\varepsilon(F)\}.$$

Then $\tilde{M}_n^\varepsilon(F)$ and $\mathcal{O}'$ are dense open subsets of $M_n^\varepsilon(F)$ and $\mathcal{O}$, respectively.

Suppose $\mathcal{O}$ is prehomogeneous, then up to a closed subset, $\mathcal{O}$ has finite number of $M_{\mathcal{O}}-$orbits, so does $\mathcal{O}'$. Let $\{L(X_i, Y_i) | i \in I\}$ be the set of representatives of these finite orbits of $\mathcal{O}'$. For any $L(X,Y) \in \mathcal{O}'$, there is an $m = \mathrm{diag}(g, h, \varepsilon(g)^{-1}) \in M$, such that $\mathrm{Ad}(m) \circ L(X,Y) = L(X_i, Y_i)$ for a certain i. i.e., $gXh^{-1} = X_i, gY\varepsilon(g) = Y_i$. Therefore, $gXX'\varepsilon(g) = X_i X_i'$.

For any $L(X,Y) \in \mathcal{O}'$, if we define $F(L(X,Y)) = \det(Y)$ and $G(L(X,Y)) = \det(XX' + Y)$. Then by the proof of Theorem 3.11 in [23], both F and G are non-constant polynomial functions in terms of the entries of $L(X,Y)$, and it's easily seen that

$$\frac{F(L(X,Y))}{G(L(X,Y))} = \frac{F(L(X_i,Y_i))}{G(L(X_i,Y_i))}. \tag{4.2}$$

In particular, for any fixed $\tilde{Y}$ and constant t such that both $F(L(t\tilde{X}, \tilde{Y}))$ and $G(L(t\tilde{X}, \tilde{Y}))$ are nonzero, one must have:

$$\frac{F(L(t\tilde{X},\tilde{Y}))}{G(L(t\tilde{X},\tilde{Y}))} = \frac{F(L(X_i,Y_i))}{G(L(X_i,Y_i))} \tag{4.3}$$

for some certain i. Moreover, it is easily seen that both numerator and denominator on the left side of equation (4.3) are polynomials in t.

Let $L_1 = L(\tilde{X}, 0) \in \mathfrak{n}_1$, and define $\mathcal{W} = \mathrm{span}\{L_1\} \oplus \mathfrak{n}_2$. Then $\mathcal{W}' = \mathcal{W} \cap \mathcal{O}' \subset \mathcal{O}'$ is a dense open subset of $\mathcal{W}$. Therefore, there must exist one $\tilde{Y} \in \tilde{M}_n^\varepsilon(F)$ such that $L(t\tilde{X}, \tilde{Y}) \in \mathcal{O}'$ for infinitely many t. Furthermore, since there are only finitely many number of open orbits of $\mathcal{O}'$ under $\mathrm{Ad}(M_{\mathcal{O}})$, there is at least one i, such that $L(t\tilde{X}, \tilde{Y})$ belongs to the orbit represented by $L(X_i, Y_i)$ for infinitely many t. In other words, there are infinitely many t satisfying equation (4.3), which is obviously absurd!

In particular, $\mathfrak{n}$ is prehomogeneous under $\mathrm{Ad}(M)$ only if $XX' = 0$ for all $L(X,Y) \in \mathfrak{n}$, but this could happen only if $n = 1$ or l. When $n = l$, $\mathfrak{n}$ is abelian and it is known that $\mathfrak{n}$ is prehomogeneous under $\mathrm{Ad}(M)$(cf [15, 21]). If $n = 1$, $\mathfrak{n}$ is prehomogeneous by Theorem 4.2 in [23]. □

Theorem 4.6. *Suppose G is type B_l and $\mathcal{V}$ is a subspace of $\mathfrak{n}$ containing $\mathfrak{n}_2$. If n is even, then $\mathcal{V}$ is prehomogeneous if and only if for any $L(X,Y) \in \mathcal{V}$, $XX' = 0$. If n is odd, then $\mathcal{V}$ is prehomogeneous if and only if for any $L(X,Y) \in \mathcal{V}$, rank$(XX') \leq 1$. In particular, $\mathfrak{n}$ is prehomogeneous if and only if $n = 1$ or l.*

