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GENERALIZED FRAMES IN HILBERT SPACES

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Abstract. Here, we develop the generalized frame theory. We introduce two methods for generating g-frames of a Hilbert space $\mathcal{H}$. The first method uses bounded linear operators between Hilbert spaces. The second method uses bounded linear operators on ℓ_2 to generate g-frames of $\mathcal{H}$. We characterize all the bounded linear mappings that transform g-frames into other g-frames. We also characterize similar and unitary equivalent g-frames in term of the range of their linear analysis operators. Finally, we generalize the fundamental frame identity to g-frames and derive some new results.

1. Introduction

Through out this paper, $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces and $\{\mathcal{H}_i : i \in I\}$ is a sequence of separable Hilbert spaces, where I is a subset of $\mathbb{Z}$. $\mathcal{L}(\mathcal{H}, \mathcal{H}_i)$ is the collection of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}_i$, and $\mathbf{\Lambda} = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$, $\mathbf{\Theta} = \{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$.

$\mathbf{\Lambda}$ is called a generalized frame or simply g-frame of the Hilbert space $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ if for any vector $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \tag{1.1}$$

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where the *g-frame bounds* A and B are positive constants. $\mathbf{\Lambda}$ is called a *Parseval g-frame* of $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ if $A = B = 1$ in (1.1). We say a sequence $\{\,\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{K}) : i \in I\,\}$ is a g-frame of $\mathcal{H}$ with respect to $\mathcal{K}$ whenever $\mathcal{H}_i = \mathcal{K}$, for each $i \in I$. We also simply say a g-frame for $\mathcal{H}$ whenever the space sequence $\{\mathcal{H}_i : i \in I\}$ is clear. This notation has been introduced by W. Sun in [6]. It is an extension of frames that conclude all previous extensions of frames. Specially, if $\mathbf{\Lambda}$ is a g-frame of $\mathcal{H}$, then any vector $f \in \mathcal{H}$ can be represented as [6]:

$$f = \sum_{i\in I} \Lambda_i^* \Lambda_i S^{-1} f, \tag{1.2}$$

where S^{-1} is the inverse of the positive linear operator S on $\mathcal{H}$, defined by:

$$Sf := \sum_{i\in I} \Lambda_i^* \Lambda_i f. \tag{1.3}$$

S is called the *g-frame operator* for $\mathbf{\Lambda}$.

Definition 1.1. *Let $\mathbf{\Lambda}$ be a g-frame of $\mathcal{H}$. A g-frame $\mathbf{\Theta}$ of $\mathcal{H}$ is called a dual g-frame of $\mathbf{\Lambda}$ if it satisfies:*

$$f = \sum_{i\in I} \Lambda_i^* \Theta_i f, \quad \forall f \in \mathcal{H}.$$

It is easy to show that if $\mathbf{\Theta}$ is a dual g-frame of $\mathbf{\Lambda}$, then $\mathbf{\Lambda}$ will be a dual g-frame of $\mathbf{\Theta}$.

Let $\mathbf{\Lambda}$ be a g-frame of $\mathcal{H}$ with g-frame operator S. Then, (1.2) shows that $\{\,\Lambda_i S^{-1} \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ is a dual g-frame of $\mathbf{\Lambda}$. $\{\,\Lambda_i S^{-1} \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ is called *canonical dual* g-frame of $\mathbf{\Lambda}$. Among all dual g-frames of $\mathbf{\Lambda}$, the canonical dual g-frame has the following property [6].

Proposition 1.2. *Let $\mathbf{\Lambda}$ be a g-frame of $\mathcal{H}$ and $\Lambda_i^\circ = \Lambda_i S^{-1}$, for all $i \in I$. Then, for any $g_i \in \mathcal{H}_i$ satisfying $f = \sum_{i\in I} \Lambda_i^* g_i$, we have,*

$$\sum_{i\in I} \|g_i\|^2 = \sum_{i\in I} \|\Lambda_i^\circ f\|^2 + \sum_{i\in I} \|g_i - \Lambda_i^\circ f\|^2.$$

2. Mapping from $\mathcal{H}$ to $\mathcal{K}$ for the construction of g-frames

For a given g-frame $\mathbf{\Lambda}$ of $\mathcal{H}$, we will obtain g-frames of $\mathcal{K}$. One approach is to construct a sequence $\{\,\Theta_i = \Lambda_i U^* \in \mathcal{L}(\mathcal{K}, \mathcal{H}_i) : i \in I\,\}$, where U is a bounded linear operator from $\mathcal{H}$ to $\mathcal{K}$. The following theorem gives us a necessary and sufficient condition for $\{\,\Theta_i = \Lambda_i U^* \in \mathcal{L}(\mathcal{K}, \mathcal{H}_i) : i \in I\,\}$ to be a g-frame of $\mathcal{K}$.

The following results generalize the results in Aldroubi [1] in the case of g-frames with analogous proofs, and we omit the details.

