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# EMBEDDING MEASURE SPACES

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ABSTRACT. For a given measure space $(X,\mathscr{B},\mu)$ we construct all measure spaces $(Y,\mathscr{C},\lambda)$ in which $(X,\mathscr{B},\mu)$ is embeddable. The construction is modeled on the ultrafilter construction of the Stone–Čech compactification of a completely regular topological space. Under certain conditions the construction simplifies. Examples are given when this simplification occurs.

**Keywords:** Ultrafilter, thick subset, set of full outer measure, topological measure space, Baire measure, Stone–Čech compactification, realcompactification.
**MSC(2010):** Primary: 28A05; Secondary: 54D35, 54D60, 28C15.

## 1. Introduction

A measurable space $(X,\mathscr{B})$ is said to be *embedded* in a measurable space $(Y,\mathscr{C})$ (denoted by $(X,\mathscr{B})\subseteq(Y,\mathscr{C})$) if $X\subseteq Y$ and

$$\mathscr{B}=\{C\cap X : C\in\mathscr{C}\}.$$

A measure space $(X,\mathscr{B},\mu)$ is said to be *embedded* in a measure space $(Y,\mathscr{C},\lambda)$ (denoted by $(X,\mathscr{B},\mu)\subseteq(Y,\mathscr{C},\lambda)$) if $(X,\mathscr{B})\subseteq(Y,\mathscr{C})$ and $\mu(C\cap X)=\lambda(C)$ for each $C\in\mathscr{C}$. In this note, for a given measure space $(X,\mathscr{B},\mu)$, we construct all measure spaces $(Y,\mathscr{C},\lambda)$ in which $(X,\mathscr{B},\mu)$ is embedded. Equivalently, for a given measure space $(X,\mathscr{B},\mu)$, we construct all measure spaces $(Y,\mathscr{C},\lambda)$ which contains $(X,\mathscr{B},\mu)$ as a thick subspace. (Recall that a subset $X$ of a measure space $(Y,\mathscr{C},\lambda)$ is said to be *thick* (or *of full outer measure*) if $\lambda(C)=0$ for each $C\in\mathscr{C}$ such that $C\subseteq Y\backslash X$, equivalently, if $\lambda_*(Y\backslash X)=0$, where $\lambda_*$ denotes

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the inner measure induced by $\lambda$. If $X$ is a thick subset of $(Y, \mathscr{C}, \lambda)$ and if

$$\mathscr{B} = \{C \cap X : C \in \mathscr{C}\}$$

and $\mu(C \cap X) = \lambda(C)$ for each $C \in \mathscr{C}$, then $(X, \mathscr{B}, \mu)$ is a measure space which is embedded in $(Y, \mathscr{C}, \lambda)$. Conversely, if $(X, \mathscr{B}, \mu)$ is embedded in $(Y, \mathscr{C}, \lambda)$, then $X$ is a thick subset of $(Y, \mathscr{C}, \lambda)$; see e.g. Theorem 17.A of [4].) Our construction here is analogous to the ultrafilter construction of the Stone–Čech compactification of a completely regular topological space $X$. (Completely regular topological spaces are always assumed to be Hausdorff.)

We recall some basic facts, definitions and notation. For details we refer the reader to [1], [3] and [4]. Let $(X, \mathscr{B})$ be a measurable space. A non-empty $\mathscr{A} \subseteq \mathscr{B}$ is called a *filter-base* in $\mathscr{B}$ if for every $A, B \in \mathscr{A}$ there exists a non-empty $C \in \mathscr{A}$ such that $C \subseteq A \cap B$. A *filter* $\mathscr{F}$ in $\mathscr{B}$ is a filter-base such that $B \in \mathscr{F}$ whenever $B \in \mathscr{B}$ and $F \subseteq B$ for some $F \in \mathscr{F}$. An *ultrafilter* in $\mathscr{B}$ is a maximal (with respect to $\subseteq$) filter. An ultrafilter is called *free* if it has empty intersection, otherwise, it is called *fixed*. An ultrafilter is said to have *the countable intersection property* (*c.i.p.*, in short) if every countable number of its elements has a non-empty intersection. It is known that every filter-base in $\mathscr{B}$ is contained in some ultrafilter in $\mathscr{B}$, and that a filter-base $\mathscr{A}$ in $\mathscr{B}$ is an ultrafilter if and only if for each $B \in \mathscr{B}$ if $B$ meets every element of $\mathscr{A}$ then $B \in \mathscr{A}$. Note that an ultrafilter $\mathscr{F}$ in $\mathscr{B}$ has c.i.p. if and only if it is $\sigma$-*complete*, i.e., it is closed under countable intersections.

Let $X$ be a topological space. By a *zero-set* in $X$ we mean a set of the form $f^{-1}(0)$ where $f : X \to [0, 1]$ is continuous; the complement of a zero-set is called a *cozero-set*; denote $\mathrm{Z}(f) = f^{-1}(0)$ and $\mathrm{Coz}(f) = X \backslash \mathrm{Z}(f)$. Let $\mathscr{Z}(X)$ and $\mathrm{Coz}(X)$ denote the set of all zero-sets and the set of all cozero-sets of $X$, respectively. Let $X$ be a completely regular topological space. A *compactification* of $X$ is a compact Hausdorff topological space which contains $X$ as a dense subspace. We denote by $\beta X$ the *Stone–Čech compactification* of $X$, which always exists, and is characterized by either of the following properties:

- Every continuous function from $X$ to a compact space is continuously extendible over $\beta X$.
- Every continuous function from $X$ to $[0, 1]$ is continuously extendible over $\beta X$.

- For every $Z, S \in \mathscr{Z}(X)$ such that $Z \cap S = \emptyset$ we have
$$\mathrm{cl}_{\beta X} Z \cap \mathrm{cl}_{\beta X} S = \emptyset.$$
- For every $Z, S \in \mathscr{Z}(X)$ we have
$$\mathrm{cl}_{\beta X}(Z \cap S) = \mathrm{cl}_{\beta X} Z \cap \mathrm{cl}_{\beta X} S.$$

Note, in particular, this implies that disjoint zero-sets (and thus disjoint closed-open subsets) in $X$ have disjoint closures in $\beta X$. For a completely regular topological space $X$ the *Hewitt realcompactification* $\upsilon X$ of $X$ is the subspace of $\beta X$ defined by
$$\upsilon X = \bigcap \big\{ C : C \in \mathrm{Coz}(\beta X) \text{ and } X \subseteq C \big\}.$$
A topological space is said to be *realcompact* if it is homeomorphic to a closed subspace of some topological product of the real line. Every regular Lindelöf topological space is realcompact. It is known that a completely regular topological space $X$ is realcompact if and only if $X = \upsilon X$ if and only if for every $p \in \beta X \backslash X$ there exists a zero-set $Z$ in $\beta X$ such that $p \in Z$ and $Z \cap X = \emptyset$.

A *topological measurable space* is a triple $(X, \mathscr{O}, \mathscr{B})$ where $(X, \mathscr{B})$ is a measurable space and $(X, \mathscr{O})$ is a topological space such that $\mathscr{O} \subseteq \mathscr{B}$, i.e., every open set (and thus every Borel set) is measurable.

This note is organized as follows. In Section 2 we construct all measure spaces $(Y, \mathscr{C}, \lambda)$ in which a given measure space $(X, \mathscr{B}, \mu)$ is embedded. In Section 3 we simplify the construction under certain additional conditions on $(X, \mathscr{B}, \mu)$. Indeed, we prove that if the points of $X$ are separated by measurable sets in $\mathscr{B}$ and there is no free ultrafilter in $\mathscr{B}$ with c.i.p., then $(X, \mathscr{B}, \mu)$ is embeddable in $(Y, \mathscr{C}, \lambda)$ if and only if $(Y, \mathscr{C}, \lambda)$ is obtained from $(X, \mathscr{B}, \mu)$ by "blowing" certain points of $X$ up and "pasting" a certain measurable space to $X$ in a certain way. In Section 4 we provide examples satisfying the assumption of the theorem in Section 3, i.e., we find examples of measure spaces $(X, \mathscr{B}, \mu)$ with no free ultrafilter in $\mathscr{B}$ having c.i.p. It turns out that the class of such measure spaces $(X, \mathscr{B}, \mu)$ is reasonably large (e.g., it contains the class of all first-countable realcompact topological measure spaces, thus in particular, containing all $n$-dimensional Lebesgue measure spaces) and behaves very nicely in connection with the standard operations on measure spaces (e.g., we show that for any $\sigma$-finite measure spaces $(X, \mathscr{B}, \mu)$ and $(Y, \mathscr{C}, \lambda)$ such that in each of which singletons are measurable, considering the measure space $(X \times Y, \mathscr{B} \times \mathscr{C}, \mu \times \lambda)$, there is no free ultrafilter in $\mathscr{B} \times \mathscr{C}$ with c.i.p. if and only if there is a free ultrafilter with c.i.p. neither in $\mathscr{B}$ nor in

$\mathscr{C}$.) Finally, in Section 5 we give examples of measure spaces $(X, \mathscr{B}, \mu)$ having arbitrarily large number of free ultrafilter in $\mathscr{B}$ with c.i.p. We leave some problems open which are formally stated.

## 2. The construction of measure spaces in which a given measure space $(X, \mathscr{B}, \mu)$ is embeddable

The following lemma is well known.

**Lemma 2.1.** *Let $(X, \mathscr{B})$ be a measurable space. Let $\mathscr{U}$ be an ultrafilter in $\mathscr{B}$.*

(1) *For any $B \in \mathscr{B}$ either $B \in \mathscr{U}$ or $X \backslash B \in \mathscr{U}$.*
(2) *Suppose that $\mathscr{U}$ has c.i.p. and $B_1, B_2, \ldots \in \mathscr{B}$. Then*

$$\bigcup_{n=1}^{\infty} B_n \in \mathscr{U} \tag{2.1}$$

*if and only if $B_n \in \mathscr{U}$ for some $n \in \mathbb{N}$.*

*Proof.* To show (1), note that if $B \notin \mathscr{U}$ for some $B \in \mathscr{B}$, then, since $\mathscr{U}$ is an ultrafilter, we have $B \cap U = \emptyset$ for some $U \in \mathscr{U}$. Thus $U \subseteq X \backslash B$, which implies that $X \backslash B \in \mathscr{U}$.