Proof. Let $\hat{M}_n^{\varepsilon}(F) = \{A | A \in \bar{M}_n^{\varepsilon}(F), \mathrm{rank}(A) = n'\}$. Where n' is defined in Theorem 4.3. Then $\hat{M}_n^{\varepsilon}(F)$ is a dense open subset of $\bar{M}_n^{\varepsilon}(F)$. Consequently, if we let $\mathcal{V}' = \{L(X,Y) | L(X,Y) \in \mathcal{V}, Y \in \hat{M}_n^{\varepsilon}(F)\}$, then $\mathcal{V}'$ is a dense open subset of $\mathcal{V}$. Moreover, by Lemma 3.7 in [23], for any $Y \in \hat{M}_n^{\varepsilon}(F)$, there is $g \in G'$, such that $gY\varepsilon(g) = B_n$.

Suppose first n is even, then $n' = n$. We may define F, G to be the same functions on $\mathcal{V}'$ as in last theorem, then both F and G are nonconstant polynomial functions in terms of the entries of $L(X,Y)$. One may apply the same technique to obtain a proof for this case.

Suppose n is odd, we will prove that if there is an $L(X,Y) \in \mathcal{V}$ such that $\mathrm{rank}(XX') \geq 2$, then $\mathcal{V}$ is not prehomogeneous. First we need the following:

Lemma 4.7. *Suppose G is as above and n is odd, $X \in M_{n\times(2m+1)}(F)$ with $r = $ rank$(XX') \geq 1$. Then there is a dense open subset $\mathcal{O}_2$ of $\bar{M}_n^{\varepsilon}(F)$ such that for every $Y \in \mathcal{O}_2$, det$(Y + (tX)(tX'))$ is a polynomial of t^2 with degree k and the coefficient of t^2 is nonzero, where $k = $ min(n, m). In particular, for any fixed $0 \neq t_0 \in F$, the set of Y satisfying det$(Y + (t_0X)(t_0X')) \neq 0$ is a dense open subset of $\bar{M}_n^{\varepsilon}(F)$.*

Proof. Define $M_n^s(F) = \{A \in M_n(F) | A = {}^tA\}$ and $M_n^{ss}(F) = \{A \in M_n(F) | A = -{}^tA\}$ to be the set of symmetric and skew-symmetric matrices in $M_n(F)$, respectively. It is well known that $M_n(F) = M_n^s(F) \bigoplus M_n^{ss}(F)$. Then

$$\begin{aligned} \det(Y + (tX)(tX')) &= \det(Y - t^2 X w_{2m+1} {}^tX w_n) \\ &= \det(w_n)\det(Y w_n - t^2 X w_{2m+1} {}^tX). \end{aligned}$$

Here one must have $Yw_n \in M_n^{ss}(F)$, and $Xw_{2m+1}{}^tX \in M_n^s(F)$.

Since $\mathrm{rank}(XX') = r$ by assumption, $\mathrm{rank}(Xw_{2m+1}{}^tX) = r$. By Gaussian Elimination in linear algebra, there is $g_1 \in G'$ such that

$$g_1(Xw_{2m+1}{}^tX)^t g_1 = \begin{pmatrix} A_r & 0 \\ 0 & 0 \end{pmatrix}.$$

It's obvious that $A_r \in M_r^s(F) \cap \mathrm{GL}_r(F)$, since the left side above is symmetric.

Set $Y_1 = g_1 Y w_n{}^t g_1 \in M_n^{ss}(F)$. By Lemma 3.7 in [23], we can choose a proper open dense subset $\mathcal{O}_2$ of $\bar{M}_n^{\varepsilon}(F)$ such that for any $Y \in \mathcal{O}_2$, there is $g_2 \in G'$, satisfying $g_2 Y_1({}^tg_2) = B_n w_n$. Moreover, g_2 acting on Y_1 from left (${}^tg_2$ acting on Y_1 from right) resulting in eliminating the rows (columns by ${}^tg_2$) of Y_1 from bottom to top. Therefore,

$$g_2(g_1(Xw_{2m+1}{}^tX)^t g_1)^t g_2 = \begin{pmatrix} A'_r & 0 \\ 0 & 0 \end{pmatrix}$$

for a suitable $A'_r \in M_r^s(F) \cap \mathrm{GL}_r(F)$. Let $g = g_2 g_1$, then

$$g(Yw_n - t^2 Xw_{2m+1}{}^tX)^t g = \begin{pmatrix} B_p w_p + t^2 A'_p & & & & & & \\ & 0 & 1 & & & & \\ & -1 & 0 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}.$$

