Bulletin of the Iranian Mathematical Society Vol. 35 *No. 1* (*2009*), *pp 111-117.*

A MEAN ERGODIC THEOREM FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE AFFINE MAPPINGS IN BANACH SPACES SATISFYING OPIAL'S CONDITION

S. SAEIDI

Abstract. Our purpose is to prove that if C is a weakly compact convex subset of a Banach space E satisfying Opial's condition and $T : C \to C$ is an asymptotically quasi-nonexpansive affine mapping such that $F(T) \neq \varnothing$, then for all $x \in C$, $\{T^n x\}$ is weakly almost-convergent to some $z \in F(T)$.

1. Introduction

Let E be a real normed space, C be a nonempty subset of E and T be a self mapping on C. T is said to be nonexpansive provided that $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. Denote by $F(T)$, the set of fixed points of T. The first nonlinear ergodic theorem for nonexpansive mappings with bounded domains in a Hilbert space was established by Baillon [1]: Let C be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesaro means $S_n(x) = \frac{1}{n}\sum_{k=0}^{n} T^k x$ converge weakly to some $y \in F(T)$. This theorem was extended to a uniformly convex Banach space whose norm is Frechet differentiable by Bruck in [2]. Gornicki [6] proved that

MSC(2000): Primary: 47H09, 47H10

Keywords: Affine mapping, asymptotically quasi nonexpansive mapping, fixed point, weakly almost-convergence, nonlinear ergodic, retraction.

Received: 10 November 2007, Accepted: 8 July 2008

if E is a uniformly convex Banach space satisfying Opial's condition, C is a nonempty bounded closed convex subset of E, and $T : C \to C$ is an asymptotically nonexpansive mapping, then for each $x \in C$, the Cesaro means $\frac{1}{n}\sum_{j=0}^{n-1} T^{j+k}x$ converges weakly to a fixed point of T uniformly in $k \in \mathbb{N} \cup \{0\}$. Since the nonlinear mean ergodic theorem for contractions in a Hilbert space has been proved by Baillon, other proofs and generalizations of the theorem have been presented by many authors (see, for example, [2, 7, 9, 10, 11, 6] and the references therein).

Here, we prove the mean ergodic theorem for asymptotically quasi nonexpansive affine mappings in a Banach space satisfying Opial's condition. However, our mappings are assumed to be affine, but our results extend the similar mean ergodic theorems to more general spaces.

2. Preliminaries

Let E be a real normed space and let C be a closed convex nonempty subset of E. Consider a mapping $T : C \to C$; T is said to be *quasi-nonexpansive* [3] provided that $\|Tx - f\| \leq \|x - f\|$, for all $x \in C$ and $f \in F(T)$; T is called *asymptotically nonexpansive* [5] if there exists a sequence $\{u_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\|$, for all $x, y \in C$ and $n \in \mathbb{N}$; T is called asymptotically quasi-nonexpansive if there exists a sequence $\{u_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $\|T^n x - f\| \leq (1 + u_n)\|x - f\|$, for all $x \in C$, $f \in F(T)$ and $n \in \mathbb{N}$; T is said to be *affine* if $T(\alpha x + (1-\alpha)y) = \alpha Tx + (1-\alpha)Ty$, for all $\alpha \in [0, 1]$ and $x, y \in C$; T is said to be *semi-compact* if, for any sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ $(n \to \infty)$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x^* \in C$; T is said to be *retraction* if $T^2 = T$. From the above definitions, it follows that a nonexpansive mapping must be asymptotically nonexpansive, and an asymptotically nonexpansive mapping with fixed point must be asymptotically quasi-nonexpansive, but the converse does not hold. A Banach space E is said to satisfy Opial's condition if whenever $\{x_n\}$ is a sequence in E which converges weakly to x, then,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad \text{for all } \; y \in E, \; y \neq x.$$

It is well known that every Hilbert space satisfies Opial's property [8].

3. Mean convergence theorems

The proof of the following lemma is elementary.

Lemma 3.1. *Let C be a nonempty bounded convex subset of a normed space E and $T : C \to C$ be an affine mapping. Then,*

$$\lim_{n\to\infty} \|S_n(x) - TS_n(x)\| = 0,$$

uniformly in $x \in C$, where $S_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k$.