To show (2), note that (2.1) holds trivially if $B_n \in \mathscr{U}$ for some $n \in \mathbb{N}$. To show the converse, suppose that (2.1) holds, while $B_n \notin \mathscr{U}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ (since $\mathscr{U}$ is an ultrafilter) there exists some $U_n \in \mathscr{U}$ such that $B_n \cap U_n = \emptyset$. Now

$$\bigcap_{i=1}^{\infty} U_i \cap \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \Big( \bigcap_{i=1}^{\infty} U_i \cap B_n \Big) \subseteq \bigcup_{n=1}^{\infty} (U_n \cap B_n) = \emptyset$$

contradicting the fact that $\mathscr{U}$ has c.i.p. $\square$

**Theorem 2.2.** *Let $(X, \mathscr{B}, \mu)$ be a measure space. Then $(Y, \mathscr{C}, \lambda)$ is a measure space in which $(X, \mathscr{B}, \mu)$ is embedded if and only if there exists a measurable space $(Z, \mathscr{D})$, a collection $\{\mathscr{D}_B : B \in \mathscr{B}\}$ of non-empty subsets of $\mathscr{D}$ such that*

(1) *$\emptyset \in \mathscr{D}_\emptyset$;*
(2) *if $B \in \mathscr{B}$, then*

$$\{Z \backslash D : D \in \mathscr{D}_B\} \subseteq \mathscr{D}_{X \backslash B};$$

(3) *if* $B_1, B_2, \ldots \in \mathscr{B}$, *then*

$$\Big\{ \bigcup_{n=1}^{\infty} D_n : D_n \in \mathscr{D}_{B_n} \Big\} \subseteq \mathscr{D}_{\bigcup_{n=1}^{\infty} B_n};$$

*and a collection* $\{S_{\mathscr{U}} : \mathscr{U} \in \mathbb{U}\}$ *of pairwise disjoint non-empty sets, bijectively indexed by a collection* $\mathbb{U}$ *of ultrafilters in* $\mathscr{B}$ *with c.i.p., where the sets* $X$, $S_{\mathscr{U}}$ *for any* $\mathscr{U} \in \mathbb{U}$, *and* $Z$ *are pairwise disjoint, such that*

$$Y = X \cup \bigcup_{\mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup Z,$$

$$\mathscr{C} = \Big\{ B \cup \bigcup_{B \in \mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup D : B \in \mathscr{B} \text{ and } D \in \mathscr{D}_B \Big\},$$

*and* $\lambda : \mathscr{C} \to [0, \infty]$ *is given by*

$$\lambda\Big( B \cup \bigcup_{B \in \mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup D \Big) = \mu(B)$$

*for each* $B \in \mathscr{B}$ *and* $D \in \mathscr{D}_B$.

*Proof.* Suppose that $Y$, $\mathscr{C}$ and $\lambda$ are defined as in the statement of the theorem. We show that $(Y, \mathscr{C}, \lambda)$ is a measure space in which $(X, \mathscr{B}, \mu)$ is embedded. First, we verify that $\mathscr{C}$ is a $\sigma$-algebra on $Y$. By (1) we have $\emptyset \in \mathscr{C}$. Let $C \in \mathscr{C}$. We show that $Y \backslash C \in \mathscr{C}$. Let

$$C = B \cup \bigcup_{B \in \mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup D$$

for some $B \in \mathscr{B}$ and $D \in \mathscr{D}_B$. Note that for each $\mathscr{U} \in \mathbb{U}$ we have $B \notin \mathscr{U}$ if and only if $X \backslash B \in \mathscr{U}$; this is because if $B \notin \mathscr{U}$ then $X \backslash B \in \mathscr{U}$ by Lemma 2.1; the converse is trivial. Therefore

$$\begin{aligned} Y \backslash C &= \Big( X \cup \bigcup_{\mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup Z \Big) \backslash \Big( B \cup \bigcup_{B \in \mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup D \Big) \\ &= (X \backslash B) \cup \Big( \bigcup_{\mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \backslash \bigcup_{B \in \mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \Big) \cup (Z \backslash D) \\ &= (X \backslash B) \cup \bigcup_{B \notin \mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup (Z \backslash D) \\ &= (X \backslash B) \cup \bigcup_{X \backslash B \in \mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup (Z \backslash D). \end{aligned}$$

By (2) we have $Z\backslash D \in \mathscr{D}_{X\backslash B}$. Thus $Y\backslash C \in \mathscr{C}$. Now, to show that $\mathscr{C}$ is closed under countable unions, let $C_1, C_2, \ldots \in \mathscr{C}$. Then

$$C_n = B_n \cup \bigcup_{B_n \in \mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup D_n$$

where $B_n \in \mathscr{B}$ and $D_n \in \mathscr{D}_{B_n}$ for each $n \in \mathbb{N}$. Using Lemma 2.1, we have

$$\begin{aligned}
\bigcup_{n=1}^{\infty} C_n &= \bigcup_{n=1}^{\infty} \Big( B_n \cup \bigcup_{B_n \in \mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup D_n \Big) \\
&= \bigcup_{n=1}^{\infty} B_n \cup \bigcup_{n=1}^{\infty} \bigcup_{B_n \in \mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup \bigcup_{n=1}^{\infty} D_n \\
&= \bigcup_{n=1}^{\infty} B_n \cup \bigcup_{\bigcup_{n=1}^{\infty} B_n \in \mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup \bigcup_{n=1}^{\infty} D_n .
\end{aligned}$$

By (3) we have

$$\bigcup_{n=1}^{\infty} D_n \in \mathscr{D}_{\bigcup_{n=1}^{\infty} B_n}.$$

Thus

$$\bigcup_{n=1}^{\infty} C_n \in \mathscr{C}.$$

This shows that $\mathscr{C}$ is a $\sigma$-algebra on $Y$. Next, we show that $\lambda$ is a measure on $\mathscr{C}$. Note that $\lambda(\emptyset) = 0$. If $C_1, C_2, \ldots \in \mathscr{C}$ are disjoint, then using the above results and notation we have

$$\lambda\Big(\bigcup_{n=1}^{\infty} C_n\Big) = \mu\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} \lambda(C_n).$$

This shows that $(Y, \mathscr{C}, \lambda)$ is a measure space. Now we show that $(X, \mathscr{B}, \mu)$ is embedded in $(Y, \mathscr{C}, \lambda)$. Obviously, by our definitions we have $X \subseteq Y$ and $C \cap X \in \mathscr{B}$ for each $C \in \mathscr{C}$. Conversely, for each $B \in \mathscr{B}$, since by our assumption $\mathscr{D}_B$ is non-empty, we have $B = C \cap X$ for some $C \in \mathscr{C}$. Thus

$$\mathscr{B} = \{C \cap X : C \in \mathscr{C}\}.$$

Also, it is obvious that $\lambda(C) = \mu(C \cap X)$ for each $C \in \mathscr{C}$. Therefore $(X, \mathscr{B}, \mu)$ is embedded in $(Y, \mathscr{C}, \lambda)$.

Now, suppose that $(Y, \mathscr{C}, \lambda)$ is a measure space in which $(X, \mathscr{B}, \mu)$ is embedded. We show that $(Y, \mathscr{C}, \lambda)$ can be constructed as in the previous part. Note that $X \subseteq Y$. Define

$$Z = \{p \in Y \backslash X : p \in C \subseteq Y \backslash X \text{ for some } C \in \mathscr{C}\}$$

and

$$\mathscr{D} = \{C \cap Z : C \in \mathscr{C}\}.$$

Then obviously $(Z, \mathscr{D})$ is a measurable space. Define

$$\mathscr{D}_B = \{C \cap Z : C \in \mathscr{C} \text{ and } C \cap X = B\}$$

for each $B \in \mathscr{B}$. Obviously $\mathscr{D}_B \subseteq \mathscr{D}$ and $\mathscr{D}_B$ is non-empty for each $B \in \mathscr{B}$. We verify that conditions (1)–(3) of the theorem hold. Condition (1) holds trivially. To show condition (2), note that if $D \in \mathscr{D}_B$ for some $B \in \mathscr{B}$ then $D = C \cap Z$, where $C \in \mathscr{C}$ and $C \cap X = B$. Thus

$$Z \backslash D = Z \backslash (C \cap Z) = Z \cap (Y \backslash C).$$

Now, since

$$(Y \backslash C) \cap X = X \backslash (C \cap X) = X \backslash B$$

we have $Z \backslash D \in \mathscr{D}_{X \backslash B}$. Therefore

$$\{Z \backslash D : D \in \mathscr{D}_B\} \subseteq \mathscr{D}_{X \backslash B}.$$

To show condition (3), let $B_n \in \mathscr{B}$ and $D_n \in \mathscr{D}_{B_n}$ for each $n \in \mathbb{N}$. Then $D_n = C_n \cap Z$ where $C_n \in \mathscr{C}$ and $C_n \cap X = B_n$ for each $n \in \mathbb{N}$. We have

$$\bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty} (Z \cap C_n) = Z \cap \bigcup_{n=1}^{\infty} C_n$$

and

$$X \cap \bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} (X \cap C_n) = \bigcup_{n=1}^{\infty} B_n$$

where

$$\bigcup_{n=1}^{\infty} C_n \in \mathscr{C}.$$

Thus

$$\bigcup_{n=1}^{\infty} D_n \in \mathscr{D}_{\bigcup_{n=1}^{\infty} B_n},$$

i.e.,

$$\Big\{ \bigcup_{n=1}^{\infty} D_n : D_n \in \mathscr{D}_{B_n} \Big\} \subseteq \mathscr{D}_{\bigcup_{n=1}^{\infty} B_n}.$$

This shows conditions (1)–(3). For each $p \in (Y\backslash X)\backslash Z$ let

$$\mathscr{U}_p = \{C \cap X : p \in C \in \mathscr{C}\}.$$

**Claim 1.** *For each $p \in (Y\backslash X)\backslash Z$ the set $\mathscr{U}_p$ is an ultrafilter in $\mathscr{B}$ which has c.i.p.*

*Proof of Claim 1.* First note that

(2.2) $(Y\backslash X)\backslash Z = \{y \in Y\backslash X : C \cap X \neq \emptyset \text{ for each } C \in \mathscr{C} \text{ with } y \in C\}$.

Let $p \in (Y\backslash X)\backslash Z$. By (2.2) we have $\emptyset \notin \mathscr{U}_p$. It is obvious that $\emptyset \neq \mathscr{U}_p \subseteq \mathscr{B}$ and that $\mathscr{U}_p$ is closed under finite intersections. Now, suppose that $U \subseteq B$ for some $U \in \mathscr{U}_p$ and $B \in \mathscr{B}$. Then $U = C \cap X$ for some $C \in \mathscr{C}$ such that $p \in C$, and $B = G \cap X$ for some $G \in \mathscr{C}$. If $p \notin G$ then $p \in Y\backslash G \in \mathscr{C}$. Thus $p \in C \cap (Y\backslash G) \in \mathscr{C}$ and therefore by (2.2) and the choice of $p$ the set $C \cap (Y\backslash G) \cap X$ is non-empty. But this is a contradiction, as

$$C \cap X \cap (Y\backslash G) = U \cap (Y\backslash G) \subseteq B \cap (Y\backslash G) = (G \cap X) \cap (Y\backslash G) = \emptyset.$$

Thus $p \in G$ and therefore $B = G \cap X \in \mathscr{U}_p$. This shows that $\mathscr{U}_p$ is a filter in $\mathscr{B}$. To show that $\mathscr{U}_p$ is an ultrafilter, let $B \in \mathscr{B}$ be such that $B \cap U$ is non-empty for each $U \in \mathscr{U}_p$. Let $B = C \cap X$ for some $C \in \mathscr{C}$. If $p \notin C$ then $p \in Y\backslash C$ and thus $(Y\backslash C) \cap X \in \mathscr{U}_p$, which is not possible, as $(Y\backslash C) \cap X$ misses $B$. Therefore $p \in C$ and thus $B = C \cap X \in \mathscr{U}_p$. To show that $\mathscr{U}_p$ has c.i.p., let $U_1, U_2, \ldots \in \mathscr{U}_p$. Then $U_n = C_n \cap X$ where $p \in C_n \in \mathscr{C}$ for each $n \in \mathbb{N}$. Then

$$p \in \bigcap_{n=1}^{\infty} C_n \in \mathscr{C}$$

and thus

$$\bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} (X \cap C_n) = X \cap \bigcap_{n=1}^{\infty} C_n \in \mathscr{U}_p.$$

Therefore

$$\bigcap_{n=1}^{\infty} U_n \neq \emptyset.$$

This proves the claim.

