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(m, n)-ALGEBRAICALLY COMPACTNESS FOR MODULES AND (m, n)-PURE INJECTIVITY

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Abstract. In this paper, we introduce the notion of (m, n)-*algebraically compact modules* as an analogue of algebraically compact modules and then we show that (m, n)-algebraically compactness and (m, n)-pure injectivity for modules coincide. Moreover, further characterizations of a (m, n)-pure injective module over a commutative ring are given.
Keywords: (m, n)-pure exact, (m, n)-pure injective, (m, n)-algebraically compact, pure injective, algebraically compact.
MSC(2010): Primary: 16D50; Secondary: 13C11, 16D40, 13C10.

1. Introduction

All rings in this paper are associative with unity, and all modules are unital. The concept of algebraic compactness and purity have already played major roles in the context of abelian group theory, module theory and also model theory (see [2], [8, p. 167-170] and [12]). For a historical survey of algebraic compactness and purity, we refer the reader to [13, p. 708]. There are several variants of the notion of purity (see [3, 6] and [13]). More generally, let m and n be two positive integers. A right R-module M is said to be (m, n)-*presented* if it is the factor module of a free right module of rank m modulo an n-generated submodule. A short exact sequence (ε) of left R-modules is called (m, n)-*pure* if it remains exact when tensoring it with any (m, n)-presented right R-module. Recall that (ε) is $(\aleph_0, n)$-*pure exact* (respectively $(m, \aleph_0)$-*pure*

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exact) if for each positive integer m (respectively n), (ε) is (m,n)-pure exact. Also, recall that (ε) is $(\aleph_0,\aleph_0)$-*pure exact* if for all positive integers m and n, (ε) is (m,n)-pure exact. The $(\aleph_0,\aleph_0)$-pure exact sequences are the pure exact sequences in the Cohn sense. Moreover, a submodule A of a left R-module B is called (m,n)-*pure submodule* (B is called (m,n)-*pure extension of* A) if the exact sequence $0 \longrightarrow A \overset{\iota}{\hookrightarrow} B \longrightarrow B/A \longrightarrow 0$ is (m,n)-pure. Also, a left R-module M is said to be (m,n)-*pure injective* if M has the injective property relative to each (m,n)-pure exact sequence (see [1, 4] and [17]). For a left R-module M, for $r_1,\ldots,r_n \in R$ and $m \in M$, an *equation* in M is an expression $r_1x_1+\cdots+r_nx_n = m$, where the x_i's are to be thought of as unknowns. A *solution* in M is an indexed sequence $b_1,\ldots,b_n \in M$ such that $r_1b_1+\cdots+r_nb_n = m$; we write $x_1 = b_1,\ldots,x_n = b_n$. Suppose that J and I are finite or infinite index sets, and that (r_{ji}) is a $J\times I$ matrix with entries $r_{ji} \in R$ such that each column has only a finite number of nonzero entries. Let $m_j \in M$, $j \in J$. Then the set of expressions

$$\sum_{i\in I} r_{ji}x_i = m_j,\ j \in J\ (*)$$

is called a *system of equations* in M with unknowns $\{x_i \mid i \in I\}$. Similarly, an indexed set $\{b_i \in M \mid i \in I\}$ is a solution of $(*)$ if $\sum_{i\in I} r_{ji}b_i = m_j$ for all $j \in J$. In this case, we say that $x_i = b_i \in M$, $i \in I$, is a solution of $(*)$. The system of equations $(*)$ is called *compatible* whenever for any choice of $s_j \in R$, $j \in J$, where only a finite number of s_j's are nonzero, if $\sum_{j\in J} s_jr_{ji} = 0$ for each $i \in I$, then $\sum_{j\in J} s_jm_j = 0$ (see [5, Chapter 18] for more details on systems system of equations). Throughout this paper, all systems of equations are assumed to be compatible.