Theorem 2.1. *Let $\mathbf{\Lambda}$ be a g-frame of $\mathcal{H}$ with g-frame bounds A and B satisfying $0 < A \leq B < \infty$. If $U : \mathcal{H} \to \mathcal{K}$ is a bounded linear operator, then $\{\,\Lambda_i U^* \in \mathcal{L}(\mathcal{K}, \mathcal{H}_i) : i \in I\,\}$ is a g-frame of $\mathcal{K}$ if and only if there exists $\delta > 0$ such that for any $f \in \mathcal{K}$, $\|U^* f\| \geq \delta \|f\|$.*

Corollary 2.2. *Let $\mathbf{\Lambda}$ be a g-frame of $\mathcal{H}$ with g-frame operator S. If $K \subseteq \mathcal{H}$ is a closed subspace and if $P : \mathcal{H} \to K$ is the orthogonal projection, then $\{\,\Lambda_i P \in \mathcal{L}(K, \mathcal{H}_i) : i \in I\,\}$ and $\{\,\Lambda_i S^{-1} P \in \mathcal{L}(K, \mathcal{H}_i) : i \in I\,\}$ are dual g-frames of K. Moreover, the g-frame bounds A and B of $\mathbf{\Lambda}$ are also g-frame bounds for $\{\,\Lambda_i P \in \mathcal{L}(K, \mathcal{H}_i) : i \in I\,\}$.*

Corollary 2.3. *Let $\mathbf{\Lambda}$ be a g-frame for $\mathcal{H}$ with g-frame bounds satisfying $0 < A \leq B < \infty$. If $U : \mathcal{H} \to \mathcal{K}$ is co-isometry, then $\{\,\Lambda_i U^* \in \mathcal{L}(\mathcal{K}, \mathcal{H}_i) : i \in I\,\}$ is a g-frame for $\mathcal{K}$ with the same bounds.*

3. Mapping on ℓ_2 for the construction of g-frames of $\mathcal{H}$

Let $\{\,\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{K}) : i \in I\,\}$ be a g-frame of $\mathcal{H}$. We want to know which conditions on the numbers $(u_{ij})_{i,j \in I}$ will imply that the linear operators,

$$\Theta_i : \mathcal{H} \to \mathcal{K}, \quad \Theta_i f = \sum_{j \in I} u_{ij} \Lambda_j f, \quad \forall i \in I, \tag{3.1}$$

are well defined and constitute a g-frame for $\mathcal{H}$. Aldroubi in [1] has answered this question about frames.

Notation 3.1. Let us define,

$$L^2(\mathcal{H}, I) = \Big\{ \{f_i\}_{i \in I} : f_i \in \mathcal{H} \quad \text{and} \quad \sum_{i \in I} \|f_i\|^2 < \infty \Big\},$$

with the inner product given by $\big\langle \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \big\rangle = \sum_{i \in I} \langle f_i, g_i \rangle$. It is clear that $L^2(\mathcal{H}, I)$ is a separable Hilbert space with respect to the pointwise operations.

Definition 3.2. Let $\mathbf{\Lambda}$ be a g-frame for $\mathcal{H}$. The *synthesis operator* for $\mathbf{\Lambda}$ is the linear operator,

$$T_{\mathbf{\Lambda}} : \Big(\sum_{i\in I} \bigoplus \mathcal{H}_i\Big)_{\ell_2} \to \mathcal{H}, \quad T_{\mathbf{\Lambda}}(\{f_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^*(f_i).$$

We call the adjoint $T_{\mathbf{\Lambda}}^*$ of the synthesis operator, the *analysis operator*. The analysis operator is the linear operator,

$$T_{\mathbf{\Lambda}}^* : \mathcal{H} \to \Big(\sum_{i\in I} \bigoplus \mathcal{H}_i\Big)_{\ell_2}, \quad T_{\mathbf{\Lambda}}^*(f) = \{\Lambda_i f\}_{i\in I}.$$

Throughout this paper, for a given g-frame $\mathbf{\Lambda}$ of $\mathcal{H}$, we denote by $T_{\mathbf{\Lambda}}$ and $T_{\mathbf{\Lambda}}^*$, respectively, the synthesis and analysis operators for $\mathbf{\Lambda}$.

The following proposition is similar to a result of [1] with an analogous proof, and we omit the details.

Proposition 3.3. *Let $\{\, \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{K}) : i \in I \,\}$ be a g-frame of $\mathcal{H}$ and assume that the bi-infinite matrix $U = (u_{ij})_{i,j\in I}$ defines a bounded linear operator on $L^2(\mathcal{K}, I)$. Then, the linear operators $\{\, \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{K}) : i \in I \,\}$ in (3.1) are well defined and constitute a g-frame for $\mathcal{H}$ if and only if there exists a constant $\delta > 0$ such that*

$$\|Ux\|_2^2 \geq \delta \|x\|_2^2, \quad \forall x \in \mathcal{R}_{T_\Lambda^*}, \tag{3.2}$$

where T_Λ is the synthesis operator of $\{\, \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{K}) : i \in I \,\}$.