Let

$$\mathbb{U} = \big\{\mathscr{U}_p : p \in (Y\backslash X)\backslash Z\big\}.$$

Then $\mathbb{U}$ is a collection of ultrafilters in $\mathscr{B}$ with c.i.p. For each $\mathscr{U} \in \mathbb{U}$ define

$$S_{\mathscr{U}} = \big\{p \in (Y \backslash X) \backslash Z : \mathscr{U}_p = \mathscr{U}\big\}.$$

Note that $S_{\mathscr{U}}$, for each $\mathscr{U} \in \mathbb{U}$, is non-empty, as $\mathscr{U} = \mathscr{U}_p$ for some $p \in (Y \backslash X) \backslash Z$ and thus $p \in S_{\mathscr{U}}$. Also, for any distinct $\mathscr{U}, \mathscr{V} \in \mathbb{U}$ we have $S_{\mathscr{U}} \cap S_{\mathscr{V}} = \emptyset$, as $p \in S_{\mathscr{U}} \cap S_{\mathscr{V}}$ implies that $\mathscr{U} = \mathscr{U}_p = \mathscr{V}$. Thus

$$\{S_{\mathscr{U}} : \mathscr{U} \in \mathbb{U}\}$$

is a bijectively indexed collection of pairwise disjoint non-empty sets. Note that by our definitions the sets $X$, $S_{\mathscr{U}}$ where $\mathscr{U} \in \mathbb{U}$ and $Z$ are pairwise disjoint. Let

$$Y' = X \cup \bigcup_{\mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup Z$$

$$\mathscr{C}' = \Big\{B \cup \bigcup_{B \in \mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup D : B \in \mathscr{B} \text{ and } D \in \mathscr{D}_B\Big\}$$

and $\lambda' : \mathscr{C}' \to [0, \infty]$ be given by

$$\lambda'\Big(B \cup \bigcup_{B \in \mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup D\Big) = \mu(B)$$

where $B \in \mathscr{B}$ and $D \in \mathscr{D}_B$. By the first part we know that $(Y', \mathscr{C}', \lambda')$ is a measure space in which $(X, \mathscr{B}, \mu)$ is embedded. We verify that

$$(Y, \mathscr{C}, \lambda) = (Y', \mathscr{C}', \lambda').$$

By our definition it is obvious that $Y' \subseteq Y$. To show the reverse inclusion, let $p \in Y$. If either $p \in X$ or $p \in Z$ then $p \in Y'$. If $p \in (Y \backslash X) \backslash Z$, then since $\mathscr{U}_p \in \mathbb{U}$ and by our definition $p \in S_{\mathscr{U}_p}$, we have $p \in Y'$. Thus $Y \subseteq Y'$ and therefore $Y = Y'$. Next, we verify that $\mathscr{C} = \mathscr{C}'$.

**Claim 2.** *Let $C \in \mathscr{C}$ and $B = C \cap X$. Then*

$$\bigcup_{B \in \mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} = C \cap \big((Y \backslash X) \backslash Z\big).$$

*Proof of Claim 2.* Suppose that $p \in S_{\mathscr{U}}$ for some $\mathscr{U} \in \mathbb{U}$ such that $B \in \mathscr{U}$. If $p \notin C$ then $p \in Y \backslash C \in \mathscr{C}$ and thus

$$(Y \backslash C) \cap X \in \mathscr{U}_p = \mathscr{U}.$$

But this is not possible, as $(Y \backslash C) \cap X$ misses $C \cap X = B$. Thus $p \in C$. Also, since $S_{\mathscr{U}} \subseteq (Y \backslash X) \backslash Z$ it is obvious that $p \in (Y \backslash X) \backslash Z$. To show the reverse inclusion, note that for each $p \in C \cap ((Y \setminus X) \setminus Z)$ since $p \in C$ we have $B = C \cap X \in \mathscr{U}_p$ and $p \in S_{\mathscr{U}_p}$. This proves the claim.

Now, let $C' \in \mathscr{C}'$. Then

$$C' = B \cup \bigcup_{B \in \mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup D.$$

for some $B \in \mathscr{B}$ and $D \in \mathscr{D}_B$. Thus, by the way we have defined $\mathscr{D}_B$ we have $D = C \cap Z$, for some $C \in \mathscr{C}$ such that $C \cap X = B$. By Claim 2 we have

$$C' = B \cup \bigcup_{B \in \mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup D = (C \cap X) \cup \big(C \cap \big((Y \setminus X) \setminus Z\big)\big) \cup (C \cap Z) = C \in \mathscr{C}.$$

Therefore $\mathscr{C}' \subseteq \mathscr{C}$. To show the reverse inclusion let $C \in \mathscr{C}$. Let $B = C \cap X \in \mathscr{B}$ and $D = C \cap Z$. Then $D \in \mathscr{D}_B$, by the way we have defined $\mathscr{D}_B$, and thus by Claim 2 we have

$$C = (C \cap X) \cup \big(C \cap \big((Y \setminus X) \setminus Z\big)\big) \cup (C \cap Z) = B \cup \bigcup_{B \in \mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup D \in \mathscr{C}'.$$

Therefore $\mathscr{C} \subseteq \mathscr{C}'$, which together with the above shows that $\mathscr{C} = \mathscr{C}'$. The fact that $\lambda = \lambda'$ is trivial, as by the above for each $C \in \mathscr{C}$ we have

$$\lambda(C) = \mu(C \cap X) = \lambda'(C).$$

This completes the proof. □

Let $(Y, \mathscr{C}, \lambda)$ be a measure space in which $(X, \mathscr{B}, \mu)$ is embedded. Assume the representation and notation given in Theorem 2.2. Then

$$Y = X \cup \bigcup_{\mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} \cup Z$$

where (by the proof of Theorem 2.2)

$$Z = \{p \in Y \setminus X : p \in C \subseteq Y \setminus X \text{ for some } C \in \mathscr{C}\}.$$

Thus

$$\begin{aligned} \bigcup_{\mathscr{U} \in \mathbb{U}} S_{\mathscr{U}} &= (Y \setminus X) \setminus Z \\ &= \{p \in Y \setminus X : C \cap X \neq \emptyset \text{ for each } C \in \mathscr{C} \text{ with } p \in C\}. \end{aligned}$$

We verify that

$$\bigcup_{\mathscr{U} \in \mathbb{U} \text{ is free}} S_{\mathscr{U}} =$$

$$\big\{p \in (Y \setminus X) \setminus Z : p \text{ is separated from each } x \in X \text{ by sets in } \mathscr{C}\big\}$$

and consequently

$$\bigcup_{\mathscr{U}\in\mathbb{U} \text{ is fixed}} S_{\mathscr{U}} =$$

$$\big\{p \in (Y\backslash X)\backslash Z : p \text{ is not separated from some } x \in X \text{ by sets in } \mathscr{C}\big\}.$$

To show this, let $p \in S_{\mathscr{U}}$ for some free $\mathscr{U} \in \mathbb{U}$. Let $x \in X$. Since $\mathscr{U}$ is free, we have $x \notin U$ for some $U \in \mathscr{U}$. Let $D \in \mathscr{D}_U$. Then

$$C = U \cup \bigcup_{U\in\mathscr{V}\in\mathbb{U}} S_{\mathscr{V}} \cup D \in \mathscr{C}$$

is such that $p \in C$ and $x \notin C$. Conversely, let $p \in (Y\backslash X)\backslash Z$ be such that it can be separated from each $x \in X$ by a measurable set in $\mathscr{C}$. Let $\mathscr{U} \in \mathbb{U}$ be such that $p \in S_{\mathscr{U}}$. Suppose that $\mathscr{U}$ is not free. Let $x \in \bigcap \mathscr{U}$. Let

$$C = B \cup \bigcup_{B\in\mathscr{V}\in\mathbb{U}} S_{\mathscr{V}} \cup D \in \mathscr{C},$$

where $B \in \mathscr{B}$ and $D \in \mathscr{D}_B$, be such that $p \in C$ and $x \notin C$. Then, since $p \in S_{\mathscr{U}}$, we have $B \in \mathscr{U}$, and thus $x \in B$, which is not possible, as $B \subseteq C$. Therefore $\mathscr{U}$ is free.

Thus, in the absence of free ultrafilters in $\mathscr{B}$, each $p \in Y\backslash X$ (depending on whether $p \in Z$ or $p \notin Z$) either "separates" from the whole $X$ by a (null) set in $\mathscr{C}$, or tightly "sticks" to some point $x$ of $X$ so that it cannot be separated from $x$ by any measurable set in $\mathscr{C}$. In the next section we restrict our attention to measure spaces $(X, \mathscr{B}, \mu)$ having no free ultrafilter in $\mathscr{B}$ with c.i.p. As we will see, this assumption considerably simplifies our construction.

## 3. The case of measure spaces $(X, \mathscr{B}, \mu)$ with no free ultrafilter in $\mathscr{B}$ having c.i.p.

In this section we show that for certain classes of measure spaces $(X, \mathscr{B}, \mu)$, the structure of measure spaces $(Y, \mathscr{C}, \lambda)$ in which $(X, \mathscr{B}, \mu)$ is embedded is expressible in a simpler way: They are simply obtained by "blowing" certain points of $X$ up and "pasting" a certain measurable space to $X$ in a certain way. This we prove in the next theorem. Examples of measure spaces satisfying this assumption are given in the next section.

**Theorem 3.1.** *Let $(X, \mathscr{B}, \mu)$ be a measure space. Suppose that the points of $X$ are separated by measurable sets in $\mathscr{B}$ and that there is* *no free ultrafilter in $\mathscr{B}$ with c.i.p. Then $(Y, \mathscr{C}, \lambda)$ is a measure space in which $(X, \mathscr{B}, \mu)$ is embedded if and only if there exists a measurable space $(Z, \mathscr{D})$, a collection $\{\mathscr{D}_B : B \in \mathscr{B}\}$ of non-empty subsets of $\mathscr{D}$ such that*

(1) $\emptyset \in \mathscr{D}_\emptyset$;
(2) *if $B \in \mathscr{B}$, then*

$$\{Z \backslash D : D \in \mathscr{D}_B\} \subseteq \mathscr{D}_{X \backslash B};$$

(3) *if $B_1, B_2, \ldots \in \mathscr{B}$, then*

$$\Big\{ \bigcup_{n=1}^{\infty} D_n : D_n \in \mathscr{D}_{B_n} \Big\} \subseteq \mathscr{D}_{\bigcup_{n=1}^{\infty} B_n};$$

*and a collection $\{T_u : u \in U\}$ of pairwise disjoint non-empty sets, bijectively indexed by a subset $U$ of $X$, where the sets $X$, $T_u$ for any $u \in U$, and $Z$ are pairwise disjoint, such that*

$$Y = X \cup \bigcup_{u \in U} T_u \cup Z,$$

$$\mathscr{C} = \Big\{ B \cup \bigcup_{u \in B \cap U} T_u \cup D : B \in \mathscr{B} \text{ and } D \in \mathscr{D}_B \Big\}$$