As in [11], a left R-module M is called *algebraically compact* if any system of equations

$$\sum_{i} r_{ji}x_i = d_j \in M \text{ where } r_{ji} \in R, i \in I, j \in J \text{ (I)}$$

which is finitely solvable in M, has a global solution in M; see [10] and [16] (note that the system (I) is called *finitely solvable* if for every finite subset $F \subseteq J$, there exist $b_i \in M$, $i \in I$, such that for $j \in F$ only $\sum_i r_{ji}b_i = d_j$. For this, only a finite set $\{b_i \mid r_{ji} \neq 0,\ j \in F\}$ of nonzero b_i's is required; see [5, Definition 18-1.1]). In [13] Warfield proved that a left R-module M is algebraically compact if and only if it is pure injective.

In this paper, we first introduce the notion of (m, n)-algebraic compactness of modules as an analogue of the notion of algebraic compactness of modules and then we show that an R-module M is (m, n)-algebraically compact if and only if it is (m, n)-pure injective (see Theorem 2.8). In Section 2, we give further characterizations of (n, m)-pure injective modules over commutative rings. In fact, we show that a module M over a commutative ring R is (n, m)-pure injective if and only if M is a direct summand of direct product of submodules of E^n of the form $E[A]$, where E is an injective cogenerator R-module, E^n is the direct sum of n copies of E, $A = (r_{ji})$ is an $m \times n$ matrix with entries $r_{ji} \in R$ and, $E[A]$ is the following submodule of E^n:

$$E[A] := \{(m_1, \ldots, m_n) \in E^n \mid \sum_{i=1}^{n} r_{ji} m_i = 0,\ 1 \leq j \leq m\}.$$

In this paper, all definitions and results can be given by replacing n or m by $\aleph_0$.

2. (m,n)-algebraically compact modules

The following lemma offers several characterizations of (m, n)-pure exact sequences.

Lemma 2.1. ([1, Theorem 1.1]) *Let* $\varepsilon : 0 \longrightarrow A \hookrightarrow B \longrightarrow C \longrightarrow 0$ *be an exact sequence of left* R*-modules. The following statements are equivalent:*

(1) ε *is* (m, n)*-pure.*

(2) *For each* (n, m)*-presented left* R*-module* M*, the sequence* $\mathrm{Hom}(M, (\varepsilon))$ *is exact.*

(3) *Every system of* m *equations* $\sum_{i=1}^{n} r_{ji} x_i = a_j \in A,\ r_{ji} \in R, 1 \leq j \leq m$ *is solvable in* A *whenever it is solvable in* B*.*

Moreover, if R *is commutative and* E *is an injective cogenerator, then the above statements are equivalent to the following:*

(4) $\mathrm{Hom}_R((\epsilon), E)$ *is* (n, m)*-pure.*

We have the following lemma about (m, n)-pure injective modules.

Lemma 2.2. ([15, Theorem 33.7]) *Let* M *be a left* R*-module. The following statements are equivalent:*

(1) *M is (m,n)-pure injective.*

(2) *Every (m,n)-pure exact sequence $0 \longrightarrow M \longrightarrow M_1 \longrightarrow M_2 \longrightarrow 0$ splits, where M_1 and M_2 are arbitrary left R-modules.*

Next, we will introduce the notion of (m,n)-algebraic compactness as an analogue of the notion of algebraic compactness.

Definition 2.3. Let R be a ring, M be a left R-module and

$$\sum_i r_{ji}x_i = d_j \in M \ , \ r_{ji} \in R, i \in I, j \in J \qquad (*)$$

be a system of equations in M. We say that X is an R-linear combination of x_i's, whenever there exists $\{s_i\}_{i\in I} \subseteq R$, where only a finite number of s_i's are nonzero and $X = \sum_i s_i x_i$. Also, we say that $\sum_{l=1}^{n} c_l X_l = m$ is an R-linear combination of equations in $(*)$, whenever there exists $\{z_j\}_{j\in J} \subseteq R$, where only a finite number of z_j are nonzero and we have the following relations in M and in the free left R-module F with the basis $\{x_i \mid i \in I\}$:

$$\sum_{l=1}^{n} c_l X_l = \sum_{j\in J} z_j (\sum_{i\in I} r_{ji}x_i),\ m = \sum_{j\in J} z_j d_j.$$