Theorem 3.2. *Let C be a nonempty convex subset of a normed space E and $T : C \to C$ be an asymptotically quasi-nonexpansive affine mapping with $F(T) \neq \varnothing$. Then, for all $x \in C$ and $z \in F(T)$, $\lim\limits_{n\to\infty} \|\frac{1}{n}\sum_{j=0}^{n-1} T^{j+k}x - z\|$ exists uniformly in k in $\mathbb{N} \cup \{0\}$.*

Proof. Put $u = \max u_i$ for which $\|T^i y - z\| \leq (1 + u_i)\|y - z\|$, for all $y \in C$ and $z \in F(T)$. Let z be an arbitrary element of $F(T)$ and consider x in C. Set $D = \{y \in C : \|y - z\| \leq (1 + u)\|x - z\|\}$. We note that $x, z, T^i x \in D$, $\forall i \in \mathbb{N}$, $T(D) \subset D$ and D is a bounded convex subset of C. So, we can assume that C is bounded. Fix $\varepsilon > 0$ and set $M_0 = \sup\{\|y\| : y \in C\}$. Consider two sequences $\{k_m\}$ and $\{l_n\}$ in $\mathbb{N} \cup \{0\}$. We have,

$$\frac{1}{m}\sum_{j=0}^{m-1} T^{j+k_m+l_n}x = \frac{1}{mn}\sum_{j=1}^{n-1}(n-j)(T^{j+k_m+l_n-1}x - T^{j+k_m+l_n+m-1}x)$$
$$+\frac{1}{m}\sum_{j=0}^{m-1}\frac{1}{n}\sum_{h=0}^{n-1}T^{j+h+k_m+l_n}x.$$

Then,

$$\|\frac{1}{m}\sum_{j=0}^{m-1} T^{j+k_m+l_n}x - z\| \leq \|\frac{1}{mn}\sum_{j=1}^{n-1}(n-j)(T^{j+k_m+l_n-1}x - T^{j+k_m+l_n+m-1}x)\|$$
$$+\|\frac{1}{m}\sum_{j=0}^{m-1}\frac{1}{n}\sum_{h=0}^{n-1}T^{j+h+k_m+l_n}x - z\|$$
$$\leq \frac{1}{mn}\sum_{j=1}^{n-1}(n-j)2M_0 + \frac{1}{m}\sum_{j=0}^{m-1}\|T^{j+k_m}(\frac{1}{n}\sum_{h=0}^{n-1}T^{h+l_n}x) - z\|$$

$$\leq \frac{M_0 n}{m} + (1 + \frac{1}{m}\sum_{j=0}^{m-1} u_{j+k_m})\|\frac{1}{n}\sum_{h=0}^{n-1} T^{h+l_n}x - z\|.$$

Note that, for each n, we have,

$$\lim_{m\to\infty} \|\frac{1}{m}\sum_{j=0}^{l_n-1} T^{j+k_m}x\| \leq \lim_{m\to\infty} \frac{l_n M_0}{m} = 0,$$

and

$$\lim_{m\to\infty} \|\frac{1}{m}\sum_{j=m}^{m+l_n-1} T^{j+k_m}x\| \leq \lim_{m\to\infty} \frac{l_n M_0}{m} = 0.$$

Thus, for each n,

$$\limsup_{m\to\infty} \|\frac{1}{m}\sum_{j=0}^{m-1} T^{j+k_m}x - z\|$$

$$= \limsup_{m\to\infty} \|\frac{1}{m}\sum_{j=0}^{l_n-1} T^{j+k_m}x + \frac{1}{m}\sum_{j=l_n}^{m+l_n-1} T^{j+k_m}x - \frac{1}{m}\sum_{j=m}^{m+l_n-1} T^{j+k_m}x - z\|$$

$$= \limsup_{m\to\infty} \|\frac{1}{m}\sum_{j=l_n}^{m+l_n-1} T^{j+k_m}x - z\|$$

$$= \limsup_{m\to\infty} \|\frac{1}{m}\sum_{j=0}^{m-1} T^{j+k_m+l_n}x - z\| \leq \|\frac{1}{n}\sum_{h=0}^{n-1} T^{h+l_n}x - z\|.$$

Therefore, we have shown,

$$\limsup_{m\to\infty} \|\frac{1}{m}\sum_{j=0}^{m-1} T^{j+k_m}x - z\| \leq \liminf_{n\to\infty} \|\frac{1}{n}\sum_{h=0}^{n-1} T^{h+l_n}x - z\|,$$

for arbitrary sequences $\{k_m\}$ and $\{l_n\}$ in $\mathbb{N}\cup\{0\}$. This leads to the desired conclusion. $\square$