*and $\lambda : \mathscr{C} \to [0, \infty]$ is given by*

$$\lambda\Big( B \cup \bigcup_{u \in B \cap U} T_u \cup D \Big) = \mu(B)$$

*for each $B \in \mathscr{B}$ and $D \in \mathscr{D}_B$.*

*Proof.* Suppose that $(Y, \mathscr{C}, \lambda)$ is a measure space in which $(X, \mathscr{B}, \mu)$ is embedded. Assume the representation given for $(Y, \mathscr{C}, \lambda)$ in Theorem 2.2. Assume the notation of Theorem 2.2. By our assumption for each $\mathscr{U} \in \mathbb{U}$ the set $\bigcap \mathscr{U}$ is non-empty. Note that $\bigcap \mathscr{U}$ is a singleton, as if $x, z \in \bigcap \mathscr{U}$ and $x \neq z$, then by our assumption $x \in B$ and $z \notin B$ for some $B \in \mathscr{B}$. Now $X \backslash B \in \mathscr{B}$ intersects each element of $\mathscr{U}$, thus $X \backslash B \in \mathscr{U}$. This contradicts the fact that $x \notin X \backslash B$. Let

$$\bigcap \mathscr{U} = \{u_{\mathscr{U}}\}.$$

Define

$$U = \{u_{\mathscr{U}} : \mathscr{U} \in \mathbb{U}\}.$$

**Claim 1.** *For each $\mathscr{U} \in \mathbb{U}$ we have*

$$\mathscr{U} = \{B \in \mathscr{B} : u_{\mathscr{U}} \in B\}.$$

*Proof of Claim 1.* Let $\mathscr{U} \in \mathbb{U}$. Obviously, $u_\mathscr{U} \in B$ for each $B \in \mathscr{U}$. Conversely, if $B \in \mathscr{B}$ is such that $u_\mathscr{U} \in B$ then $B \in \mathscr{U}$. As otherwise, $B \cap G = \emptyset$ for some $G \in \mathscr{U}$. Thus $G \subseteq X \backslash B \in \mathscr{B}$ which implies that $X \backslash B \in \mathscr{U}$. This contradicts the fact that $u_\mathscr{U} \notin X \backslash B$ and proves the claim.

For each $u \in U$ there exists some $\mathscr{U} \in \mathbb{U}$ such that $u = u_\mathscr{U}$. Note that by Claim 1 such a $\mathscr{U}$ is unique; let $T_u = S_\mathscr{U}$. The collection

$$\{T_u : u \in U\}$$

consists of non-empty sets which are pairwise disjoint and bijectively indexed (as any distinct $u, v \in U$ are of the form $u = u_\mathscr{U}$ and $v = u_\mathscr{V}$ for some distinct $\mathscr{U}, \mathscr{V} \in \mathbb{U}$).

**Claim 2.** *For each $B \in \mathscr{B}$ we have*

$$\bigcup_{u \in B \cap U} T_u = \bigcup_{B \in \mathscr{U} \in \mathbb{U}} S_\mathscr{U}.$$

*Proof of Claim 2.* Suppose that $B \in \mathscr{U} \in \mathbb{U}$. By Claim 1 we have $u_\mathscr{U} \in B \cap U$. Note that $T_{u_\mathscr{U}} = S_\mathscr{U}$. To show the reverse inclusion, let $u \in B \cap U$. Let $\mathscr{U} \in \mathbb{U}$ be such that $u = u_\mathscr{U}$. Then, by our definition $T_u = S_\mathscr{U}$. Note that by Claim 1 we have $B \in \mathscr{U}$. This proves the claim.

Note that if $B = X$ then Claim 2 implies that

$$\bigcup_{u \in U} T_u = \bigcup_{\mathscr{U} \in \mathbb{U}} S_\mathscr{U}.$$

From the above the desired representation of $(Y, \mathscr{C}, \lambda)$ follows.

Conversely, suppose that $Y$, $\mathscr{C}$ and $\lambda$ are given as in the statement of the theorem. Assume the notation of the theorem. For each $u \in U$ define

$$\mathscr{U}_u = \{B \in \mathscr{B} : u \in B\}.$$

**Claim 3.** *For each $u \in U$ the set $\mathscr{U}_u$ is an ultrafilter in $\mathscr{B}$ with c.i.p.*

*Proof of Claim 3.* Let $u \in U$. Obviously, $\emptyset \neq \mathscr{U}_u \subseteq \mathscr{B}$, $\emptyset \notin \mathscr{U}_u$ and $\mathscr{U}_u$ is closed under finite intersections. Also, note that if $G \subseteq B$ for some $G \in \mathscr{U}_u$ and $B \in \mathscr{B}$, then $B \in \mathscr{U}_u$. Thus $\mathscr{U}_u$ is a filter in $\mathscr{B}$. To show that $\mathscr{U}_u$ is an ultrafilter, suppose that $B \in \mathscr{B}$ is such that $B \cap G$ is non-empty for each $G \in \mathscr{U}_u$. If $B \notin \mathscr{U}_u$ then $u \notin B$. Thus $u \in X \backslash B \in \mathscr{B}$

and $X\backslash B \in \mathscr{U}_u$. But $X\backslash B$ misses $B$, which is a contradiction. Therefore $B \in \mathscr{U}_u$. The fact that $\mathscr{U}_u$ has c.i.p. is obvious. This proves the claim.

Let

$$\mathbb{U} = \{\mathscr{U}_u : u \in U\}.$$

Note that each $\mathscr{U} \in \mathbb{U}$ is of the form $\mathscr{U}_u$ for some unique $u \in U$. This is because if $\mathscr{U}_u = \mathscr{U} = \mathscr{U}_v$ for some distinct $u, v \in U$, then by our assumption $u \in B$ and $v \notin B$ for some $B \in \mathscr{B}$. Thus $B \in \mathscr{U}_u\backslash\mathscr{U}_v$. Therefore $\mathscr{U}_u \neq \mathscr{U}_v$, which is a contradiction. For each $\mathscr{U} \in \mathbb{U}$ define $S_{\mathscr{U}} = T_u$, where $u \in U$ is such that $\mathscr{U} = \mathscr{U}_u$. The collection

$$\{S_{\mathscr{U}} : \mathscr{U} \in \mathbb{U}\}$$

consists of non-empty sets which are pairwise disjoint and bijectively indexed (as distinct elements of $\mathbb{U}$ are assigned to distinct elements of $U$).

**Claim 4.** *For each $B \in \mathscr{B}$ we have*

$$\bigcup_{B\in\mathscr{U}\in\mathbb{U}} S_{\mathscr{U}} = \bigcup_{u\in B\cap U} T_u.$$

*Proof of Claim 4.* Let $u \in B \cap U$. Then, by our definition of $\mathscr{U}_u$ we have $B \in \mathscr{U}_u \in \mathbb{U}$. Also, by our definition $T_u = S_{\mathscr{U}_u}$. To show the reverse inclusion, let $B \in \mathscr{U} \in \mathbb{U}$. Let $u \in U$ be such that $\mathscr{U} = \mathscr{U}_u$. Then, by our definition $S_{\mathscr{U}} = T_u$. But since $B \in \mathscr{U}_u$, by our definition we have $u \in B$, and thus $u \in B \cap U$. This proves the claim.

Note that if $B = X$ then Claim 4 implies that

$$\bigcup_{\mathscr{U}\in\mathbb{U}} S_{\mathscr{U}} = \bigcup_{u\in U} T_u.$$

From the above and Theorem 2.2 the result follows. $\square$

## 4. Examples of measure spaces $(X, \mathscr{B}, \mu)$ with no free ultrafilter in $\mathscr{B}$ having c.i.p.

In this section we give examples of measure spaces $(X, \mathscr{B}, \mu)$ for which the assumption of Theorem 3.1 holds, i.e., measure spaces $(X, \mathscr{B}, \mu)$ for which there is no free ultrafilter in $\mathscr{B}$ with c.i.p. The following gives some equivalent ways to express this condition. The equivalence of conditions (1) and (2) in the following proposition is well known; we include the proof in here for the sake of completeness.

**Proposition 4.1.** *Let $(X,\mathscr{B})$ be a measurable space. Then the following are equivalent:*

(1) *There is no free ultrafilter in $\mathscr{B}$ with c.i.p.*
(2) *For every $\{0,1\}$-valued measure $\mu$ on $\mathscr{B}$ whose null sets cover $X$ we have $\mu \equiv 0$.*
(3) *For every $\{0,1\}$-valued measure $\mu$ on $\mathscr{B}$ whose null sets cover $X$, if $\mathscr{C} \subseteq \mathscr{B}$ is non-empty and such that $\bigcup\mathscr{C} \in \mathscr{B}$, then*

$$\mu\Big(\bigcup\mathscr{C}\Big) = \sup_{C\in\mathscr{C}} \mu(C).$$

*Proof.* That (2) implies (3) is trivial. (1) *implies* (2). Let $\mu$ be a non-trivial $\{0,1\}$-valued measure on $\mathscr{B}$ whose null sets cover $X$. Define

$$\mathscr{F} = \big\{B \in \mathscr{B} : \mu(B) = 1\big\}.$$

We show that $\mathscr{F}$ is a free ultrafilter in $\mathscr{B}$ with c.i.p. If $F, G \in \mathscr{F}$, then since

$$\mu(F \cup G) = \mu(F\backslash G) + \mu(G)$$

and $\mu(G) = 1$, we have $\mu(F\backslash G) = 0$. Therefore

$$\mu(F \cap G) = \mu(F\backslash G) + \mu(F \cap G) = \mu(F) = 1$$

and thus $F \cap G \in \mathscr{F}$. Obviously, if $F \subseteq B$ for some $F \in \mathscr{F}$ and $B \in \mathscr{B}$, then $B \in \mathscr{F}$. Therefore, $\mathscr{F}$ is a filter in $\mathscr{B}$. To show that $\mathscr{F}$ is an ultrafilter, let $B \in \mathscr{B}$ be such that $B \cap F$ is non-empty for each $F \in \mathscr{F}$. If $B \notin \mathscr{F}$, then $\mu(B) = 0$, and thus, since $\mu(X\backslash B) = 1$, we have $X\backslash B \in \mathscr{F}$. But this is not possible, as $B$ misses $X\backslash B$. Therefore $B \in \mathscr{F}$. To show that $\mathscr{F}$ has c.i.p., let $F_1, F_2, \ldots \in \mathscr{F}$. Without any loss of generality we may assume that $F_1 \supseteq F_2 \supseteq \cdots$. If

$$\bigcap_{n=1}^{\infty} F_n \notin \mathscr{F}$$

then

$$\mu\Big(\bigcap_{n=1}^{\infty} F_n\Big) = 0.$$

Thus (with the empty intersection interpreted as $X$) we have

$$1 = \mu\Big(X\backslash \bigcap_{n=1}^{\infty} F_n\Big) = \mu\Big(\bigcup_{n=1}^{\infty} (F_{n-1}\backslash F_n)\Big) = \sum_{n=1}^{\infty} \mu(F_{n-1}\backslash F_n). \tag{4.1}$$

Now, each $G_n = F_{n-1} \backslash F_n$, where $n \in \mathbb{N}$, misses $F_n \in \mathscr{F}$ and thus $G_n \notin \mathscr{F}$; therefore $\mu(G_n) = 0$. This contradicts (4.1) and shows that

$$\bigcap_{n=1}^{\infty} F_n \in \mathscr{F}.$$

To show that $\mathscr{F}$ is free, let $x \in X$. By our assumption $x \in B$ for some $B \in \mathscr{B}$ such that $\mu(B) = 0$. Thus $\mu(X \backslash B) = 1$. But $X \backslash B \in \mathscr{F}$ and $x \notin X \backslash B$. Therefore $\bigcap \mathscr{F} = \emptyset$.