Where $p = r$ if r is odd; $p = r+1$ if r is even, and

$$A'_p = \begin{pmatrix} A'_r & 0_{(p-r)\times 1} \\ 0_{1\times(p-r)} & 0 \end{pmatrix}.$$

Suppose $A'_p = (a'_{i,j})_{p\times p}$, from the construction of g_1 and g_2, it can be seen that their entries are rational functions of those of Yw_n and $XX'w_n$, so are the entries of A'_p. Therefore, by choosing $\mathcal{O}_2$ properly, we can always assume $a'_{1,1} \neq 0$ up to a closed subset of $\mathfrak{n}_2$. By a simple calculation, $a'_{1,1}$ is the coefficient of t^2 in $\det(g(Yw_n - t^2 Xw_{2m+1}{}^tX)^t g)$.

The rest of the lemma follows trivially. □

Continue the proof of Theorem 4.6. Suppose $\mathcal{V}$ is prehomogeneous, then so is $\mathcal{V}'$ (both under $\mathrm{Ad}(M_{\mathcal{V}})$). Let $L(X_i, Y_i)$ be the generators of these orbits with each $Y_i \in \hat{M}_n^{\varepsilon}(F)$. If $L(X_\circ, Y_\circ) \in \mathcal{V}$ such that $r =$

$\operatorname{rank}(X_\circ X'_\circ) \geq 2$. Then $\operatorname{rank}(X_\circ) \geq 2$. Moreover, $L(X_\circ, Y) \in \mathcal{V}$ for any $Y \in \bar{M}_n^\varepsilon(F)$, since $\mathfrak{n}_2 \subset \mathcal{V}$ and $\mathcal{V}$ is a vector space.

Since $\mathcal{V}'$ is a dense open subset of $\mathcal{V}$, there is $Y \in \hat{M}_n^\varepsilon(F)$ such that for infinitely many $t \in F$, $L(tX_\circ, Y) \in \mathcal{V}'$. By Lemma 4.7, we can choose such $Y_\circ$ so that $\det(Y_\circ + (tX_\circ)(tX'_\circ))$ is a polynomial of degree r in variable t^2. Since there are only finitely many number of orbits, there must be one orbit in $\mathcal{V}'$, say, represented by $L(X_i, Y_i)$, that contains $L(tX_\circ, Y)$ for infinitely many $t \in F$. Therefore, for every such t, there must be $m_t = \operatorname{diag}(g_t, h_t, \varepsilon(g_t)^{-1}) \in M_{\mathcal{V}}$, so that $\operatorname{Ad}(m_t) \circ L(tX_\circ, Y_\circ) = L(X_i, Y_i)$. Hence, $g_t(Y_\circ + (tX_\circ)(tX_\circ)')\varepsilon(g_t) = Y_i + X_i X'_i$. But since both M_n^ε and $\bar{M}_n^\varepsilon$ are invariant under the action of $\varepsilon(G')$, we have:

$$g_t Y_\circ \varepsilon(g_t) = Y_i, \qquad g_t(tX_\circ)(tX_\circ)'\varepsilon(g_t) = X_i X'_i.$$

Therefore, for a variable z,

$$g_t(Y_\circ + (tzX_\circ)(tzX_\circ)')\varepsilon(g_t) = Y_i + (zX_i)(zX'_i). \tag{4.4}$$

Suppose

$$\det(Y_\circ + (zX_\circ)(zX_\circ)') = a_r z^{2r} + a_{r-1} z^{2(r-1)} + \cdots + a_1 z^2,$$

(Since $\det(Y) = 0$, there is no constant term), and

$$\det(Y_i + (zX_i)(zX_i)') = b_r z^{2r} + b_{r-1} z^{2(r-1)} + \cdots + b_1 z^2.$$

By Lemma 4.7, we can further assume that $a_r, a_1 \neq 0$ by choosing $\mathcal{V}'$ properly. Since

$$\det(g_t(Y_\circ + (tzX_\circ)(tzX_\circ)')\varepsilon(g_t)) = \\ \det(g_t)^2\{a_r(tz)^{2r} + a_{r-1}(tz)^{2(r-1)} + \cdots + a_1(tz)^2\}.$$

By equation (4.4), we must have:

$$b_r = \det(g_t)^2(a_r t^{2r}), \qquad b_1 = \det(g_t)^2(a_1 t^2).$$

Thus

$$\frac{b_1}{b_r} = \frac{a_1 t^2}{a_r t^{2r}}$$

holds for infinitely many t, which is obviously impossible!