Corollary 3.3. *Let C be a nonempty closed convex subset of a normed space E and $T : C \to C$ be a semi-compact asymptotically quasi-nonexpansive affine mapping such that either C is bounded and T is continuous or $F(T) \neq \varnothing$. Then, for each $x \in C$, the Cesaro means $\frac{1}{n}\sum_{j=0}^{n-1} T^{j+k}x$ converge strongly to some $z \in F(T)$, uniformly in k in $\mathbb{N}\cup\{0\}$.*

Proof. We first assume that C is bounded. Then, using Lemma 3.1, the semi-compactness and continuity of T, $S_n(x)$ has a limit point which is a fixed point of T. Now, we apply Theorem 3.2 to conclude the result. Also, if we consider the case $F(T) \neq \varnothing$, then it is easy to show that T is continuous and, as in the proof of Theorem 3.2 with no lose of generality, we may assume that C is bounded. Therefore, both conditions lead to the same conclusion. $\square$

A more general case of the following lemma is proved in [4]. For the sake of convenience, we give a proof.

Lemma 3.4. *(Demiclosedness Principle). Assume that C is a closed convex subset of a normed space E and $T : C \to E$ is a continuous affine mapping. Then, $I - T$ is demiclosed; that is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$.*

Proof. Let $\{x_n\}$ be a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y. By considering the mapping $T_y z = Tz + y$, for all $z \in C$, without lose of generality, we can assume that $y = 0$. Then, $\|x_n - Tx_n\| \to 0$, as $n \to \infty$. Since T is continuous and affine, then $F_\epsilon(T) = \{x \in C : \|x - Tx\| \leq \epsilon\}$ is closed and convex for all $\epsilon > 0$. Therefore, $x \in F_\epsilon(T)$ for each $\epsilon > 0$, and then $x \in F(T)$. That is, $(I - T)x = 0$. $\square$

The following theorem is our main result.

Theorem 3.5. *Suppose C is a locally weakly compact convex subset of a Banach space E satisfying Opial's condition and $T : C \to C$ is an asymptotically quasi-nonexpansive affine mapping such that $F(T) \neq \varnothing$. Then, for each $x \in C$, the Cesaro means $\frac{1}{n}\sum_{j=0}^{n-1} T^{j+k}x$ converge weakly to some $z \in F(T)$ uniformly in $k \in \mathbb{N} \cup \{0\}$. That is, $\{T^n x\}$ is weakly almost-convergent to some $z \in F(T)$.*

Proof. Since $F(T) \neq \varnothing$, then T is continuous and, as in the proof of Theorem 3.2, we may assume that C is bounded and then weakly compact. Pick an arbitrary sequence $\{k_n\} \subset \mathbb{N} \cup \{0\}$. We show that $\Phi_n = \frac{1}{n}\sum_{j=0}^{n-1} T^{j+k_n}x$ converges weakly to a fixed point of T. Let $\Phi_{n_k} \rightharpoonup f_1$, $\Phi_{m_k} \rightharpoonup f_2$ and $f_1 \neq f_2$. Applying Lemmas 3.1 and 3.4, we have $f_1, f_2 \in F(T)$. Then, considering Theorem 3.2 we put $r_1 :=$

$\lim_{n\to\infty}\|\Phi_n - f_1\|$ and $r_2 := \lim_{n\to\infty}\|\Phi_n - f_2\|$. By Opial's condition, we conclude,

$$r_1 = \lim_{k\to\infty}\|\Phi_{n_k} - f_1\| < \lim_{k\to\infty}\|\Phi_{n_k} - f_2\| = r_2$$

$$= \lim_{k\to\infty}\|\Phi_{m_k} - f_2\| < \lim_{k\to\infty}\|\Phi_{m_k} - f_1\| = r_1,$$

which is a contradiction. It means that $f_1 = f_2$. This leads to the weak convergence, of Φ_n to a fixed point of T. To prove the uniform convergence we show for a fixed z in $F(T)$ and arbitrary sequences $\{k_n\} \subset \mathbb{N} \cup \{0\}$ that $\frac{1}{n}\sum_{j=0}^{n-1} T^{j+k_n}x$ converges weakly to z. We consider two sequences $\{k_n\}$ and $\{l_n\}$ in $\mathbb{N} \cup \{0\}$. By the first part of the proof, $\frac{1}{n}\sum_{j=0}^{n-1} T^{j+k_n}x$ and $\frac{1}{n}\sum_{j=0}^{n-1} T^{j+l_n}x$ converge weakly to fixed points of T, say f and g, respectively. We show that $f = g$. To this end, let $f \neq g$. Then, using Theorem 3.2 and the Opial's condition, we have,