(3) *implies* (1). Suppose that there exists a free ultrafilter $\mathscr{F}$ in $\mathscr{B}$ with c.i.p. Define $\mu : \mathscr{B} \to \{0, 1\}$ such that $\mu(B) = 1$ if $B \in \mathscr{F}$ and $\mu(B) = 0$ if $B \notin \mathscr{F}$. To show that $\mu$ is a measure, first note that $\mu(\emptyset) = 0$. Let $B_1, B_2, \ldots \in \mathscr{B}$ be pairwise disjoint. Suppose that

$$\bigcup_{n=1}^{\infty} B_n \notin \mathscr{F}.$$

Then $B_n \notin \mathscr{F}$ for some $n \in \mathbb{N}$. Therefore

$$\mu\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \sum_{n=1}^{\infty} \mu(B_n) \tag{4.2}$$

as each side is identical to 0. Suppose that

$$\bigcup_{n=1}^{\infty} B_n \in \mathscr{F}.$$

By Lemma 2.1 this implies that $B_n \in \mathscr{F}$ for some $n \in \mathbb{N}$. Note that for each $n \neq i \in \mathbb{N}$, since $B_i \cap B_n = \emptyset$ we have $B_i \notin \mathscr{F}$. Thus (4.2) holds, as in this case each side is identical to 1. This shows that $\mu$ is a measure. Since $\mathscr{F}$ is free, for each $x \in X$ there exists some $F \in \mathscr{F}$ such that $x \notin F$. Thus $x \in X \backslash F$, and since $X \backslash F \notin \mathscr{F}$ we have $\mu(X \backslash F) = 0$. Therefore the null sets of $\mathscr{B}$ cover $X$. Now, let

$$\mathscr{C} = \{X \backslash F : F \in \mathscr{F}\}.$$

Then since $\mathscr{F}$ is free, we have $\bigcup \mathscr{C} = X$, and therefore by our assumption

$$\sup_{F \in \mathscr{F}} \mu(X \backslash F) = \sup_{C \in \mathscr{C}} \mu(C) = \mu\Big(\bigcup \mathscr{C}\Big) = \mu(X) = 1.$$

But this is not possible, as if $F \in \mathscr{F}$ then $X \backslash F \notin \mathscr{F}$, as it misses $F$, and thus $\mu(X \backslash F) = 0$. $\square$

**Theorem 4.2.** *Let $(X, \mathscr{B})$ be a measurable space. If $X$ is countable then there is no free ultrafilter in $\mathscr{B}$ with c.i.p.*

*Proof.* Let $X = \{x_1, x_2, \ldots\}$ and let $\mathscr{F}$ be a free ultrafilter in $\mathscr{B}$. Since $\mathscr{F}$ is free for each $n \in \mathbb{N}$ there is some $F_n \in \mathscr{F}$ such that $x_n \notin F_n$. Then

$$\bigcap_{n=1}^{\infty} F_n = \emptyset$$

and thus $\mathscr{F}$ does not have c.i.p. $\square$

**Theorem 4.3.** *Let $(Y, \mathscr{C})$ be a measurable space. Let $X \in \mathscr{C}$ and*

$$\mathscr{B} = \{C \in \mathscr{C} : C \subseteq X\},$$

*i.e., $X \in \mathscr{C}$ and $(X, \mathscr{B})$ is embedded in $(Y, \mathscr{C})$. If there is no free ultrafilter in $\mathscr{C}$ with c.i.p. then there is no free ultrafilter in $\mathscr{B}$ with c.i.p.*

*Proof.* Simply note that if $\mathscr{F}$ is a free ultrafilter in $\mathscr{B}$ with c.i.p., then

$$\mathscr{G} = \{G \in \mathscr{C} : G \supseteq F \text{ for some } F \in \mathscr{F}\}$$

is a filter in $\mathscr{C}$ which is an ultrafilter (as if $C \in \mathscr{C}$ meets each $G \in \mathscr{G}$ then, since $\mathscr{F} \subseteq \mathscr{G}$, the set $(X \cap C) \cap F = C \cap F$ is non-empty for each $F \in \mathscr{F}$ and thus $X \cap C \in \mathscr{F}$ which implies that $C \in \mathscr{G}$, as $X \cap C \subseteq C$) and it is free (as $\mathscr{F} \subseteq \mathscr{G}$ and thus $\bigcap \mathscr{G} \subseteq \bigcap \mathscr{F}$ and $\mathscr{F}$ is free) and has c.i.p. (as $\mathscr{F}$ has, and if $G_1, G_2, \ldots \in \mathscr{G}$ then

$$\bigcap_{n=1}^{\infty} G_n \supseteq \bigcap_{n=1}^{\infty} F_n,$$

where for each $n \in \mathbb{N}$, the element $F_n \in \mathscr{F}$ is such that $F_n \subseteq G_n$). $\square$

**Theorem 4.4.** *Let $(X, \mathscr{B}) \subseteq (Y, \mathscr{C})$ and let $Y \backslash X$ be countable. If there is no free ultrafilter in $\mathscr{B}$ with c.i.p. then there is no free ultrafilter in $\mathscr{C}$ with c.i.p.*

*Proof.* Let

$$Y \backslash X = \{y_1, y_2, \ldots\}.$$

Let $\mathscr{H}$ be a free ultrafilter in $\mathscr{C}$ with c.i.p. Since $\mathscr{H}$ is free, for any $n \in \mathbb{N}$ there exists some $H_n \in \mathscr{H}$ such that $y_n \notin H_n$. Let

$$\mathscr{A} = \{H \cap X : H \in \mathscr{H}\} \subseteq \mathscr{B}.$$

Note that $\emptyset \notin \mathscr{A}$, as otherwise $H \cap X = \emptyset$ for some $H \in \mathscr{H}$. Now

$$H \cap \bigcap_{n=1}^{\infty} H_n = \emptyset,$$

as

$$H \cap \bigcap_{n=1}^{\infty} H_n \subseteq (X \cap H) \cup \Big((Y \backslash X) \cap \bigcap_{n=1}^{\infty} H_n\Big) = \emptyset$$

contradicting the fact that $\mathscr{H}$ has c.i.p. Thus $\mathscr{A}$ is a filter-base in $\mathscr{B}$, as $\mathscr{A}$ is closed under finite intersections. Let $\mathscr{F}$ be an ultrafilter in $\mathscr{B}$ such that $\mathscr{A} \subseteq \mathscr{F}$. Then since

$$\bigcap \mathscr{F} \subseteq \bigcap \mathscr{A} = \bigcap \mathscr{H} \cap X$$

(and $\mathscr{H}$ is free), $\mathscr{F}$ is free. To show that $\mathscr{F}$ has c.i.p., let $F_1, F_2, \ldots \in \mathscr{F}$. Let $F_n = C_n \cap X$ where $C_n \in \mathscr{C}$ for each $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. For each $H \in \mathscr{H}$, since $H \cap X \in \mathscr{A} \subseteq \mathscr{F}$ we have

$$C_n \cap H \cap X = F_n \cap H \cap X \in \mathscr{F}.$$

Therefore $H \cap C_n$ is non-empty and thus (since $\mathscr{H}$ is an ultrafilter) $C_n \in \mathscr{H}$. Now, since

$$\begin{aligned} \bigcap_{n=1}^{\infty} C_n \cap \bigcap_{n=1}^{\infty} H_n \quad &\subseteq \quad \Big(X \cap \bigcap_{n=1}^{\infty} C_n\Big) \cup \Big((Y \backslash X) \cap \bigcap_{n=1}^{\infty} H_n\Big) \\ &= \quad \bigcap_{n=1}^{\infty} (X \cap C_n) = \bigcap_{n=1}^{\infty} F_n \end{aligned}$$

and $\mathscr{H}$ has c.i.p., we have

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

Therefore $\mathscr{F}$ has c.i.p. □

If $(X, \mathscr{B})$ and $(Y, \mathscr{C})$ are measurable spaces, we denote by $\mathscr{B} \times \mathscr{C}$ the smallest $\sigma$-algebra on $X \times Y$ containing the set

$$\{B \times C : B \in \mathscr{B} \text{ and } C \in \mathscr{C}\}$$

of all measurable rectangles in $X \times Y$.

**Theorem 4.5.** *Let $(X, \mathscr{B})$ and $(Y, \mathscr{C})$ be measurable spaces such that in each of them singletons are measurable. Then the following are equivalent:*

(1) *There is no free ultrafilter in $\mathscr{B} \times \mathscr{C}$ with c.i.p.*
(2) *Neither there is a free ultrafilter in $\mathscr{B}$ with c.i.p. nor there is a free ultrafilter in $\mathscr{C}$ with c.i.p.*

*Proof.* (1) *implies* (2). Let $\mathscr{F}$ be a free ultrafilter in $\mathscr{B}$ with c.i.p. Fix some $y \in Y$ and let

$$\mathscr{A} = \big\{F \times \{y\} : F \in \mathscr{F}\big\}.$$

Then $\mathscr{A}$ is a filter-base in $\mathscr{B} \times \mathscr{C}$, as $\emptyset \notin \mathscr{A}$ and $\mathscr{A}$ is closed under finite intersections. Let $\mathscr{H}$ be an ultrafilter in $\mathscr{B} \times \mathscr{C}$ such that $\mathscr{A} \subseteq \mathscr{H}$. Then

$$\bigcap \mathscr{H} \subseteq \bigcap \mathscr{A} = \Big(\bigcap \mathscr{F}\Big) \times \{y\} = \emptyset$$

and $\mathscr{H}$ is also free. We verify that $\mathscr{H}$ has c.i.p. Let $H_1, H_2, \ldots \in \mathscr{H}$. Let

$$H_n^y = \big\{x \in X : (x, y) \in H_n\big\}$$

for each $n \in \mathbb{N}$. Then $H_n^y$, for each $n \in \mathbb{N}$, being the $y$-section of the measurable set $H_n$ of $\mathscr{B} \times \mathscr{C}$ is measurable in $X$. Note that for each $n \in \mathbb{N}$ and $F \in \mathscr{F}$, since $F \times \{y\} \in \mathscr{H}$, we have

$$H_n \cap \big(F \times \{y\}\big) \neq \emptyset,$$

and thus $H_n^y \cap F$ is non-empty. Therefore, since $\mathscr{F}$ is an ultrafilter, we have $H_n^y \in \mathscr{F}$ for each $n \in \mathbb{N}$. Since $\mathscr{F}$ has c.i.p., we have

$$\bigcap_{n=1}^{\infty} H_n^y \neq \emptyset.$$

Now since

$$\Big(\bigcap_{n=1}^{\infty} H_n^y\Big) \times \{y\} \subseteq \bigcap_{n=1}^{\infty} H_n$$

it follows that

$$\bigcap_{n=1}^{\infty} H_n \neq \emptyset.$$

A similar argument can be used in the case when there is a free ultrafilter in $\mathscr{C}$ with c.i.p.

(2) *implies* (1). Let $\mathscr{H}$ be a free ultrafilter in $\mathscr{B} \times \mathscr{C}$ with c.i.p. Let

$$\mathscr{A} = \{B \in \mathscr{B} : B \times C \in \mathscr{H} \text{ for some } C \in \mathscr{C}\}.$$

Then $\mathscr{A}$ is a filter-base in $\mathscr{B}$, as $\emptyset \notin \mathscr{A}$ and it is closed under finite intersections. Let $\mathscr{F}$ be an ultrafilter in $\mathscr{B}$ such that $\mathscr{A} \subseteq \mathscr{F}$.