To complete the rest of this theorem, we have yet to prove that if $\mathcal{V}$ has the property that for any $L(X, Y) \in \mathcal{V}$, $\operatorname{rank}(XX') = 1$, then $\mathcal{V}$ is prehomogeneous. We start with the following lemma:

Lemma 4.8. *Suppose G, n and $\mathcal{V}$ are as in Lemma 4.7 such that for any $L(X_0, Y_0) \in \mathcal{V}$, $rank(X_0X_0') \leq 1$. Then for any $L(X, Y) \in \mathcal{V}'$, there is $m = diag(g, h, \varepsilon(g)^{-1}) \in M$, such that $Ad(m) \circ L(X, Y)$ has the form $L(E_r^i, gY\varepsilon(g))$, where*

$$E_r^i = \begin{cases} \begin{pmatrix} I_r & 0 & H_i \\ 0 & 0 & 0 \end{pmatrix} \in M_{n\times\kappa}(F), & \text{if} \quad rank(XX') = 1; \\ \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \in M_{n\times\kappa}(F), & \text{if} \quad rank(XX') = 0. \end{cases}$$

Here $r = rrank(X)$, and

$$H_i = \begin{pmatrix} 0 & \cdots & 0 & -\varepsilon_i/2 \\ 0 & \cdots & 0 & 0 \\ . & \cdots & . & . \\ 0 & \cdots & 0 & 0 \end{pmatrix} \in M_r(F),$$

with $\varepsilon_i \in F^*/(F^*)^2$. In particular, $\mathrm{rrank}(X) = \mathrm{rank}(X)$.

Proof. We first prove that all XX' are linearly dependent. Since $\mathcal{V}$ contains $\mathfrak{n}_2$ as a vector space, $L(X, 0) \in \mathcal{V}$ for any $L(X, Y) \in \mathcal{V}$. If X_1X_1' and X_2X_2' are linearly independent for some $L(X_1, Y_1), L(X_2, Y_2) \in \mathcal{V}$, then $L(X_1, 0), L(X_2, 0) \in \mathcal{V}$ for the reason mentioned above. Therefore, $L(aX_1 + bX_2, 0) \in \mathcal{V}$ for any $a, b \in F$. But then $\mathrm{rank}\{(aX_1 + bX_2)(aX_1 + bX_2)'\} \geq 2$ for some $a, b \in F$, which is a contradiction.

If $XX' = 0$, the it will fall into Theorem 4.3, so we may assume $\mathrm{rank}(XX') = 1$. Since $XX'w_n \in M_n^s(F)$, there is $g_1 \in G'$, such that

$$g_1(XX'w_n)^t g_1 = \begin{pmatrix} \varepsilon_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ . & . & \cdots & . \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Therefore,

$$g_1(XX')\varepsilon(g_1) = \begin{pmatrix} 0 & \cdots & 0 & \varepsilon_i \\ 0 & \cdots & 0 & 0 \\ . & \cdots & . & . \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Let $X_1 = g_1X$, and $\mathcal{W}$ be the first row of X_1. By Lemma 3.3 in [22] or Lemma 4.2 in [19], there is $h_1 \in G_m$, such that $\mathcal{W}h_1^{-1} = (c_0, 0, \cdots, 0, c_1) \in$

F^{2m+1}. By the fact that $X_1X_1' = (X_1h_1^{-1})(X_1h_1^{-1})'$, we have: $2c_0c_1 = -\varepsilon_i \neq 0$. Let $h_2 = \mathrm{diag}(c_0, 1, \cdots, 1, c_0^{-1}) \in G_m$ and $h = h_1h_2$, then

$$\mathcal{W}h^{-1} = \left(1, 0, \cdots, 0, -\frac{\varepsilon_i}{2}\right) \in F^{2m+1}.$$