The condition (3.2) can also be written as:

$$\sum_{i\in I} \Big\| \sum_{j\in J} u_{ij} \Lambda_j f \Big\|^2 \geq \delta \sum_{j\in J} \|\Lambda_j f\|^2, \quad \forall f \in \mathcal{H}.$$

The proof of the next proposition is a modification of the analogous proof for frames [see 4, Prop. 5.5.8].

Proposition 3.4. *Let $\{\, \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{K}) : i \in I \,\}$ be a g-frame of $\mathcal{H}$ with g-frame bounds A and B. If the numbers $(u_{ij})_{i,j\in I}$ satisfy the two* *conditions,*

$$b := \sup_{k\in I} \sum_{j\in I} \Big| \sum_{i\in I} u_{ik}\overline{u_{ij}} \Big| < \infty,$$
$$a := \inf_{k\in I} \left(\sum_{i\in I} |u_{ik}|^2 - \sum_{j\neq k} \Big| \sum_{i\in I} u_{ik}\overline{u_{ij}} \Big| \right) > 0,$$

then $\{\, \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{K}) : i \in I \,\}$ *defined by (3.1) is a g-frame of* $\mathcal{H}$ *with g-frame bounds* aA *and* bB*.*

Proof. Let $f \in \mathcal{H}$. Then,

$$\begin{aligned}\sum_{i\in I} \Big\| \sum_{j\in I} u_{ij}\Lambda_j f \Big\|^2 &= \sum_{i\in I} \Big(\sum_{k\in I} \sum_{j\in I} u_{ij}\overline{u_{ik}} \langle \Lambda_j f, \Lambda_k f\rangle \Big) \\ &= \sum_{i\in I} \sum_{k\in I} |u_{ik}|^2 \|\Lambda_k f\|^2 + \sum_{i\in I} \sum_{k\in I} \sum_{j\neq k} u_{ij}\overline{u_{ik}} \langle \Lambda_j f, \Lambda_k f\rangle.\end{aligned}$$

Let

$$(*) := \sum_{i\in I} \sum_{k\in I} \sum_{j\neq k} u_{ij}\overline{u_{ik}} \langle \Lambda_j f, \Lambda_k f\rangle.$$

Then, by Cauchy-Schwarz's inequality we get,

$$|(*)| \leq \sum_{k\in I} \sum_{j\neq k} \|\Lambda_k f\|^2 \Big| \sum_{i\in I} u_{ij}\overline{u_{ik}} \Big|.$$

Therefore,

$$\begin{aligned}\sum_{i\in I} \Big\| \sum_{j\in I} u_{ij}\Lambda_j f \Big\|^2 &\geq \sum_{i\in I} \sum_{k\in I} |u_{ik}|^2 \|\Lambda_k f\|^2 - |(*)| \\ &\geq \sum_{k\in I} \|\Lambda_k f\|^2 \left(\sum_{i\in I} |u_{ik}|^2 - \sum_{j\neq k} \Big| \sum_{i\in I} u_{ik}\overline{u_{ij}} \Big| \right). \\ &\geq aA\|f\|^2.\end{aligned}$$

For the upper g-frame bound we have,

$$\begin{aligned}\sum_{i\in I}\Big\|\sum_{j\in I}u_{ij}\Lambda_j f\Big\|^2 &\le \sum_{i\in I}\sum_{k\in I}|u_{ik}|^2\|\Lambda_k f\|^2 + |(*)| \\ &\le \sum_{k\in I}\|\Lambda_k f\|^2\left(\sum_{i\in I}|u_{ik}|^2 + \sum_{j\neq k}\Big|\sum_{i\in I}u_{ik}\overline{u_{ij}}\Big|\right) \\ &= \sum_{k\in I}\|\Lambda_k f\|^2\sum_{j\in I}\Big|\sum_{i\in I}u_{ik}\overline{u_{ij}}\Big| \\ &\le bB\|f\|^2.\end{aligned}$$

The converse of Proposition 3.3 is also true; i.e., any two g frames of $\mathcal{H}$ are related by a linear operator U on $\left(\sum_{i\in I}\bigoplus\mathcal{H}_i\right)_{\ell_2}$ that satisfies condition (3.2).