**Claim.** *For each $F \in \mathscr{F}$ we have $F \times Y \in \mathscr{H}$.*

*Proof of the claim.* Otherwise, if $F \times Y \notin \mathscr{H}$ for some $F \in \mathscr{F}$, then since $\mathscr{H}$ is an ultrafilter, we have $(F \times Y) \cap H = \emptyset$ for some $H \in \mathscr{H}$. Thus $H \subseteq (X \backslash F) \times Y$ and therefore $(X \backslash F) \times Y \in \mathscr{H}$. But this implies that $X\backslash F \in \mathscr{A}$, and thus $X\backslash F \in \mathscr{F}$, which is not possible, as it misses $F \in \mathscr{F}$.

Now, we show that $\mathscr{F}$ has c.i.p. Let $F_1, F_2, \ldots \in \mathscr{F}$. By the above $F_n \times Y \in \mathscr{H}$ for each $n \in \mathbb{N}$ and thus, since $\mathscr{H}$ has c.i.p.,

$$\bigcap_{n=1}^{\infty} (F_n \times Y) \neq \emptyset.$$

But

$$\bigcap_{n=1}^{\infty} (F_n \times Y) = \Big( \bigcap_{n=1}^{\infty} F_n \Big) \times Y.$$

Therefore

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

By our assumption $\mathscr{F}$ is not free, i.e., $\bigcap \mathscr{F}$ is non-empty. Let $p \in \bigcap \mathscr{F}$. Then $\{p\} \in \mathscr{B}$ meets each $F \in \mathscr{F}$, and thus, since $\mathscr{F}$ is an ultrafilter we have $\{p\} \in \mathscr{F}$. By the claim $\{p\} \times Y \in \mathscr{H}$. Similarly, there exists some $q \in Y$ such that $X \times \{q\} \in \mathscr{H}$. Now

$$\big\{(p,q)\big\} = \big(\{p\} \times Y\big) \cap \big(X \times \{q\}\big) \in \mathscr{H}$$

and thus, since $\{(p,q)\} \cap H$ is non-empty for each $H \in \mathscr{H}$, we have $(p,q) \in \bigcap \mathscr{H}$, which is a contradiction. □

Let $X$ be a completely regular topological space. Recall that *the $\sigma$-algebra of Baire subsets of* $X$ (denoted by $\mathscr{B}^*(X)$) is the smallest $\sigma$-algebra in $X$ containing $\mathscr{Z}(X)$. By a *Baire measure on* $X$ we mean a finite measure on $\mathscr{B}^*(X)$. The *support* of a Baire measure $\mu$ on $X$ is defined to be the set

$$\big\{x \in X : \mu(U) > 0 \text{ for every } U \in \mathrm{Coz}(X) \text{ such that } x \in U\big\}$$

and is denoted by $\mathrm{supp}(\mu)$.

The following Lemma is well known. (See [2].)

**Lemma 4.6.** *Let $X$ be a completely regular topological space. Then the following are equivalent:*

(1) *$X$ is realcompact.*
(2) *Each $\{0,1\}$-valued non-trivial Baire measure on $X$ has a non-empty support.*
(3) *Each ultrafilter in $\mathscr{Z}(X)$ with c.i.p. is fixed.*

**Theorem 4.7.** *Let $(X, \mathscr{O}, \mathscr{B})$ be a first-countable realcompact topological measurable space. Then there is no free ultrafilter in $\mathscr{B}$ with c.i.p.*

*Proof.* We prove the theorem in two different ways. Our first approach, which is rather direct, is more topological; our second approach makes use of the characterization given in Lemma 4.6.

*First approach.* Suppose to the contrary that there exists a free ultrafilter $\mathscr{F}$ in $\mathscr{B}$ with c.i.p. Note that the collection $\mathscr{B}$ of all measurable sets can be considered as a base for a topology on $X$, as it is closed under finite intersections and covers $X$. Denote by $\mathscr{O}_{\mathscr{B}}$ the topology generated by $\mathscr{B}$ on $X$. Since $(X, \mathscr{O})$ is Hausdorff and $\mathscr{O} \subseteq \mathscr{B}$, the topological space $(X, \mathscr{O}_{\mathscr{B}})$ is Hausdorff and therefore completely regular, as the elements of $\mathscr{B}$ are closed-open in $(X, \mathscr{O}_{\mathscr{B}})$. Let

$$\phi : \beta(X, \mathscr{O}_{\mathscr{B}}) \to \beta(X, \mathscr{O})$$

continuously extend

$$\mathrm{id}_X : (X, \mathscr{O}_{\mathscr{B}}) \to (X, \mathscr{O}).$$

Since

$$\{\mathrm{cl}_{\beta(X,\mathscr{O}_{\mathscr{B}})} F : F \in \mathscr{F}\}$$

has the finite intersection property, as $\mathscr{F}$ has, by compactness we have

$$G = \bigcap \{\mathrm{cl}_{\beta(X,\mathscr{O}_{\mathscr{B}})} F : F \in \mathscr{F}\} \neq \emptyset.$$

Let $p \in G$. Note that since $\mathscr{F}$ is free we have $p \notin X$, as otherwise

$$p \in X \cap \mathrm{cl}_{\beta(X,\mathscr{O}_{\mathscr{B}})} F = F$$

for each $F \in \mathscr{F}$.

**Claim.** *If $V$ is an open neighborhood of $\phi(p)$ in $\beta(X, \mathscr{O})$ then $V \cap X \in \mathscr{F}$.*

*Proof of the claim.* Suppose the contrary, i.e., suppose that $V \cap X \notin \mathscr{F}$. Note that $V \cap X \in \mathscr{O} \subseteq \mathscr{B}$. Since $\mathscr{F}$ is an ultrafilter, $V \cap X \cap F = \emptyset$ for some $F \in \mathscr{F}$. Since $V \cap X \in \mathscr{B}$ and $F \in \mathscr{B}$, the sets $V \cap X$ and $F$ are closed-open in $(X, \mathscr{O}_{\mathscr{B}})$, and therefore

$$\mathrm{cl}_{\beta(X,\mathscr{O}_{\mathscr{B}})}(V \cap X) \cap \mathrm{cl}_{\beta(X,\mathscr{O}_{\mathscr{B}})} F = \mathrm{cl}_{\beta(X,\mathscr{O}_{\mathscr{B}})}(V \cap X \cap F) = \emptyset.$$

By the choice of $p$ we have

$$p \notin \mathrm{cl}_{\beta(X,\mathscr{O}_{\mathscr{B}})}(V \cap X).$$

Let $W$ be an open neighborhood of $p$ in $\beta(X, \mathscr{O}_{\mathscr{B}})$ such that $W \cap V \cap X = \emptyset$. By continuity of $\phi$ there exists an open neighborhood $U$ of $p$ in $\beta(X, \mathscr{O}_{\mathscr{B}})$ such that $\phi(U) \subseteq V$. Then (since $\phi|X = \mathrm{id}_X$) we have

$$U \cap W \cap X = \phi(U \cap W \cap X) \subseteq \phi(U) \cap W \cap X \subseteq V \cap W \cap X = \emptyset.$$

But this is a contradiction, as $U \cap W$, being a non-empty open subset of $\beta(X, \mathscr{O}_{\mathscr{B}})$, meets $X$. Thus $V \cap X \in \mathscr{F}$.

Next, we show that $\phi(p) \notin X$. Suppose the contrary. By our assumption, there exists a countable base

$$\{V_n : n \in \mathbb{N}\}$$

at $\phi(p)$ in $(X, \mathscr{O})$. By the claim, $V_n \in \mathscr{F}$ for each $n \in \mathbb{N}$. Now, since $\mathscr{F}$ has c.i.p., for each $F \in \mathscr{F}$ we have

$$F \cap \big\{\phi(p)\big\} = F \cap \bigcap_{n=1}^{\infty} V_n \neq \emptyset.$$

Thus $\phi(p) \in \bigcap \mathscr{F}$, which is a contradiction, as $\mathscr{F}$ is free. Therefore $\phi(p) \in \beta(X, \mathscr{O}) \backslash X$. Since, by our assumption $(X, \mathscr{O})$ is realcompact, there exists a zero-set $Z$ in $\beta(X, \mathscr{O})$ such that $\phi(p) \in Z$ and $Z \cap X = \emptyset$. Let $Z = f^{-1}(0)$ for some continuous $f : \beta(X, \mathscr{O}) \to [0, 1]$. Now, for each $n \in \mathbb{N}$ the set $f^{-1}([0, 1/n))$ is an open neighborhood of $\phi(p)$ in $\beta(X, \mathscr{O})$, thus by the claim

$$f^{-1}\big([0, 1/n)\big) \cap X \in \mathscr{F}.$$

Since $\mathscr{F}$ has c.i.p. we have

$$Z \cap X = f^{-1}(0) \cap X = \bigcap_{n=1}^{\infty} f^{-1}\big([0, 1/n)\big) \cap X \neq \emptyset$$

which contradicts the choice of $Z$.

*Second approach.* Suppose to the contrary that there exists a free ultrafilter $\mathscr{F}$ in $\mathscr{B}$ with c.i.p. Define $\nu : \mathscr{B} \to \{0, 1\}$ such that $\nu(B) = 0$ if $B \notin \mathscr{F}$ and $\nu(B) = 1$ if $B \in \mathscr{F}$. Then (arguing as in the proof of Proposition 4.1 (3)⇒(1)) $\nu$ is a measure. Note that $\mathscr{B}$ contains the set $\mathscr{B}^*(X)$ of Baire subsets of $X$ (as each zero-set in $X$, being a $G_\delta$, is contained in $\mathscr{B}$). Denote

$$\mu = \nu|\mathscr{B}^*(X) : \mathscr{B}^*(X) \to \{0, 1\}.$$

Then $\mu$ is a non-trivial Baire measure on $X$, and since we are assuming that $X$ is realcompact, by Lemma 4.6 it has non-empty support. Let $x \in \mathrm{supp}(\mu)$ and let $\{C_n : n \in \mathbb{N}\}$ be a local base at $x$ in $X$. Without any loss of generality (since $X$ is completely regular) we may assume that $C_n \in \mathrm{Coz}(X)$ for each $n \in \mathbb{N}$ and $C_1 \supseteq C_2 \supseteq \cdots$. Then

$$\mu(C_n) \to \mu\Big(\bigcap_{n=1}^{\infty} C_n\Big).$$

But this is a contradiction, as (since $x \in \mathrm{supp}(\mu)$) $\mu(C_n) = 1$ for each $n \in \mathbb{N}$, and since $\mathscr{F}$ is free,

$$\bigcap_{n=1}^{\infty} C_n = \{x\} \notin \mathscr{F};$$

as otherwise, $x \in F$ for each $F \in \mathscr{F}$, as $\{x\} \cap F$ is non-empty, and thus (by our definition of $\nu$)

$$\mu\Big(\bigcap_{n=1}^{\infty} C_n\Big) = 0.$$

□

Obviously, every $n$-dimensional Euclidean space $\mathbb{R}^n$, where $n \in \mathbb{N}$, is realcompact. Thus, from Theorem 4.7 we obtain the following.