Definition 2.4. Let R be a ring and M be a left R-module. We say that a system of equations

$$\sum_i r_{ji}x_i = d_j \in M \ , \ r_{ji} \in R, i \in I, j \in J \qquad (*)$$

is (m,n)-*solvable*, whenever every system of m equations

$$\sum_{l=1}^{n} c_{kl}X_l = m_k \in M,\ 1 \le k \le m,\ c_{kl} \in R,$$

where for each $l \in \{1,\dots,n\}$, X_l is an R-linear combination of x_i's and for each k, $\sum_{l=1}^{n} c_{kl}X_l = m_k$ is an R-linear combination of equations in $(*)$, is solvable for X_l in M. Also, we say that M is (m,n)-*algebraically compact* if any (m,n)-solvable system of equations of the form $(*)$ has a global solution.

Remark 2.5. A left R-module M is algebraically compact if and only if it is (m,n)-algebraically compact, for all positive integers m and n. To see this, it suffices to show that every finitely solvable system of equations $\sum_i r_{ji}x_i = d_j \in M,\ r_{ji} \in R, i \in I, j \in J$ is (m,n)-solvable, for all positive integers m and n. Assume that $\sum_{l=1}^{n} c_{kl}X_l = m_k \in M, 1 \le k \le m$ is a system of m equations where $X_l = t_{l_1}x_{l_1} + \cdots + t_{l_n}x_{l_n}$ for $t_{l_1},\dots,t_{l_n} \in R$, $l_1,\dots,l_n \in I$ and for each k, $\sum_{l=1}^{n} c_{kl}X_l = m_k \in M$ is an R-linear combination of equations $\sum_i r_{ji}x_i = d_j \in M$; i.e., for each $k \in \{1,\dots,m\}$, there exists $\{z_{kj}\}_{j\in J} \subseteq R$, where only a finite number of z_{kj} are nonzero and we have the following relations in M and in the free left R-module F with the basis $\{x_i \mid i \in I\}$:

$$\textstyle\sum_{l=1}^n c_{kl}X_l = \sum_{j\in J} z_{kj}(\sum_{i\in I} r_{ji}x_i),\ m_k = \sum_{j\in J} z_{kj}d_j.$$

Suppose that $\Upsilon \subseteq J$ is the set of all element j of J that there exists $k \in \{1,\dots,m\}$ for which $z_{kj} \neq 0$. Since Υ is a finite set, there is a solution $\{b_i\}_{i\in I} \subseteq M$ of equations $\sum_i r_{ji}x_i = d_j$, for each $j \in \Upsilon$. Set $Y_l = t_{l_1}b_{l_1}+\cdots+t_{l_n}b_{l_n}$, for each $l \in \{1,\dots,n\}$. Therefore, $\{Y_l\}_{l=1}^n \subset M$ is a solution of the system of equations $\sum_{l=1}^n c_{kl}X_l = m_k \in M$, $1 \le k \le n$.

Proposition 2.6. *Let M be a left R-module and*

$$\textstyle\sum_i r_{ji}x_i = d_j \in M, r_{ji} \in R, i \in I, j \in J \qquad (*)$$

be an (m,n)-solvable system of equations. Then there exists an (m,n)-pure extension B of M such that the system of equations $(*)$ has a solution in B.

Proof. Set $B = (M \oplus F)/S$, where F is the free left R-module with the basis $\{x_i\}_{i\in I}$ and

$$S = \{(\sum_j z_jd_j, -\sum_j z_j \sum_i r_{ji}x_i) \mid z_j \in R,\ z_j = 0 \text{ for almost all } j \in J\}.$$

Since the system of equations $(*)$ is compatible, one can easily see that the map $\alpha : M \longrightarrow B$ defined by $\alpha(a) = (a,0) + S$ is a monomorphism for each $a \in M$ and $\{(0,x_i) + S\}_{i\in I} \subseteq B$ is a solution of the following system of equations

$$\sum_i r_{ji}y_i = (d_j,0) + S \in B.$$

Thus it suffices to show that $\alpha(M)$ is an (m,n)-pure submodule of B. To see this, let

$$\sum_{l=1}^n c_{kl}x_l = (b_k,0) + S \in \alpha(M)\ (1 \le k \le m)$$

be a system of equations with the solution $(a_l,X_l)+S \in B$, where $b_k, a_l \in M$, $X_l \in F$ and $c_{kl} \in R$ for each $k \in \{1,\dots,m\}$. Then $(\sum_{l=1}^n c_{kl}a_l - b_k, \sum_{l=1}^n c_{kl}X_l) \in S$ for each $k \in \{1,\dots,m\}$. Therefore, for each $k \in$