Proposition 3.5. *Let* $\mathbf{\Lambda}$ *and* $\mathbf{\Theta}$ *be two g-frames for* $\mathcal{H}$*. Then, there is a bounded linear operator,*

$$U:\Big(\sum_{i\in I}\bigoplus\mathcal{H}_i\Big)_{\ell_2}\to\Big(\sum_{i\in I}\bigoplus\mathcal{H}_i\Big)_{\ell_2},$$

such that for any $f\in\mathcal{H}$*,*

$$T^*_{\mathbf{\Theta}}f = UT^*_{\mathbf{\Lambda}}f.$$

Proof. Since $\mathbf{\Lambda}$ is a g-frame, then $X = T^*_{\mathbf{\Lambda}}(\mathcal{H})$ is a closed subspace of $\left(\sum_{i\in I}\bigoplus\mathcal{H}_i\right)_{\ell_2}$. Therefore, $T^*_{\mathbf{\Lambda}}:\mathcal{H}\to X$ is bijective and so it is invertible. By the open mapping Theorem, $(T^*_{\mathbf{\Lambda}})^{-1}$ is bounded. Let $U_0 = T^*_{\mathbf{\Theta}}(T^*_{\mathbf{\Lambda}})^{-1}: X\to\left(\sum_{i\in I}\bigoplus\mathcal{H}_i\right)_{\ell_2}$. It is obvious that U_0 is bounded on X and we can extend it on $\left(\sum_{i\in I}\bigoplus\mathcal{H}_i\right)_{\ell_2}$ by:

$$U(x) := \begin{cases} U_0(x) & \text{if } x\in X \\ 0 & \text{if } x\in X^{\perp}\end{cases}$$

Then, for each $f\in\mathcal{H}$ we have,

$$UT^*_{\mathbf{\Lambda}}f = T^*_{\mathbf{\Theta}}(T^*_{\mathbf{\Lambda}})^{-1}T^*_{\mathbf{\Lambda}}f = T^*_{\mathbf{\Theta}}f.$$

4. **Similar and unitary equivalence g-frames**

The definitions of similar and unitary equivalent frames give rise to definitions of similar and unitary equivalent g-frames.

Definition 4.1. Let $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ be two g-frames of $\mathcal{H}$.

(1) We say that $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ are *similar* if there is a bounded linear invertible operator $T:\mathcal{H}\to\mathcal{H}$ such that $\Theta_i=\Lambda_i T$, for all $i\in I$.
(2) We say that $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ are *unitary equivalent* if there is a unitary linear operator $T:\mathcal{H}\to\mathcal{H}$ such that $\Theta_i=\Lambda_i T$, for all $i\in I$.
(3) We say that $\mathbf{\Lambda}$ is *isometrically equivalent* to $\mathbf{\Theta}$ if there is an isometric linear operator $T:\mathcal{H}\to\mathcal{H}$ such that $\Theta_i=\Lambda_i T$, for all $i\in I$.

The following proposition characterizes unitary equivalence Parseval g-frames. It generalizes a result of Balan[2] in the case of Parseval g-frames

Proposition 4.2. *Let $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ be two Parseval g-frames of $\mathcal{H}$. Then,* (i) $\mathcal{R}_{T^*_{\mathbf{\Theta}}}\subseteq\mathcal{R}_{T^*_{\mathbf{\Lambda}}}$ *if and only if $\mathbf{\Lambda}$ is isometrically equivalent to $\mathbf{\Theta}$. Furthermore, if $U:\mathcal{H}\to\mathcal{H}$ is an isometry such that $\Theta_i=\Lambda_i U$, for $i\in I$, then,*

$$\ker U^*=T_{\mathbf{\Lambda}}(\mathcal{R}_{T^*_{\mathbf{\Lambda}}}\cap(\mathcal{R}_{T^*_{\mathbf{\Theta}}})^{\perp}). \tag{4.1}$$

(ii) $\mathcal{R}_{T^*_{\mathbf{\Theta}}}=\mathcal{R}_{T^*_{\mathbf{\Lambda}}}$ *if and only if $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ are unitary equivalent.*

Proof. (i) (*Necessity*). Suppose that $\mathcal{R}_{T^*_{\mathbf{\Theta}}}\subseteq\mathcal{R}_{T^*_{\mathbf{\Lambda}}}$. It is clear that $\mathcal{R}_{T^*_{\mathbf{\Theta}}}$ and $\mathcal{R}_{T^*_{\mathbf{\Lambda}}}$ are closed subspaces of $\left(\sum_{i\in I}\bigoplus\mathcal{H}_i\right)_{\ell_2}$. Let $P=T^*_{\mathbf{\Lambda}}T_{\mathbf{\Lambda}}$ and $Q=T^*_{\mathbf{\Theta}}T_{\mathbf{\Theta}}$. So, $\mathcal{R}_P=\mathcal{R}_{T^*_{\mathbf{\Lambda}}}$ and $\mathcal{R}_Q=\mathcal{R}_{T^*_{\mathbf{\Theta}}}$. Since $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ are Parseval g-frames of $\mathcal{H}$, then P and Q are orthogonal projections from $\left(\sum_{i\in I}\bigoplus\mathcal{H}_i\right)_{\ell_2}$ onto $\mathcal{R}_{T^*_{\mathbf{\Lambda}}}$ and $\mathcal{R}_{T^*_{\mathbf{\Theta}}}$, respectively. Let $U:=T_{\mathbf{\Lambda}}T^*_{\mathbf{\Theta}}:\mathcal{H}\to\mathcal{H}$ and let $f\in\mathcal{H}$. Then,

$$U^*Uf=T_{\mathbf{\Theta}}T^*_{\mathbf{\Lambda}}T_{\mathbf{\Lambda}}T^*_{\mathbf{\Theta}}f=T_{\mathbf{\Theta}}T^*_{\mathbf{\Theta}}f=f.$$