Let $m_1 = \mathrm{diag}(g_1, h, \varepsilon(g_1)^{-1}) \in M$, $\bar{X} = g_1Xh^{-1}$ and $\bar{Y} = g_1Y\varepsilon(g_1)$. Then $\mathrm{Ad}(m_1) \circ L(X,Y) = L(\bar{X}, \bar{Y})$. By Lemma 4.1, we may assume the first r columns of $\bar{X}$ are linearly independent. By classical Gaussian Elimination, we can find $g_2 \in G'$, such that

$$g_2\bar{X} = \begin{pmatrix} I_r & B & A \\ 0 & B_1 & A_1 \end{pmatrix}.$$

Where $r = \mathrm{rrank}(\bar{X}) = \mathrm{rrank}(X), B \in M_{r\times(\kappa-2r)}(F), A \in M_{r\times r}(F), B_1 \in M_{(n-r)\times(\kappa-2r)}(F)$, $A_1 \in M_{(n-r)\times r}(F)$ and the first row of $g\bar{X}$ equals to that of $\bar{X}$, which is $\mathcal{W}h^{-1}$.

Let

$$X_2 = \begin{pmatrix} I_k & B & A - H_i \\ 0 & B_1 & A_1 \end{pmatrix}.$$

Then it can be verified that $X_2X_2' = 0$. By a similar proof as Lemma 4.2, there is $h_3 \in G_m$ such that $X_2h_3^{-1} = E_r$. Moreover, by the construction of h_3 there, it can be checked that

$$\begin{pmatrix} 0 & 0 & H_i \\ 0 & 0 & 0 \end{pmatrix} h_3^{-1} = \begin{pmatrix} 0 & 0 & H_i \\ 0 & 0 & 0 \end{pmatrix}.$$

Now let $g' = g_2g_1, h' = h_3h, m_2 = \mathrm{diag}(g', h', \varepsilon(g')^{-1}) \in M$. Then $\mathrm{Ad}(m_2)\circ L(X,Y) = L(E_r^i, g'Y\varepsilon(g'))$. By the proof in Theorem 4.3, there exists $g_3 \in G'$ so that $\varepsilon(g_3) \circ [g'Y\varepsilon(g')] = B_n$ and $\varepsilon(g_3)$ fixes $E_r^i(E_r^i)'$. Thus by Lemma 3.1 in [4], there is $h \in G_m$ such that $g_3E_r^ih^{-1} = E_r^i$. Let $m_3 = \mathrm{diag}(g_3, h, \varepsilon(g_3)^{-1})$, then $\mathrm{Ad}(m_3) \circ L(E_r^i, g'Y\varepsilon(g')) = L(E_r^i, B_n)$. Setting $m = m_3m_2m_1$ will finish the proof of the lemma. □

With these preparation, we may go back to finish the proof of Theorem 4.6 for the case n is odd. Let $\mathcal{V}_i = \{L(X,Y) \in \mathcal{V}|\ \mathrm{rrank}(X) = i\}$, $\mathcal{V}_i' = \mathcal{V}_i \cap \mathcal{V}'(i = 1, \cdots, q)$. Let r be the maximal integer so that $\mathcal{V}_r$ is not empty, then $\mathcal{V}_r'$ is a dense open subset of $\mathcal{V}$. For any $L(X_0, Y_0) \in \mathcal{V}_r'$, there is an $m \in M$ so that $\mathrm{Ad}(m)\circ L(X_0, Y_0) = L(E_r^i, B_n)$ by Lemma 4.8. We may assume that both $L(X_0, Y_0)$ and $L(E_r^i, B_n)$ are contained in $\mathcal{V}_r'$ by a certain adjoint action of M on $\mathcal{V}_r'$. Thus, m may be assumed to be inside $M_{\mathcal{V}}$. Since there are finite such $L(E_r^i, B_n)$, $\mathcal{V}$ is prehomogeneous. □

Theorem 4.9. *Suppose G is symplectic or G is type B_l with n even. Then every maximal prehomogeneous subgroup of N which contains N_2 is abelian, these groups are conjugate to each other under Int(M).*