$\{1,\ldots,m\}$, there exists $\{z_{kj}\}_{j\in J}\subseteq R$ with $z_{kj}=0$ for almost all $j\in J$ such that

$$\sum_{l=1}^{n} c_{kl}a_l - b_k = \sum_{j\in J} z_{kj}d_j,\ \sum_{l=1}^{n} c_{kl}X_l = -\sum_{j\in J} z_{kj}(\sum_i r_{ji}x_i).$$

Since the system of equations $(*)$ is (m,n)-solvable, the system of equations

$$\sum_{l=1}^{n} c_{kl}X_l = -\sum_{j\in J} z_{kj}d_j,\ 1\leq k\leq m,\ j\in J$$

has a solution $\tilde{a_1},\ldots,\tilde{a_n}\in M$. Therefore, for each $k\in\{1,\ldots,m\}$,

$$b_k - \sum_{l=1}^{n} c_{kl}a_l = \sum_{l=1}^{n} c_{kl}\tilde{a}_l.$$

This implies that $\sum_{l=1}^{n} c_{kl}(a_l+\tilde{a}_l) = b_k$, for each $k\in\{1,\ldots,m\}$. Thus for each $k\in\{1,\ldots,m\}$

$$\sum_{l=1}^{n} c_{kl}((a_l,X_l)+S) = (b_k,0)+S = \sum_{l=1}^{n} c_{kl}((a_l+\tilde{a}_l,0)+S)\in\alpha(M).$$

It means that $\alpha(M)$ is an (m,n)-pure submodule of B. □

Let M be a left R-module and

$$\sum_i r_{ji}x_i = d_j\in M,\ \text{where } r_{ji}\in R, i\in I, j\in J,$$

be an (m,n)-solvable system of equations. In the sequel, we use the notation

$$\langle M, b_i \mid \sum_i r_{ji}b_i = d_j,\ i\in I, j\in J\rangle$$

for the (m,n)-pure extension B of M obtained in the proof of the previous proposition.

Proposition 2.7. *Let M be a left R-module and $\aleph$ be an infinite cardinal number. Then there exists an (m,n)-pure extension B of M such that any (m,n)-solvable system of equations*

$$\textstyle\sum_i r_{ji}x_i = d_j\in M, r_{ji}\in R, i\in I, j\in J, |I|,|J|<\aleph \qquad (**)$$

has a solution in B.

Proof. Index the set of all (m,n)-solvable systems of equations over M such that containing strictly less than $\aleph$ equations and varieties by ordinals $\alpha < v$:

$$\sum_i r^{\alpha}_{ji} x^{\alpha}_i = d^{\alpha}_j \in M,\ r^{\alpha}_{ji} \in R, i \in I^{\alpha}, j \in J^{\alpha}, |J^{\alpha}|, |I^{\alpha}| < \aleph, \alpha \in [0, v).$$

where $|v| = \aleph\aleph|R||M|\aleph_0$. By Proposition 2.6, M is (m,n)-pure submodule of

$$B^0 = \langle M, b^0_i \mid \sum_i r^0_{ji} b^0_i = d^0_j \in M,\ i \in I^0, j \in J^0\rangle.$$

By the proof of Proposition 2.6, we have

$$B^1 = \langle B^0, b^1_i \mid \sum_i r^1_{ji} b^1_i = d^1_j \in M,\ i \in I^1, j \in J^1\rangle$$

such that $B^0 \leq B^1$ is (m,n)-pure and then $M \leq B^1$ is (m,n)-pure. Also, the 0-th and the 1-th system of equations have solution in B^1. By ordinal induction for all $\alpha < v$, assume that B^{γ} is defined for all $\gamma < \alpha$ and define

$$B^{\alpha} = \langle \bigcup_{\gamma<\alpha} B^{\gamma}, b^{\alpha}_i \mid \sum_i r^{\alpha}_{ji} b^{\alpha}_i = d^{\alpha}_j,\ i \in I^{\alpha}, j \in J^{\alpha}\rangle.$$