Hence, U is an isometry. Also, $f=\sum_{i\in I}\Theta_i^*\Theta_i f$ and

$$U^*f=T_{\mathbf{\Theta}}T^*_{\mathbf{\Lambda}}f=T_{\mathbf{\Theta}}(\{\Lambda_i f\}_{i\in I})=\sum_{i\in I}\Theta_i^*\Lambda_i f.$$

So, $f = \sum_i \Theta_i^* \Lambda_i U f$. By Proposition 1.2,

$$\sum_{i\in I} \|\Lambda_i U f\|^2 = \sum_{i\in I} \|\Theta_i f\|^2 + \sum_{i\in I} \|\Lambda_i U f - \Theta_i f\|^2.$$

Since $\sum_{i\in I} \|\Lambda_i U f\|^2 = \|Uf\|^2 = \|f\|^2$ and $\sum_{i\in I} \|\Theta_i f\|^2 = \|f\|^2$, then $\Lambda_i U f = \Theta_i f$, for any $i \in I$. Therefore, $\Lambda_i U = \Theta_i$ and consequently $T_{\mathbf{\Lambda}}^* U = T_{\mathbf{\Theta}}^*$.

(*Sufficiency*). Suppose that there exists an isometry $U : \mathcal{H} \to \mathcal{H}$ such that $\Theta_i = \Lambda_i U$, for $i \in I$. Then, for any $f \in \mathcal{H}$,

$$T_{\mathbf{\Theta}}^* f = \{\Theta_i f\}_{i\in I} = \{\Lambda_i U f\}_{i\in I} = T_{\mathbf{\Lambda}}^* U f,$$

and so $T_{\mathbf{\Theta}}^* = T_{\mathbf{\Lambda}}^* U$. Therefore, $\mathcal{R}_{T_{\mathbf{\Theta}}^*} \subseteq \mathcal{R}_{T_{\mathbf{\Lambda}}^*}$.

For the second part of (i), since $T_{\mathbf{\Lambda}}^*$ is isometric, then we have,

$$\mathcal{R}_{T_{\mathbf{\Lambda}}^*} = T_{\mathbf{\Lambda}}^*(\ker U^* \oplus \mathcal{R}_U) = T_{\mathbf{\Lambda}}^*(\ker U^*) \oplus \mathcal{R}_{T_{\mathbf{\Lambda}}^* U}.$$

So, we have,

$$\ker U^* = T_{\mathbf{\Lambda}}(\mathcal{R}_{T_{\mathbf{\Lambda}}^*} \cap (\mathcal{R}_{T_{\mathbf{\Theta}}^*})^{\perp}). \tag{4.2}$$

(ii) If $\mathcal{R}_{T_{\mathbf{\Theta}}^*} = \mathcal{R}_{T_{\mathbf{\Lambda}}^*}$, then by (4.2) we obtain that U is invertible. Since U is isometric, then U is unitary. Conversely, let $U : \mathcal{H} \to \mathcal{H}$ be a unitary linear operator such that $\Theta_i = \Lambda_i U$, for $i \in I$. Since $T_{\mathbf{\Lambda}}^* U = T_{\mathbf{\Theta}}^*$, then $\mathcal{R}_{T_{\mathbf{\Theta}}^*} = \mathcal{R}_{T_{\mathbf{\Lambda}}^*}$.

For the general case, we have the following proposition.

Proposition 4.3. *Let $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ be two g-frames of $\mathcal{H}$. Then,*
(i) $\mathcal{R}_{T_{\mathbf{\Theta}}^*} \subseteq \mathcal{R}_{T_{\mathbf{\Lambda}}^*}$ *if and only if there exists a bounded linear operator $U : \mathcal{H} \to \mathcal{H}$ such that $\Theta_i = \Lambda_i U$ for $i \in I$. Furthermore (4.1) holds.*
(ii) $\mathcal{R}_{T_{\mathbf{\Theta}}^*} = \mathcal{R}_{T_{\mathbf{\Lambda}}^*}$ *if and only if $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ are similar.*

Proof. Let us denote by $S_{\mathbf{\Lambda}}$ and $S_{\mathbf{\Theta}}$ the g-frame operators of $\mathbf{\Lambda}$ and $\mathbf{\Theta}$, respectively.