Proof. Let $X_1 = \{(X, 0) \in M_{n\times(2m)}(F) | X \in M_{n\times m}(F)\}$,

$$I = \left\{ n(X,Y) | X \in X_1,\ Y \in \left\{ \begin{array}{ll} \bar{M}_n^{\varepsilon}(F), & \text{if } G = \mathrm{SO}_{2l+1}(F) \\ M_n^{\varepsilon}(F), & \text{if } G = \mathrm{S}p_{2l}(F) \end{array} \right\} \right\}.$$

Then I is obviously abelian, it is a maximal prehomogeneous subgroup of N by Theorem 4.3, 4.5 and 4.6. Suppose I' is another maximal prehomogeneous subgroup of N containing N_2, we'll show that I' is conjugate to I under Int (M).

Let $\Lambda = \mathrm{Lie}(I')$, the Lie subalgebra of I'. Choose $L(X_\circ, Y_\circ) \in \Lambda$ such that $\mathrm{rank}(X_\circ) = \max\{\mathrm{rank}(X) | L(X,Y) \in \Lambda\} = i$ ($i \le k$). Since Λ contains $\mathfrak{n}_2$ and is prehomogeneous, by Theorem 4.5 and 4.6, $\Lambda \subset \mathcal{E}$. Then by Lemma 4.2, there is $m \in M$, such that

$$\mathrm{Ad}(m) \circ L(X_\circ, Y_\circ) = L(E_i, Y_0')$$

for a suitable $Y_0' \in M_n^{\varepsilon}(F)$ or $\bar{M}_n^{\varepsilon}(F)$.

Let $\Lambda' = \mathrm{Ad}(m) \circ \Lambda$. We first prove that for any element $L(X,Y) \in \Lambda'$, X has the form

$$X = \begin{pmatrix} * & * & * & * \\ * & 0 & 0 & 0 \end{pmatrix}.$$

Suppose otherwise there is an element $L(X,Y) \in \Lambda'$, with

$$X = \begin{pmatrix} * & * & * & * \\ * & \diamond & \diamond & \diamond \end{pmatrix}.$$

Where the diamond blocks are not all zero. Since Λ' is a vector space, there is $L(X', Y') \in \Lambda'$, with

$$X' = \begin{pmatrix} xI_i + * & * & * & * \\ * & \diamond & \diamond & \diamond \end{pmatrix}$$

for any $x \in F$. But it's obvious that $\mathrm{rank}(X') > i$, which is a contradiction to the choice of $L(X_\circ, Y_\circ)$.

For the same reason, all elements in Λ' must be in the form as:

$$X = \begin{pmatrix} * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix},$$

or

$$X = \begin{pmatrix} * & * & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the first case, $\Lambda' \subset \mathrm{Lie}(I)$ and since Λ' is a maximal prehomogeneous space, one must have $\Lambda' = \mathrm{Lie}(I)$. Therefore $I = \mathrm{Int}(m) \circ I'$.

In the second case, first of all, any element $L(X, Y)$ in Λ' must have its first component X satisfy the following form:

$$X = \begin{pmatrix} * & * & * & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since otherwise there will be an element $L(\bar{X}, \bar{Y}) \in \Lambda'$, with

$$\bar{X} = \begin{pmatrix} xI_i + * & * & * & \diamond \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Where the block marked by diamond is nonzero. Then it contradicts to $\bar{X}\bar{X}' = 0$.

Secondly, by the same reason as above, if an element $L(X, Y) \in \Lambda'$ with some nonzero columns in the blocks marked by diamonds below:

$$X = \begin{pmatrix} * & \diamond & \diamond & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the conjugate columns of the above ones in any $\bar{X}$ must be zero, where $L(\bar{X}, \bar{Y}) \in \Lambda'$ is arbitrary. Therefore, all columns in any X can be nonzero only within conjugate irrelevant ones. By a same proof as in Lemma 4.1, we can choose $s \in G_m$ such that for every $L(X, Y) \in \Lambda'$, $\mathrm{Ad}(s) \circ L(X, Y) = L(\tilde{X}, \tilde{Y})$, with $\tilde{X}$ carrying the following form:

$$\tilde{X} = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

i,e. $\mathrm{Ad}(s) \circ \Lambda' \subset \mathrm{Lie}(I)$. Thus, by the maximality of I', $\mathrm{Ad}(s) \circ \Lambda' = \mathrm{Lie}(I)$. □