Obviously, for each $\gamma \leq \alpha$, the γ-th system of equations has a solution in B^{α}. We first prove that $M \leq B^{\alpha}$ is (m,n)-pure. $M \leq B^0$ is (m,n)-pure and by ordinal induction assume that $M \leq B^{\gamma}$ is (m,n)-pure for any $\gamma < \alpha$, then $M \leq \bigcup_{\gamma<\alpha} B^{\gamma}$ is (m,n)-pure. Since by Proposition 2.6 and definition of B^{α}, $\bigcup_{\gamma<\alpha} B^{\gamma} \leq B^{\alpha}$ is (m,n)-pure, so is $M \leq B^{\alpha}$. Now, define $B = \bigcup\{B^{\alpha} \mid \alpha < v\}$. Obviously, $M \leq B$ is (m,n)-pure and also any (m,n)-solvable system of equations of the form $(**)$ has a global solution in B. $\square$

We conclude this section with the following main result which is an analogue of algebraic compactness.

Theorem 2.8. *Let M be a left R-module. The following conditions are equivalent:*

(1) *M is (m,n)-pure injective.*

(2) *M is (m,n)-algebraically compact.*

Proof. (1) ⇒ (2). Assume that

$$\textstyle\sum_i r_{ji} x_i = d_j \in M,\ r_{ji} \in R,\ i \in I, j \in J \qquad (*)$$

is an (m, n)-solvable system of equations. By Lemma 2.6, there exists an (m, n)-pure extension B of M such that the system of equations $(*)$ has a solution $\{b_i\}_{i\in I} \subseteq B$. By Proposition 2.2, there exists a submodule A of B such that $B = M \oplus A$. Therefore, for each $i \in I$, there exist $m_i \in M$ and $a_i \in A$ such that $b_i = m_i + a_i$. Since $\sum_i r_{ji} b_i = d_j$, we conclude that

$$\sum_i r_{ji} m_i - \sum_i r_{ji} b_i = \sum_i r_{ji} m_i - d_j = \sum_i r_{ji} a_i \in A \cap M = 0.$$

Therefore, $\{m_i\}_{i\in I} \subseteq M$ is a solution of the system of equations $(*)$.
(2) $\Rightarrow$ (1). Let B be a left R-module, A be an (m, n)-pure submodule of B and $f : A \longrightarrow M$ be an R-module homomorphism. Assume that $\{b_i\}_{i\in I} \subseteq B$ such that the set A together with the b_i's generate B. Suppose that $\sum_i r_{ji} b_i = a_j \in A,\ r_{ji} \in R,\ i \in I, j \in J$ include all relationships among b_i's and A. Therefore, $\{b_i\}_{i\in I}$ is a solution of the compatible system of equations $\sum_i r_{ji} x_i = a_j \in A,\ r_{ji} \in R,\ i \in I, j \in J$. We denote this system of equations by Υ. Next, we consider the reduced set $\bar{\Upsilon} : \sum_i r_{ji} y_i = f(a_j) \in M,\ r_{ji} \in R,\ i \in I, j \in J$ of equations over M obtained from Υ. We claim that $\bar{\Upsilon}$ is (m, n)-solvable. To see this, assume that $\sum_{l=1}^n c_{kl} Y_l = m_k \in M,\ k \in \{1, \ldots, m\}$ is a system of m equations where $Y_l = t_{l_1} y_{l_1} + \cdots + t_{l_n} y_{l_n}$ for $t_{l_1}, \ldots, t_{l_n} \in R, l_1, \ldots, l_n \in I$. Also, assume that for each $k \in \{1, \ldots, m\}$, $\sum_{l=1}^n c_{kl} Y_l = m_k \in M$ is an R-linear combination of equations $\sum_i r_{ji} y_i = f(a_j) \in M$; i.e., for each $k \in \{1, \ldots, m\}$, there exists $\{z_{kj}\}_{j\in J} \subseteq R$, where only a finite number of z_{kj} are nonzero and

$$\textstyle\sum_{l=1}^n c_{kl} Y_l = \sum_{j\in J} z_{kj} (\sum_{i\in I} r_{ji} y_i),\ m_k = \sum_{j\in J} z_{kj} f(a_j).$$