(i) (*Necessity*). Suppose that $\mathcal{R}_{T_{\mathbf{\Theta}}^*} \subseteq \mathcal{R}_{T_{\mathbf{\Lambda}}^*}$. Since $\mathcal{R}_{T_{\mathbf{\Lambda}}^*}$ and $\mathcal{R}_{T_{\mathbf{\Theta}}^*}$ are closed subspaces of $\left(\sum_{i\in I} \bigoplus \mathcal{H}_i\right)_{\ell_2}$, then $\mathcal{R}_{T_{\mathbf{\Lambda}}^*} = [\ker T_{\mathbf{\Lambda}}]^{\perp}$ and $\mathcal{R}_{T_{\mathbf{\Theta}}^*} = [\ker T_{\mathbf{\Theta}}]^{\perp}$. Therefore, $\ker T_{\mathbf{\Lambda}} \subseteq \ker T_{\mathbf{\Theta}}$. Denote by $T_{\mathbf{\Lambda}'}$ and $T_{\mathbf{\Theta}'}$, the synthesis operators for Parseval g-frames $\mathbf{\Lambda}' := \{\Lambda_i' = \Lambda_i S_{\mathbf{\Lambda}}^{-\frac{1}{2}}\}_{i\in I}$ and $\mathbf{\Theta}' := \{\Theta_i' = \Theta_i S_{\mathbf{\Theta}}^{-\frac{1}{2}}\}_{i\in I}$, respectively. Then, $T_{\mathbf{\Theta}'} = S_{\mathbf{\Theta}}^{-\frac{1}{2}} T_{\mathbf{\Theta}}$ and $T_{\mathbf{\Lambda}'} = S_{\mathbf{\Lambda}}^{-\frac{1}{2}} T_{\mathbf{\Lambda}}$. So, $\ker T_{\mathbf{\Theta}'} = \ker T_{\mathbf{\Theta}}$ and $\ker T_{\mathbf{\Lambda}'} = \ker T_{\mathbf{\Lambda}}$. By Proposition

4.2, there exists an isometry $V : \mathcal{H} \to \mathcal{H}$ such that $\Theta'_i = \Lambda'_i V$, for all $i \in I$. Hence $\Theta_i = \Lambda_i S_{\mathbf{\Lambda}}^{-\frac{1}{2}} V S_{\mathbf{\Theta}}^{\frac{1}{2}}$. Therefore, the result follows by letting $U = S_{\mathbf{\Lambda}}^{-\frac{1}{2}} V S_{\mathbf{\Theta}}^{\frac{1}{2}}$.
(*Sufficiency.*) It is straightforward.

For the second part of (i), by Proposition 4.2, we have,

$$\ker U^* = S_{\mathbf{\Lambda}}^{\frac{1}{2}} \ker V^* = S_{\mathbf{\Lambda}}^{\frac{1}{2}} T_{\mathbf{\Lambda}'}(\mathcal{R}_{T^*_{\mathbf{\Lambda}'}} \cap (\mathcal{R}_{T^*_{\mathbf{\Theta}'}})^{\perp}) = T_{\mathbf{\Lambda}}(\mathcal{R}_{T^*_{\mathbf{\Lambda}}} \cap (\mathcal{R}_{T^*_{\mathbf{\Theta}}})^{\perp}).$$

(ii) (*Necessity*). If $\mathcal{R}_{T^*_{\mathbf{\Theta}}} = \mathcal{R}_{T^*_{\mathbf{\Lambda}}}$, then $\mathcal{R}_{T^*_{\mathbf{\Theta}'}} = \mathcal{R}_{T^*_{\mathbf{\Lambda}'}}$. Therefore, by part ($ii$) of Proposition 4.2, V is unitary and consequently U^* is onto. Hence, $\ker U = [\mathcal{R}_{U^*}]^{\perp} = \{0\}$. Also, by part ($i$), U is onto. Therefore, U is invertible.
(*Sufficiency*). It is straightforward.

Proposition 4.4. *Let* $\mathbf{\Lambda}$ *be a g-frame of* $\mathcal{H}$ *and let* $\mathbf{\Theta}$ *be a sequence of bounded linear operators. Suppose that there exist constants* $\lambda, \mu \in [0, 1)$ *such that*

$$\Big\| \sum_{i\in I} \Theta_i^* f_i - \Lambda_i^* f_i \Big\| \le \lambda \Big\| \sum_{i\in I} \Lambda_i^* f_i \Big\| + \mu \Big\| \sum_{i\in I} \Theta_i^* f_i \Big\|, \tag{4.3}$$

for any $\{f_i\}_{i\in I} \in \left(\sum_{i\in I} \bigoplus \mathcal{H}_i \right)_{\ell_2}$. *Then,*
(i) $\mathbf{\Theta}$ *is a g-frame of* $\mathcal{H}$.
(ii) $\mathbf{\Lambda}$ *and* $\mathbf{\Theta}$ *are similar.*

Proof. (i) See [5].
(ii) It is clear that $\ker T_{\mathbf{\Theta}} = \ker T_{\mathbf{\Lambda}}$. Therefore, (ii) follows from Proposition 4.3.