**Corollary 4.8.** *Let $(\mathbb{R}^n, \mathscr{M})$ be the Lebesgue measurable space, where $n \in \mathbb{N}$. Then there is no free ultrafilter in $\mathscr{M}$ with c.i.p.*

Recall that a cardinal $\zeta$ is said to be *measurable* if there is a non-trivial $\{0,1\}$-valued measure defined on the power set $\mathscr{P}(X)$ of a set $X$ of cardinality $\zeta$ which vanishes at singletons. The following is well known.

**Theorem 4.9.** *In the measurable space $(X, \mathscr{B})$, let $\mathscr{B} = \mathscr{P}(X)$. Then the following are equivalent:*

(1) *There is no free ultrafilter in $\mathscr{B}$ with c.i.p.*
(2) *$X$ is of non-measurable cardinality.*

Our next theorem is sort of converse to Theorem 4.7. We need, however, some definitions and some lemmas first.

Let $X$ be a completely regular topological space. For an open subset $U$ of $X$ the *extension of $U$ to $\beta X$* is defined by

$$\mathrm{Ex}_X U = \beta X \backslash \mathrm{cl}_{\beta X}(X \backslash U).$$

The following lemma is well known (see Lemma 7.1.13 of [1] or Lemma 3.1 of [6]).

**Lemma 4.10.** *Let $X$ be a completely regular topological space and let $U$ and $V$ be open subsets of $X$. Then*

(1) $X \cap \mathrm{Ex}_X U = U$, *and thus* $\mathrm{cl}_{\beta X}\mathrm{Ex}_X U = \mathrm{cl}_{\beta X} U$.
(2) $\mathrm{Ex}_X(U \cap V) = \mathrm{Ex}_X U \cap \mathrm{Ex}_X V$.

The following lemma is proved by E. G. Skljarenko in [5]. It is rediscovered by E. K. van Douwen in [6].

**Lemma 4.11.** *Let $X$ be a completely regular topological space and let $U$ be an open subset of $X$. Then*

$$\mathrm{bd}_{\beta X}\mathrm{Ex}_X U = \mathrm{cl}_{\beta X}\mathrm{bd}_X U.$$

**Lemma 4.12.** *Let $X$ be a completely regular topological space. If $B$ is a subset of $X$ with compact boundary then*

$$\mathrm{cl}_{\beta X} B \backslash X = \mathrm{Ex}_X(\mathrm{int}_X B)\backslash X.$$

*Proof.* Since $\mathrm{bd}_X B$ is compact, we have

$$\mathrm{cl}_{\beta X}\mathrm{bd}_X B \subseteq \mathrm{bd}_X B \subseteq X,$$

and therefore

$$\begin{aligned} \mathrm{cl}_{\beta X} B\backslash X &= \mathrm{cl}_{\beta X}(\mathrm{int}_X B \cup \mathrm{bd}_X B)\backslash X \\ &= (\mathrm{cl}_{\beta X}\mathrm{int}_X B \cup \mathrm{cl}_{\beta X}\mathrm{bd}_X B)\backslash X \\ &= (\mathrm{cl}_{\beta X}\mathrm{int}_X B\backslash X) \cup (\mathrm{cl}_{\beta X}\mathrm{bd}_X B\backslash X) = \mathrm{cl}_{\beta X}\mathrm{int}_X B\backslash X. \end{aligned}$$

By Lemma 4.10 we have

$$\begin{aligned} \mathrm{cl}_{\beta X}\mathrm{int}_X B\backslash X &= \mathrm{cl}_{\beta X}\mathrm{Ex}_X(\mathrm{int}_X B)\backslash X \\ &= \big(\mathrm{Ex}_X(\mathrm{int}_X B) \cup \mathrm{bd}_{\beta X}\mathrm{Ex}_X(\mathrm{int}_X B)\big)\backslash X \\ &= \big(\mathrm{Ex}_X(\mathrm{int}_X B)\backslash X\big) \cup \big(\mathrm{bd}_{\beta X}\mathrm{Ex}_X(\mathrm{int}_X B)\backslash X\big). \end{aligned}$$

But by Lemma 4.11, we have

$$\mathrm{bd}_{\beta X}\mathrm{Ex}_X(\mathrm{int}_X B) = \mathrm{cl}_{\beta X}\mathrm{bd}_X(\mathrm{int}_X B).$$

On the other hand $\mathrm{bd}_X(\mathrm{int}_X B) \subseteq \mathrm{bd}_X B$. Thus

$$\mathrm{cl}_{\beta X}\mathrm{bd}_X(\mathrm{int}_X B) \subseteq \mathrm{bd}_X B \subseteq X.$$

Combining these, we obtain the result. □

Note that if $U$ is an open subset of a completely regular topological space $X$, then (since $X$ is dense in $\beta X$) we have

$$\mathrm{cl}_{\beta X} U = \mathrm{cl}_{\beta X}(U \cap X).$$

We use this simple observation in the following.

**Theorem 4.13.** *Let $(X,\mathscr{O},\mathscr{B})$ be a completely regular topological measurable space. Suppose that each $B\in\mathscr{B}$ has compact boundary in $X$. If there is no free ultrafilter in $\mathscr{B}$ with c.i.p. then $X$ is realcompact.*

*Proof.* Suppose the contrary, i.e., suppose that $X$ is not realcompact. Then $X\neq \upsilon X$. Let $p\in\upsilon X\backslash X$. Let

$$\mathscr{F}=\{B\in\mathscr{B}:p\in \mathrm{cl}_{\beta X}B\}.$$

We show that $\mathscr{F}$ is a free ultrafilter in $\mathscr{B}$ with c.i.p., contradicting our assumption. Note that $\emptyset\neq\mathscr{F}\subseteq\mathscr{B}$ and $\emptyset\notin\mathscr{F}$. Suppose that $F,G\in\mathscr{F}$. Then, by Lemmas 4.10 and 4.12 we have

$$\begin{aligned} p &\in (\mathrm{cl}_{\beta X}F\cap \mathrm{cl}_{\beta X}G)\backslash X\\ &= (\mathrm{cl}_{\beta X}F\backslash X)\cap(\mathrm{cl}_{\beta X}G\backslash X)\\ &= \big(\mathrm{Ex}_X(\mathrm{int}_XF)\backslash X\big)\cap\big(\mathrm{Ex}_X(\mathrm{int}_XG)\backslash X\big)\\ &= \big(\mathrm{Ex}_X(\mathrm{int}_XF)\cap \mathrm{Ex}_X(\mathrm{int}_XG)\big)\backslash X\\ &= \mathrm{Ex}_X(\mathrm{int}_XF\cap\mathrm{int}_XG)\backslash X\\ &= \mathrm{Ex}_X\big(\mathrm{int}_X(F\cap G)\big)\backslash X=\mathrm{cl}_{\beta X}(F\cap G)\backslash X. \end{aligned}$$

Therefore $p\in\mathrm{cl}_{\beta X}(F\cap G)\backslash X$ and thus $F\cap G\in\mathscr{F}$. Next, suppose that $F\subseteq B$ for some $B\in\mathscr{B}$ and $F\in\mathscr{F}$. Then

$$p\in\mathrm{cl}_{\beta X}F\subseteq\mathrm{cl}_{\beta X}B$$

and thus $B\in\mathscr{F}$. This shows that $\mathscr{F}$ is a filter in $\mathscr{B}$. To show that $\mathscr{F}$ is an ultrafilter, let $B\in\mathscr{B}$ be such that $B\cap F$ is non-empty for each $F\in\mathscr{F}$. If $B\notin\mathscr{F}$ then $p\notin\mathrm{cl}_{\beta X}B$. Thus $p\in\mathrm{cl}_{\beta X}(X\backslash B)$, i.e., $X\backslash B\in\mathscr{F}$. But this is not possible, as $X\backslash B$ misses $B$. Therefore $B\in\mathscr{F}$. To show that $\mathscr{F}$ is free, let $x\in X$. Then (since $p\notin X$) there exist some disjoint open neighborhoods $U$ and $V$ of $p$ and $x$ in $\beta X$, respectively. Then

$$p\in\mathrm{cl}_{\beta X}U=\mathrm{cl}_{\beta X}(U\cap X)$$

and thus (since $U\cap X\in\mathscr{O}\subseteq\mathscr{B}$) we have $x\notin U\cap X\in\mathscr{F}$. Therefore $x\notin\bigcap\mathscr{F}$. Thus $\bigcap\mathscr{F}=\emptyset$ and $\mathscr{F}$ is free. To show that $\mathscr{F}$ has c.i.p., let $F_1,F_2,\ldots\in\mathscr{F}$. Then $p\in\mathrm{cl}_{\beta X}F_n$ for each $n\in\mathbb{N}$ and thus, since by Lemma 4.12 we have

$$\mathrm{cl}_{\beta X}F_n\backslash X=\mathrm{Ex}_X(\mathrm{int}_XF_n)\backslash X$$

it follows that $p\in\mathrm{Ex}_X(\mathrm{int}_XF_n)$. For each $n\in\mathbb{N}$, let $f_n:\beta X\to[0,1]$ be continuous and such that

$$f_n(p)=0 \text{ and } f_n|\big(\beta X\backslash\mathrm{Ex}_X(\mathrm{int}_XF_n)\big)\equiv 1.$$

Then

$$p \in Z = \bigcap_{n=1}^{\infty} \mathrm{Z}(f_n) \in \mathscr{Z}(\beta X).$$

If $Z \cap X = \emptyset$ then $Z \subseteq \beta X \backslash \upsilon X$, which is a contradiction, as $p \in \upsilon X$. Thus, using Lemma 4.10 we have

$$\begin{aligned} \emptyset \neq Z \cap X &= \bigcap_{n=1}^{\infty} \mathrm{Z}(f_n) \cap X \\ &\subseteq \bigcap_{n=1}^{\infty} \mathrm{Ex}_X(\mathrm{int}_X F_n) \cap X = \bigcap_{n=1}^{\infty} \mathrm{int}_X F_n \subseteq \bigcap_{n=1}^{\infty} F_n. \end{aligned}$$

Therefore

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

This show that $\mathscr{F}$ has c.i.p. $\square$

**Remark 4.14.** *In the case when $X$ is normal, using Lemma 4.6, one can give an alternative proof for Theorem 4.13. To show this assume that there exists a non-trivial $\{0,1\}$-valued Baire measure $\nu$ on $X$ whose support is empty. Let*

$$\mathscr{F} = \big\{B \in \mathscr{B} : \nu^*(X \backslash B) = 0\big\}$$

*in which $\nu^*$ is the outer measure induced by $\nu$. We verify that $\mathscr{F}$ is a free ultrafilter in $\mathscr{B}$ with c.i.p. Obviously, $\mathscr{F}$ is non-empty (as $X \in \mathscr{F}$) and $\emptyset \notin \mathscr{F}$ (as $\nu$ is non-trivial). Note that for any $F, G \in \mathscr{F}$ since*

$$\nu^*\big(X \backslash (F \cap G)\big) = \nu^*\big((X \backslash F) \cup (X \backslash G)\big) \leq \nu^*(X \backslash F) + \nu^*(X \backslash G) = 0$$