By Remark 2.5, Υ is (m, n)-solvable. So the system of equations

$$\sum_{l=1}^n c_{kl} X_l = d_k \in M, 1 \leq k \leq m$$

where $X_l = t_{l_1} x_{l_1} + \cdots + t_{l_n} x_{l_n}$ and $d_k = \sum_{j\in J} z_{kj} a_j$, is solvable. Therefore, $\sum_{l=1}^n c_{kl} Y_l = m_k \in M, 1 \leq k \leq m$, is solvable and hence so $\bar{\Upsilon}$ is. Thus there exists a global solution $y_i = z_i$ for $\bar{\Upsilon}$. We now define $h : B \longrightarrow M$ by $h(b_i) = z_i$ and $h|_A = f$. Obviously, h is well-defined and a homomorphism, which completes the proof. $\square$

3. **Further characterizations of (m, n)-pure injective modules over a commutative ring**

Let R be a commutative ring and D an R-module. In [14] Warfield proved that D is RD-pure injective if and only if D is isomorphic to a direct summand of a product of modules of the form $E[r] = \{e \in E \mid re = 0\}$ where $r \in R$ and E is an injective R-module (see also [9]). In [7, Theorem 1] Fakhruddin showed that D is pure injective if and only if D is isomorphic to a direct summand of a product of modules of the form $\mathrm{Hom}_R(P, \overline{\mathbb{R}})$ where P is a finitely presented module and $\overline{\mathbb{R}} = \mathbb{R}/\mathbb{Z}$ is the circle group (the injective cogenerator group of real numbers modulo the integers). Also, in [6, Theorem 3.6] Divaani-Azar and et al. proved that D is I-pure injective if and only if D is isomorphic to a direct summand of a product of modules of the form $E[I] = \{e \in E \mid Ie = 0\}$ where I is an ideal of R and E is an injective R-module. Our purpose in this section is to offer a similar characterizations for an (m, n)-pure injective module.

Let R be a commutative ring, M be an R-module and M^n be the direct sum of n copies of M. Assume that $A = (r_{ji})$ is an $m \times n$ matrix of elements of R. It is easy to see that

$$M[A] := \{(m_1, \ldots, m_n) \in M^n \mid \sum_{i=1}^{n} r_{ji} m_i = 0, 1 \leq j \leq m\}$$

is a submodule of M^n. Note that if $n = 1$, then

$$M[A] = M[I] = \{m \in M \mid Im = 0\}$$

where I is the ideal of R generated by $r_{11}, \ldots, r_{m1}$. Also, if $m = 1$ and for some $1 \leq k \leq n$, r_{1k} is a unit, then $M[A] \cong M^{n-1}$. Therefore, if R is a commutative ring and E is an injective R-module, then for each $1 \times n$-matrix $A = (r_{ji})$ such that r_{1k} is a unit for some $1 \leq k \leq n$, the R-module $E[A]$ is injective.

Lemma 3.1. *Let R be a commutative ring, M be an R-module and K be a m-generated submodule of R^n. Then there exists an $m \times n$-matrix $A = (r_{ji})$ such that*

$$M[A] \cong_R \mathrm{Hom}_R(R^n/K, M).$$

Proof. Assume that $\{e_1, \ldots, e_n\}$ is the standard basis of R^n and K is a submodule of R^n generated by $\{\Sigma_{i=1}^{n} r_{ji} e_i\}_{j=1}^{m}$. Set $A = (r_{ji})$ where

$i \in \{1,\dots,n\}$ and $j \in \{1,\dots,m\}$. Define

$\varphi : \mathrm{Hom}_R(R^n/K, M) \longrightarrow M[A]$ by $\varphi(f) = (f(e_1+K),\dots,f(e_n+K))$.