5. *Maximal Prehomegeneous Spaces on* D_l

Lemma 5.1. *Suppose G is type D_l, Ω and Ω_r are defined similarly as in section 4. Then for any $n(X, Y) \in \Omega_r$, there is an s which is a representative of the Weyl group element in G_m, such that if we set $W = Xs$, then rank$\{W_1, W_2, \cdots, W_r\} = r$ or $r - 1$.*

Proof. For any $n(X, Y) \in \Omega_r$, suppose $\{X_{i_1}, X_{i_2}, \cdots, X_{i_r}\}$ are conjugate irrelevant columns of X such that $\mathrm{rank}(X_{i_1}, X_{i_2}, \cdots, X_{i_r}) = r$. Without loss of generality, we amy assume further that $i_1 < i_2 < \cdots < i_{p-1} < m \leq i_p < \cdots < i_r$. Let $a = \left[\frac{r-p+1}{2}\right]$ and

$$s_1 = \prod_{k=1}^{a} s_{e_{i_{n+p+2k}}+e_{i_{n+p+2k+1}}} \in M.$$

Notice that multiplying X by $s_{e_{i_{n+p+2k}}+e_{i_{n+p+2k+1}}}$ from right, it results in interchanging the $i_{p+2k}-$th column of X with the conjugate column of $i_{p+2k+1}-$th and the $i_{p+2k+1}-$th column with the conjugate one of $i_{p+2k}-$th, leaving all other columns unchanged. Therefore, if we set $W' = Xs_1$, then by the fact that all these columns $\{X_{i_1}, X_{i_2}, \cdots, X_{i_r}\}$ are conjugate irrelevant, there are r linearly independent columns in the first m columns of W' if $r-p+1$ is even, and $r-1$ linearly independent ones if $r-p+1$ is odd.

We can then adopt the same method in Lemma 4.1 to choose a representative of Weyl group element s_2, such that by multiplying it to W' from right, it will result in grouping $\{X_{i_1}, X_{i_2}, \cdots, X_{i_r}\}$ to the first r columns if $r-p+1$ is even; or grouping $\{X_{i_1}, X_{i_2}, \cdots, X_{i_{r-1}}\}$ to the first $r-1$ columns, leaving X_{i_r} to be the $(m+1)-$th column, if $r-p+1$ is odd. Setting $s = s_2 s_1$ will finish the proof of the lemma. □

Lemma 5.2. *Suppose G is the same as above. Then for any $n(X,Y) \in \Omega_r$, there is $m = diag(g, h, \varepsilon(g)^{-1}) \in M$, such that $Int(m) \circ n = n(E_r, gY\varepsilon(g))$ or $n(E'_r, gY\varepsilon(g))$. Where*

$$E'_r = \begin{pmatrix} I_{r-1} & 0 & P_r \\ 0 & 0 & 0 \end{pmatrix} \in M_{n\times\kappa}(F),$$

with

$$P_r = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \cdot & \cdots & \cdot & \cdot \\ 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \end{pmatrix} \in M_r(F).$$

Proof. This is an analogue of Lemma 4.2, the proof is similar with a slight modification. □

Theorem 5.3. *Suppose G is the same as above. Then for each $r \leq q$, $\mathcal{E}_r$ is a prehomogeneous algebraic set under $Ad(M)$. In particular, $\mathcal{E}$ is a prehomogeneous algebraic set and there is only one open orbit.*

Proof. This is an analogue of Theorem 4.3 and the proof is almost the same as the part for $\mathrm{SO}_{2l+1}(F)$ case there. □

Theorem 5.4. *Let G be as above and $\mathcal{E}$ be defined as in section 4. Suppose $\mathcal{V}$ is a subspace of $\mathfrak{n}$ containing $\mathfrak{n}_2$ as a proper subspace. If n is even, then $\mathcal{V}$ is prehomogeneous if and only if for any $L(X,Y) \in \mathcal{V}$, $XX' = 0$. If n is odd, then $\mathcal{V}$ is prehomogeneous if and only if for any $L(X,Y) \in \mathcal{V}$, rank$(XX') \leq 1$. In particular, $\mathfrak{n}$ is prehomegeneous under Ad(M) if and only if it is abelian.*