$$\lim_{n\to\infty}\|\frac{1}{n}\sum_{j=0}^{n-1} T^{j+k_n}x - f\|$$

$$< \lim_{n\to\infty}\|\frac{1}{n}\sum_{j=0}^{n-1} T^{j+k_n}x - g\| = \lim_{n\to\infty}\|\frac{1}{n}\sum_{j=0}^{n-1} T^{j+l_n}x - g\|$$

$$< \lim_{n\to\infty}\|\frac{1}{n}\sum_{j=0}^{n-1} T^{j+l_n}x - f\| = \lim_{n\to\infty}\|\frac{1}{n}\sum_{j=0}^{n-1} T^{j+k_n}x - f\|.$$

This is a contradiction and consequently $f = g$. This leads to the desired conclusion. □

Proposition 3.6. *Suppose C is a locally weakly compact convex subset of a Banach space E satisfying Opial's condition and $T : C \to C$ is an asymptotically (quasi-)nonexpansive affine mapping such that $F(T) \neq \varnothing$. Then, there exists a (quasi-)nonexpansive affine retraction P form C onto $F(T)$ such that $PT = TP = P$ and $Px \in \cap_{i=0}^{\infty}\overline{co}\{T^k x : k \geq i\}$, for each $x \in C$.*

Proof. Considering Theorem 3.5, we put,

$$Px = w - \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} T^{j+k}x,$$

for each $x \in C$, for which the limit is uniform in k. Note that $Px \in F(T)$ and $Px \in \cap_{i=0}^{\infty} \overline{co}\{T^k x : k \geq i\}$. Now, consider a sequence $\{k_n\}$ such that $k_n \to \infty$. Then, using the lower semi-continuity of the norm we have,

$$\|Px - f\| \leq \lim_{n\to\infty} \|\frac{1}{n}\sum_{j=0}^{n-1} T^{j+k_n}x - f\|$$

$$\leq \lim_{n\to\infty}(1 + \frac{1}{n}\sum_{j=0}^{n-1} u_{j+k_n})\|x - f\| = \|x - f\|,$$

for each $x \in C$ and $f \in F(T)$. This means that P is quasi-nonexpansive. The proof for the nonexpansiveness case is similar. □

References

[1] J. B. Baillon, Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert, *C.R. Acad. Sci. Paris Ser. A-B* **280** (1975) 1511-1514.

[2] R. E. Bruck, A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces, *Israel J. Math.* **32** (1979) 107-116.

[3] J.B. Diaz and F.T. Metcalf, On the structure of the set of subsequential limit points of successive approximations, *Bull. Amer. Math. Soc.* **73** (1967) 516-519.

[4] J. Garcia-Falset and E. Llorens-Fuster, Fixed point theory for almost convex functions, *Nonlinear Anal.* **32** (1998) 601-608.

[5] K. Goebel and W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.* **35** (1972) 171-174.

[6] J. Gornicki, Nonlinear ergodic theorems for asymptotically nonexpansive mappings in Banach spaces satisfying Opial's condition, *J. Math. Anal. Appl.* **161**(2) (1991), 440-446.

[7] N. Hirano, A proof of the mean ergodic theorem for nonexpansive mappings in Banach space, *Proc. Amer. Math. Soc.* **78** (1980) 361-365.

[8] Z. Opial, Weakconverg ence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.* **73** (1967) 591-597.

[9] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.* **67** (1979) 274-276.

[10] S. Saeidi, Convergence of an implicit iteration for affine mappings in normed and Banach spaces, *Bull. Iran. Math. Soc.* Vol. **33**(2) (2007) 27-35.

[11] P. K. Tam and K. K. Tan, Mean ergodic theorems for affine operators, *Math. Computer Modelling* **32** (2000) 1417-1421.

Shahram Saeidi

Department of Mathematics, University of Kurdistan, Sanandaj 416, Iran.,

Email: shahram_saeidi@yahoo.com; sh.saeidi@uok.ac.ir