We claim that φ is an isomorphism. Obviously, φ is a well-defined R-monomorphism. Thus we only need to show that φ is an epimorphism. Assume that $(a_1,\dots,a_n) \in M[A]$. Define $g : R^n \to M$ by $g(e_i) = a_i$. Obviously, $K \subseteq Ker(g)$. If $\bar{g} : R^n/K \to M$ denotes the natural homomorphism induced by g ($\bar{g}(e_i+K) = g(e_i)$), then $\varphi(\bar{g}) = (a_1,\dots,a_n)$, and so φ is an epimorphism. □

Let R be a commutative ring and E be an injective R-module. By a similar way as in [4, Theorem 3.2], we have the following lemma.

Lemma 3.2. *Let R be a commutative ring and E be an injective R-module. Then $\mathrm{Hom}_R(M,E)$ is (m,n)-pure injective for every (m,n)-presented R-module M.*

Corollary 3.3. *Let R be a commutative ring and E be an injective R-module. Then for each $m \times n$-matrix $A = (r_{ji})$, the R-module $E[A]$ is (n,m)-pure injective.*

Proof. Assume that $A = (r_{ji})$ is an $m \times n$-matrix with entries in R. Let K be a submodule of R^n generated by m elements $(r_{j1},\dots,r_{jn}) \in R^n$. By Lemma 3.1, $E[A] \cong_R \mathrm{Hom}_R(R^n/K, E)$. Since $\mathrm{Hom}_R(R^n/K,E)$ is (n,m)-pure injective for each m-generated submodule K of R^n, so is $E[A]$. □

The following lemma will be useful.

Lemma 3.4. ([15, Theorem 33.5]) *Let M be a left R-module. Then there exists an (n,m)-pure exact sequence of left R-modules*

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0,$$

where P is a direct sum of (m,n)-presented left R-modules.

Proposition 3.5. *Let R be a commutative ring and E be an injective cogeneratore R-module. Then every R-module M is isomorphic to an (m,n)-pure submodule of an R-module of the form $\prod_{\lambda\in\Lambda} E_\lambda$, where for each $\lambda \in \Lambda$, there exists a $n \times m$-matrix A_λ such that $E_\lambda = E[A_\lambda]$.*

Proof. By Lemma 3.4, for the R-module $\mathrm{Hom}_R(M,E)$ there exists a family of (m,n)-presented R-modules $\{P_\lambda\}_{\lambda\in\Lambda}$ and an (n,m)-pure exact sequence

$$\varepsilon : 0 \longrightarrow N \longrightarrow P \longrightarrow \mathrm{Hom}_R(M,E) \longrightarrow 0$$

of R-modules such that $P = \bigoplus_{\lambda\in\Lambda} P_\lambda$. Thus the following exact sequence of R-modules is (m, n)-pure:

$$\varepsilon^* : 0 \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E) \to \mathrm{Hom}_R(P, E) \to \mathrm{Hom}_R(N, E) \to 0.$$

Now by [9, XIII, Lemma 2.3], M embeds in $\mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E)$ as a pure submodule and so as an (m, n)-pure submodule. Therefore, M embeds in $\mathrm{Hom}_R(P, E)$ as an (m, n)-pure submodule. Obviously, $\mathrm{Hom}_R(P, E) \cong \prod_{\lambda\in\Lambda} \mathrm{Hom}_R(P_\lambda, E)$. Since for each $\lambda \in \Lambda$, $\{P_\lambda\}_{\lambda\in\Lambda}$ is (m, n)-presented, so for each $\lambda \in \Lambda$, there exists an n-generated submodule K of R^m such that $P_\lambda = R^m/K$. Thus by Lemma 3.1, there exists an $n \times m$-matrix A_λ such that $\mathrm{Hom}_R(P_\lambda, E) \cong E[A]$, as required. □

Corollary 3.6. *Let R be a commutative ring and $\{E_\alpha\}_{\alpha\in\mathcal{A}}$ be the family of the injective envelopes of all mutually non-isomorphic simple R-modules. Then every R-module M is isomorphic to an (m, n)-pure submodule of an R-module of the form $\prod_{\lambda\in\Lambda} E_\lambda$, where for each $\lambda \in \Lambda$, there exist $\alpha \in \mathcal{A}$, and $n \times m$-matrix A such that $E_\lambda = E_\alpha[A]$.*

Proof. It follows from Proposition 3.5, since $E = \prod_{\alpha\in\mathcal{A}} E_\alpha$ is an injective cogenerator R-module. □

We conclude this section with the following characterizations of an (m, n)-pure injective module over a commutative ring.