Proof. This is a counterpart of Theorem 4.6, the proof is exactly the same. By this Theorem, $\mathfrak{n}$ is prehomogeneous only if $n = 1$ or $m = 0$. In both cases, $\mathfrak{n}$ is abelian. □

Theorem 5.5. *Suppose G is the same as above and n is even. Then every maximal prehomogeneous subgroup of N cantaining N_2 is abelian, these groups are conjugate to each other under Int(M).*

Proof. Similar to Theorem 4.9. □

Remark: The results in this paper are not coincident. In general, by using the same methods in this article, it's not hard to show that if $\mathcal{O}$ is a subspace of $\mathfrak{n}$, then $\mathcal{O}$ is prehomogeneous under its stabilizer if and only if for any $L(X,Y) \in \mathcal{O}$,

$$\mathrm{rank}(XX' + Y) = \mathrm{rank}(XX') + \mathrm{rank}(Y) \leq n.$$

This should give us a complete determination of all prehomogeneous subspaces in $\mathfrak{n}$, as mentioned at the beginning.

List of Main mathematical Symbols

F : An infinite field.
G : $F-$rational points of a split reductive connected classical orthogonal or symplectic group of split rank l defined over F.
G_m : A group with same type as G of split rank m.
P : A maximal parabolic subgroup of G.
M : The Levi subgroup of P.
N : The unipotent radical of P.
$\mathfrak{n}$: Lie algebra of N.
$\mathfrak{n}_1$: One step nilpotent subspace of $\mathfrak{n}$.
$\mathfrak{n}_2$: Two step nilpotent subspace of $\mathfrak{n}$.
X' : A linear transformation defined in Lemma 3.3.
$n(X,Y)$: An element of N with X,Y satisfying Lemma 3.3.
$L(X,Y')$: An element of $\mathfrak{n}$ with components X,Y' as in equation (3.1).
$M_n^\varepsilon(F)$: The space of $\varepsilon-$symmetric matrices in $M_n(F)$.
$\bar{M}_n^\varepsilon(F)$: The space of $\varepsilon-$skew-symmetric matrices in $M_n(F)$.
$\tilde{M}_n^\varepsilon(F)$: Subset of $M_n^\varepsilon(F)$ whose elements have rank n.
$\hat{M}_n^\varepsilon(F)$: Subset of $\bar{M}_n^\varepsilon(F)$ whose elements have rank $[\frac{n}{2}]$.
$\varepsilon(Y) := w_n\, {}^tY w_n^{-1}$.
$\varepsilon(g)\circ A := gA\varepsilon(g)$.
$\bar{\varepsilon}(g)\circ A := gA({}^tg)$.
$\mathcal{E} := \{L(X,Y')\in\mathfrak{n}|\ XX'=0\}$.
$\mathcal{E}_r := \{L(X,Y')|\ L(X,Y')\in\mathcal{E}, \mathrm{rrank}(X)=r\},\ \ r=1,2,\cdots,q$.
$\Omega := \{\exp(L(X,Y'))|\ L(X,Y')\in\mathcal{E}\}$.
$\Omega_r := \{\exp(L(X,Y'))|\ L(X,Y')\in\mathcal{E}_r\}$.
$\Omega_r' := \{n(X,A)|\ n(X,A)\in\Omega_r,\ A \text{ has maximal rank in } \mathfrak{n}_2\}$.
$\mathcal{V}$: A subspace of $\mathfrak{n}$.
$M_{\mathcal{V}}$: The stabilizer of $\mathcal{V}$ in M.
$\mathcal{V}_r := \mathcal{V}\cap\mathcal{E}_r$.
$\mathcal{V}'$: A certain dense open subset of $\mathcal{V}$.
$\mathcal{V}_r' := \mathcal{V}'\cap\mathcal{E}_r$.
$\mathcal{O}$: A subspace of $\mathfrak{n}$ containing $\mathcal{V}$.
$\mathcal{O}'$: A certain dense open subset of $\mathcal{O}$.
$M_n^s(F)$: The space of symmetric matrices in $M_n(F)$.
$M_n^{ss}(F)$: The space of skew-symmetric matrices in $M_n(F)$.

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