*we have $F \cap G \in \mathscr{F}$. Also, if $F \subseteq B$ for some $F \in \mathscr{F}$ and $B \in \mathscr{B}$ (since $X \backslash B \subseteq X \backslash F$) we have*

$$\nu^*(X \backslash B) \leq \nu^*(X \backslash F) = 0$$

*and thus $B \in \mathscr{F}$. Therefore $\mathscr{F}$ is a filter in $\mathscr{B}$. To show that $\mathscr{F}$ is an ultrafilter, let $B \in \mathscr{B}$ be such that $B \cap F$ is non-empty for each $F \in \mathscr{F}$. Since* $\mathrm{supp}(\nu) = \emptyset$, *for each $x \in X$ there exists some $U_x \in \mathrm{Coz}(X)$ such that $x \in U_x$ and $\nu(U_x) = 0$. By compactness of $\mathrm{bd}_X B$ there exist $x_1, \ldots, x_n \in X$ such that*

$$\mathrm{bd}_X B \subseteq \bigcup_{i=1}^{n} U_{x_i}.$$

*Let*

$$U=\bigcup_{i=1}^{n}U_{x_i}\in\mathrm{Coz}(X).$$

*Then*

$$\mathrm{cl}_X B\backslash U\subseteq \mathrm{cl}_X B\backslash \mathrm{bd}_X B\subseteq \mathrm{int}_X B$$

*and thus (since we are assuming that* $X$ *is normal) by the Urysohn Lemma there exists a continuous* $f:X\to[0,1]$ *such that*

$$f|(X\backslash \mathrm{int}_X B)\equiv 0 \text{ and } f|(\mathrm{cl}_X B\backslash U)\equiv 1.$$

*Now, if* $B\notin\mathscr{F}$ *(by the definition of* $\mathscr{F}$*) we have* $\nu^*(X\backslash B)=1$*. Thus, since* $X\backslash B\subseteq \mathrm{Z}(f)$ *we have*

$$\nu\big(\mathrm{Z}(f)\big)=\nu^*\big(\mathrm{Z}(f)\big)=1$$

*and therefore*

$$\nu\big(\mathrm{Coz}(f)\big)=1-\nu\big(\mathrm{Z}(f)\big)=0.$$

*Now, since*

$$B\subseteq \mathrm{cl}_X B\subseteq (\mathrm{cl}_X B\backslash U)\cup U\subseteq \mathrm{Coz}(f)\cup U=\mathrm{Coz}(f)\cup\bigcup_{i=1}^{n}U_{x_i}$$

*we have*

$$\nu^*\big(X\backslash(X\backslash B)\big)=\nu^*(B)\leq \nu\big(\mathrm{Coz}(f)\big)+\sum_{i=1}^{n}\nu(U_{x_i})=0.$$

*Therefore (by the definition of* $\mathscr{F}$*)* $X\backslash B\in\mathscr{F}$*, which is not possible, as it misses* $B$*. This contradiction shows that* $\mathscr{F}$ *is an ultrafilter. It remains to show that* $\mathscr{F}$ *has c.i.p. But this follows easily, as if* $F_1,F_2,\ldots\in\mathscr{F}$*, then since*

$$\nu^*\Big(X\backslash\bigcap_{i=1}^{\infty}F_i\Big)=\nu^*\Big(\bigcup_{i=1}^{\infty}(X\backslash F_i)\Big)\leq\sum_{i=1}^{\infty}\nu^*(X\backslash F_i)=0$$

*we have*

$$\bigcap_{i=1}^{\infty}F_i\in\mathscr{F}.$$

*Finally, note that* $\mathscr{F}$ *is free, as for each* $x\in X$ *since* $\nu(U_x)=0$ *(with* $U_x$ *as defined in the above) we have* $X\backslash U_x\in\mathscr{F}$*, and thus* $x\notin\bigcap\mathscr{F}$*.*

**Theorem 4.15.** *Let $(Y, \mathscr{U}, \mathscr{C})$ be a first-countable Hausdorff topological measurable space. Let $(X, \mathscr{B}) \subseteq (Y, \mathscr{C})$ and let $Y \backslash X$ be Lindelöf. If there is no free ultrafilter in $\mathscr{B}$ with c.i.p. then there is no free ultrafilter in $\mathscr{C}$ with c.i.p.*

*Proof.* For each $y \in Y$, let

$$\{V_n^y : n \in \mathbb{N}\}$$

be an open base at $y$ in $Y$. Suppose to the contrary that there exists a free ultrafilter $\mathscr{H}$ in $\mathscr{C}$ with c.i.p.

**Claim 1.** *For each $y \in Y$ there exist some $n_y \in \mathbb{N}$ and $H_y \in \mathscr{H}$ such that $V_{n_y}^y \cap H_y = \emptyset$.*

*Proof of Claim 1.* Suppose the contrary, i.e., suppose that for some $y \in Y$ the set $V_n^y \cap H$ is non-empty for each $n \in \mathbb{N}$ and $H \in \mathscr{H}$. Note that since $\mathscr{H}$ is an ultrafilter in $\mathscr{C}$ this implies that $V_n^y \in \mathscr{H}$ for each $n \in \mathbb{N}$. Since $Y$ is Hausdorff, we have

$$\bigcap_{n=1}^{\infty} V_n^y = \{y\},$$

and since $\mathscr{H}$ has c.i.p., we have

$$H \cap \{y\} = H \cap \bigcap_{n=1}^{\infty} V_n^y \neq \emptyset$$

i.e., $y \in H$ for each $H \in \mathscr{H}$, contradicting the fact that $\mathscr{H}$ is free. This shows Claim 1.

**Claim 2.** *$H \cap X$ is non-empty for each $H \in \mathscr{H}$.*

*Proof of Claim 2.* Suppose the contrary, i.e., suppose that $H \subseteq Y \backslash X$ for some $H \in \mathscr{H}$. Since

$$Y \backslash X \subseteq \bigcup \{V_{n_y}^y : y \in Y \backslash X\}$$

and $Y \backslash X$ is Lindelöf, we have

$$Y \backslash X \subseteq \bigcup_{i=1}^{\infty} V_{n_{y_i}}^{y_i}$$

for some $y_1, y_2, \ldots \in Y \backslash X$. Now, by Claim 1 we have

$$\begin{aligned} H \cap \bigcap_{j=1}^{\infty} H_{y_j} &\subseteq \Big(\bigcup_{i=1}^{\infty} V^{y_i}_{n_{y_i}}\Big) \cap \bigcap_{j=1}^{\infty} H_{y_j} \\ &= \bigcup_{i=1}^{\infty} \Big(V^{y_i}_{n_{y_i}} \cap \bigcap_{j=1}^{\infty} H_{y_j}\Big) \subseteq \bigcup_{i=1}^{\infty} (V^{y_i}_{n_{y_i}} \cap H_{y_i}) = \emptyset \end{aligned}$$

contrary to the fact that $\mathscr{H}$ has c.i.p.

Let

$$\mathscr{A} = \{H \cap X : H \in \mathscr{H}\}.$$

Then $\mathscr{A} \subseteq \mathscr{B}$ (as $(X, \mathscr{B}) \subseteq (Y, \mathscr{C})$ and $\mathscr{H} \subseteq \mathscr{C}$) and by Claim 2 we have $\emptyset \notin \mathscr{A}$. Since $\mathscr{A}$ is obviously closed under finite intersections, as $\mathscr{H}$ is so, $\mathscr{A}$ is a filter-base in $\mathscr{B}$. Let $\mathscr{F}$ be an ultrafilter in $\mathscr{B}$ such that $\mathscr{A} \subseteq \mathscr{F}$. Since $\mathscr{H}$ is free, we have

$$\bigcap \mathscr{F} \subseteq \bigcap \mathscr{A} = \bigcap \mathscr{H} \cap X = \emptyset$$

i.e., $\mathscr{F}$ also is free. To show that $\mathscr{F}$ has c.i.p., let $F_1, F_2, \ldots \in \mathscr{F}$. Let $F_n = C_n \cap X$ where $C_n \in \mathscr{C}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ and $H \in \mathscr{H}$, since $H \cap X \in \mathscr{A} \subseteq \mathscr{F}$, we have

$$\emptyset \neq H \cap X \cap F_n = H \cap X \cap C_n \subseteq H \cap C_n$$

and thus, since $\mathscr{H}$ is an ultrafilter in $\mathscr{C}$, we have $C_n \in \mathscr{H}$. Now, for each $H \in \mathscr{H}$, since $\mathscr{H}$ has c.i.p., we have

$$H \cap \bigcap_{n=1}^{\infty} C_n \neq \emptyset,$$

and therefore

$$\bigcap_{n=1}^{\infty} C_n \in \mathscr{H}.$$

By Claim 2 we have

$$\bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} C_n \cap X \neq \emptyset.$$

This shows that $\mathscr{F}$ is a free ultrafilter in $\mathscr{B}$ with c.i.p., which is a contradiction. □

Note that in the above proof we only need $Y$ to be first-countable at the points of $Y \backslash X$.

## 5. Examples of measure spaces $(X, \mathscr{B}, \mu)$ with an arbitrarily large number of free ultrafilters in $\mathscr{B}$ having c.i.p.

**Example 5.1.** *Let $\zeta$ be a cardinal. Then there exists a measure space $(Z, \mathscr{D}, \nu)$ having at least $\zeta$ free ultrafilters in $\mathscr{D}$ with c.i.p.*

*Proof.* Let $(X, \mathscr{B}, \mu)$ be a measure space in which $\mu$ is a non-trivial $\{0,1\}$-valued measure which (is defined and) vanishes at singletons. Let $(Y, \mathscr{C}, \lambda)$ be a $\sigma$-finite measure space in which singletons are measurable and such that $\mathrm{card}(Y) \geq \zeta$. By Proposition 4.1 there exists a free ultrafilter $\mathscr{F}$ in $\mathscr{B}$ with c.i.p. To see this, simply let

$$\mathscr{A} = \big\{\{x\} : x \in X\big\}$$

and observe that (since $\mu$ is non-trivial)

$$\mu\Big(\bigcup \mathscr{A}\Big) = \mu(X) = 1 \neq 0 = \sup_{x \in X} \mu(x) = \sup_{A \in \mathscr{A}} \mu(A).$$

For each $y \in Y$, let $\mathscr{H}_y$ be an ultrafilter in $\mathscr{B} \times \mathscr{C}$ such that

$$\big\{F \times \{y\} : F \in \mathscr{F}\big\} \subseteq \mathscr{H}_y.$$

By the proof of Theorem 4.5 the ultrafilter $\mathscr{H}_y$, for each $y \in Y$, is free and has c.i.p. Note that $\mathscr{H}_y$'s are distinct if $y \in Y$ are distinct. The measure space $(X \times Y, \mathscr{B} \times \mathscr{C}, \mu \times \lambda)$ has the desired property. $\square$

## 6. Questions

We conclude this article with the following questions.

**Question 6.1.** *In Theorem 4.7, does the converse hold? More precisely, for a first-countable topological measurable space $(X, \mathscr{O}, \mathscr{B})$ does the non-existence of any free ultrafilter in $\mathscr{B}$ with c.i.p. imply its realcompactness?*

**Question 6.2.** *It is known that each finite measure space can be embedded in a perfect measure space. Even more, each finite measure space can be embedded in a compact (in the sense of Marczewski) measure space. Find the corresponding choices of $\mathbb{U}$, $S_{\mathscr{U}}$ where $\mathscr{U} \in \mathbb{U}$, $(Z, \mathscr{D})$ and $\{\mathscr{D}_B : B \in \mathscr{B}\}$ in Theorem 2.2 for any such embeddings.*

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