Theorem 3.7. *Let R be a commutative ring, E be an injective cogenerator R-module and $\{E_\alpha\}_{\alpha\in\mathcal{A}}$ be the family of the injective envelopes of all mutually non-isomorphic simple R-modules. Then for an R-module M, the following statements are equivalent:*

(1) *M is (m, n)-pure injective.*

(2) *M is isomorphic to a direct summand of an R-module of the form $\prod_{\lambda\in\Lambda} E_\lambda$, where for each $\lambda \in \Lambda$, there exists a $n \times m$-matrix A_λ such that $E_\lambda = E[A_\lambda]$.*

(3) *M is isomorphic to a direct summand of an R-module of the form $\prod_{\lambda\in\Lambda} E_\lambda$, where for each $\lambda \in \Lambda$, there exist $\alpha \in \mathcal{A}$, an $n \times m$-matrix A_{λ_α} such that $E_\lambda = E_\alpha[A_{\lambda_\alpha}]$.*

Proof. (1) $\Leftrightarrow$ (2) follows from Proposition 3.5 and Lemma 2.2.
(1) $\Leftrightarrow$ (3) follows from Proposition 3.6 and Lemma 2.2. □

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References

[1] W. Al-Kawarit and F. Couchot, Comparison of some purities, flatnesses and injectivities, *Comm. Algebra* **39** (2011), no. 10, 3879–3896.
[2] P. Crowley, and B. Jonson, Refinements for infinite direct decomposition of algebraic systems, *Pacific J. Math.* **14** (1964) 797–855.
[3] F. Couchot, RD-flatness and RD-injectivity, *Comm. Algebra* **34** (2006), no. 10, 3675–3689.
[4] I. Crivei and S. Crivei, (m, n)-purity for modules, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **49** (2006), no. 1, 37–46.
[5] J. Dauns, Modules and Rings, Cambridge University Press, Cambridge, 1994.
[6] K. Divaani-Azar, M. A. Esmkhani and M. Tousi, Some criteria of cyclically pure injective modules, *J. Algebra* **304**, (2006), no. 1, 367–381.
[7] S. M. Fakhruddin, Pure-injective modules, *Glasgow Math. J.* **14** (1973) 120–122.
[8] L. Fuchs, Infinite Abelian Groups, I, Academic Press, New York-London, 1970.
[9] L. Fuchs and L. Salce, Modules Over Non-Noetherian domains, Mathematical Surveys and Monographs, 84, Amer. Math. Soc., Providence, 2001.
[10] O. A. S. Karamzadeh and A. A. Koochakpour, On $\aleph_0$-self injectivity of strongly regular rings, *Comm. Algebra* **27** (1999), no. 4, 1501–1513.
[11] J. Mycielski, Some compactifications of general algebras, *Colloq. Math.* **13** (1964) 1–9.
[12] M. Prest, Model Theory and Modules, London Mathematical Society Lecture Note Series, 130, Cambridge University Press, Cambridge, 1988.
[13] R. B. Warfield, Purity and algebraic compactness for modules, *Pacific J. Math.* **28** (1969) 699–719.
[14] R. B. Warfield, Relative injective modules, Unpublished Manuscript, 1969.
[15] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach Science Publishers, New York, 1991.
[16] B. H. Zimmermann, Purity, algebraic compactness, direct sum decompositions, and representation type, 331–367, Trends Math., Birkhauser, 2000.
[17] Z. Zhu, J. Chen and X. Zhang, On (m, n)-purity of modules, *East-West J. Math.* **5** (2003), no. 1, 35–44.

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