5. g-frame identity

Here, we generalize the frame identity to the situation of g-frames. We also give some results related to g-frame identity. The following identity has been introduced in [3].

Theorem 5.1. *Let* $\{f_i\}_{i\in I}$ *be a Parseval frame of* $\mathcal{H}$. *For any* $J \subseteq I$ *and all* $f \in \mathcal{H}$, *we have:*

$$\sum_{i\in J} |\langle f, f_i\rangle|^2 - \Big\| \sum_{i\in J} \langle f, f_i\rangle f_i \Big\|^2 = \sum_{i\in J^c} |\langle f, f_i\rangle|^2 - \Big\| \sum_{i\in J^c} \langle f, f_i\rangle f_i \Big\|^2.$$

Let $\mathbf{\Lambda}$ be a g-frame of $\mathcal{H}$. For any $J \subseteq I$, let $S_J : \mathcal{H} \to \mathcal{H}$ be a linear operator defined by:

$$S_J(f) = \sum_{i \in J} \Lambda_i^* \Lambda_i f, \quad \forall f \in \mathcal{H}.$$

We begin with the following key lemma.

Lemma 5.2. *Suppose that $T : \mathcal{H} \to \mathcal{H}$ is a bounded and self-adjoint linear operator. Let $a, b, c \in \mathbb{R}$ and $U = aT^2 + bT + cI$.*
(i) *If $a > 0$, then*

$$\inf_{\|f\|=1} \langle Uf, f \rangle \geq \frac{4ac - b^2}{4a}.$$

(ii) *If $a < 0$, then*

$$\sup_{\|f\|=1} \langle Uf, f \rangle \leq \frac{4ac - b^2}{4a}.$$

Proof. (i) By elementary computations, we have,

$$U = a\Big(T + \frac{b}{2a} I\Big)^2 + \frac{4ac - b^2}{4a} I.$$

Since $(T + \frac{b}{2a} I)^2 \geq 0$, then

$$U \geq \frac{4ac - b^2}{4a} I.$$

Then, for all $f \in \mathcal{H}$,

$$\langle Uf, f \rangle \geq \frac{4ac - b^2}{4a} \|f\|^2.$$

So,

$$\inf_{\|f\|=1} \langle Uf, f \rangle \geq \frac{4ac - b^2}{4a}.$$

(ii) It follows from (i).

The following theorem generalizes Theorem 5.1 for Parseval g-frames.

Theorem 5.3. *Let $\mathbf{\Lambda}$ be a Parseval g-frame of $\mathcal{H}$. Then, for any $J \subseteq I$ and all $f \in \mathcal{H}$, we have:*

$$\sum_{i \in J} \|\Lambda_i f\|^2 + \|S_{J^c} f\|^2 = \sum_{i \in J^c} \|\Lambda_i f\|^2 + \|S_J f\|^2, \tag{5.1}$$

$$\sum_{i \in J} \|\Lambda_i f\|^2 + \|S_{J^c} f\|^2 \geq \frac{3}{4}\|f\|^2, \tag{5.2}$$

$$0 \leq S_J - S_J^2 \leq \frac{1}{4} I, \tag{5.3}$$

$$\frac{1}{2} I \leq S_J^2 + S_{J^c}^2 \leq \frac{3}{2} I. \tag{5.4}$$

Proof. Let $S : \mathcal{H} \to \mathcal{H}$ be the g-frame operator of $\mathbf{\Lambda}$. Then, $S = I$ and $f = S_J f + S_{J^c} f$, for all $f \in \mathcal{H}$. Let $f \in \mathcal{H}$. Then,

$$\begin{aligned}
\sum_{i \in J} \|\Lambda_i f\|^2 + \|S_{J^c} f\|^2 &= \langle S_J f, f\rangle + \langle S_{J^c} f, S_{J^c} f\rangle \\
&= \Big\langle (S_J + S_{J^c}^2) f, f \Big\rangle \\
&= \Big\langle (I - S_J + S_J^2) f, f \Big\rangle \\
&= \langle (S_{J^c} + S_J^2) f, f\rangle \\
&= \sum_{i \in J^c} \|\Lambda_i f\|^2 + \|S_J f\|^2.
\end{aligned}$$

This proves (5.1). Since $\sum_{i \in J} \|\Lambda_i f\|^2 + \|S_{J^c} f\|^2 = \langle (S_J^2 - S_J + I) f, f\rangle$, then the inequality (5.2) follows by Lemma 5.2. To prove (5.3), we have $S_J S_{J^c} = S_{J^c} S_J$, for all $J \subseteq I$. So $0 \leq S_J S_{J^c} = S_J - S_J^2$. Also, by Lemma 5.2 we have $S_J - S_J^2 \leq \frac{1}{4} I$.

To prove (5.4), we have $S_J^2 + S_{J^c}^2 = I - 2 S_J S_{J^c} = 2 S_J^2 - 2 S_J + I$, for all $J \subseteq I$. By Lemma 5.2, we get that $S_J^2 + S_{J^c}^2 \geq \frac{1}{2} I$. Since $S_J - S_J^2 \geq 0$ (by (5.3)), then we have $S_J^2 + S_{J^c}^2 \leq I + 2 S_J - 2 S_J^2$. So, the result follows from Lemma 5.2.

Corollary 5.4. *Let* $\mathbf{\Lambda}$ *be a g-frame of* $\mathcal{H}$ *with g-frame operator* S*. Then, for any* $J \subseteq I$ *and for all* $f \in \mathcal{H}$*, we have,*

(5.5)
$$\sum_{i\in J}\|\Lambda_i f\|^2 + \sum_{i\in I}\|\Lambda_i S^{-1}S_{J^c}f\|^2 = \sum_{i\in J^c}\|\Lambda_i f\|^2 + \sum_{i\in I}\|\Lambda_i S^{-1}S_J f\|^2,$$

(5.6)
$$\sum_{i\in J}\|\Lambda_i f\|^2 + \sum_{i\in I}\|\Lambda_i S^{-1}S_{J^c}f\|^2 \geq \frac{3}{4}\langle Sf, f\rangle \geq \frac{3}{4}\|S^{-1}\|^{-1}\|f\|^2,$$

$$0 \leq S_J - S_J S^{-1} S_J \leq \frac{1}{4}S, \tag{5.7}$$

$$\frac{1}{2}S \leq S_J S^{-1} S_J - S_{J^c} S^{-1} S_{J^c} \leq \frac{3}{2}S. \tag{5.8}$$

Proof. Let $\Theta_i = \Lambda_i S^{-\frac{1}{2}}$, for each $i \in I$. Then, $\mathbf{\Theta}$ is a Parseval g-frame of $\mathcal{H}$. For any $J \subseteq I$, let $\tilde{S}_J : \mathcal{H} \to \mathcal{H}$ be a linear operator defined by:

$$\tilde{S}_J f = \sum_{i\in J}\Theta_i^*\Theta_i f, \qquad \forall f \in \mathcal{H}.$$

So, $\tilde{S}_J = S^{-\frac{1}{2}}S_J S^{-\frac{1}{2}}$. Hence, it follows from (5.1) of Theorem 5.3 that

$$\sum_{i\in J}\|\Theta_i f\|^2 + \sum_{i\in I}\|\Theta_i \tilde{S}_{J^c}f\|^2 = \sum_{i\in J^c}\|\Theta_i f\|^2 + \sum_{i\in I}\|\Theta_i \tilde{S}_J f\|^2, \tag{5.9}$$

for all $f \in \mathcal{H}$. Replacing f by $S^{\frac{1}{2}}f$ in (5.9), we get (5.5). Applying (5.2) for the Parseval g-frame $\mathbf{\Theta}$, we get,

$$\sum_{i\in J}\|\Theta_i f\|^2 + \|\tilde{S}_{J_c}f\|^2 \geq \frac{3}{4}\|f\|^2, \quad \forall f \in \mathcal{H}.$$

Since $\langle Sf, f\rangle \geq \|S^{-1}\|^{-1}\|f\|^2$, then replacing f by $S^{\frac{1}{2}}f$ in the last inequality, we get (5.6). To prove (5.7), it follows from (5.3) that $0 \leq \tilde{S}_J - \tilde{S}_J^2 \leq \frac{1}{4}I$. Hence,

$$0 \leq S^{-\frac{1}{2}}(S_J - S_J S^{-1} S_J)S^{-\frac{1}{2}} \leq \frac{1}{4}I,$$

which is equivalent to (5.7). Finally, we have from (5.4),

$$\frac{1}{2}I \leq \tilde{S}_J^2 + \tilde{S}_{J^c}^2 \leq \frac{3}{2}I. \tag{5.10}$$

Since $\tilde{S}_J = S^{-\frac{1}{2}} S_J S^{-\frac{1}{2}}$ and $\tilde{S}_{J^c} = S^{-\frac{1}{2}} S_{J^c} S^{-\frac{1}{2}}$, then we get (5.8) from (5.10). □

Corollary 5.5. *Let $\{f_i\}_{i\in I}$ be a Parseval frame of $\mathcal{H}$. Then, for any $J \subseteq I$ and all $f \in \mathcal{H}$, we have:*

$$0 \leq \sum_{i\in J} |\langle f, f_i\rangle|^2 - \Big\| \sum_{i\in J} \langle f, f_i\rangle f_i \Big\|^2 \leq \frac{1}{4}\|f\|^2,$$

$$\frac{1}{2}\|f\|^2 \leq \Big\| \sum_{i\in J} \langle f, f_i\rangle f_i \Big\|^2 + \Big\| \sum_{i\in J^c} \langle f, f_i\rangle f_i \Big\|^2 \leq \frac{3}{2}\|f\|^2.$$

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