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Author(s):

K. Haghnejad Azar

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ARENS REGULARITY OF BILINEAR FORMS AND UNITAL BANACH MODULE SPACES

K. HAGHNEJAD AZAR

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Abstract. Assume that A, B are Banach algebras and that $m : A \times B \to B$, $m' : A \times A \to B$ are bounded bilinear mappings. We study the relationships between Arens regularity of m, m' and the Banach algebras A, B. For a Banach A-bimodule B, we show that B factors with respect to A if and only if B^{**} is unital as an A^{**}-module. Let $Z_{e''}(B^{**}) = B^{**}$ where e'' is a mixed unit of A^{**}. Then B^* factors on both sides with respect to A if and only if B^{**} has a unit as A^{**}-module.
Keywords: Arens regularity, bilinear mappings, topological center, unital A-module, module action.
MSC(2010): Primary: 46L06, 46L07, 46L10; Secondary: 47L25, 47L50.

1. **Introduction**

Throughout this paper, A is a Banach algebra and A^*, A^{**}, respectively, are the first and second dual of A. Recall that a left approximate identity, abbrevaited asLAI, (respectively, right approximate identity, abbrevaited as RAI) in a Banach algebra A is a net $(e_\alpha)_{\alpha\in I}$ in A such that $e_\alpha a \to a$ (respectively, $ae_\alpha \to a$) for each $a \in A$. We say that a net $(e_\alpha)_{\alpha\in I} \subseteq A$ is a approximate identity, abbrevaited as AI, for A if it is both an LAI and a RAI for A. If $(e_\alpha)_{\alpha\in I}$ in A is bounded and is an AI for A, then we say that $(e_\alpha)_{\alpha\in I}$ is a bounded approximate identity, abbreviated as BAI, for A. For $a \in A$ and $a' \in A^*$, we denote by $a'a$ and aa' respectively, the functionals on A^* defined by

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$\langle a'a, b\rangle = \langle a', ab\rangle = a'(ab)$ and $\langle aa', b\rangle = \langle a', ba\rangle = a'(ba)$ for all $b \in A$. The Banach algebra A is embedded in its second dual via the identification $\langle a, a'\rangle$ - $\langle a', a\rangle$ for every $a \in A$ and $a' \in A^*$. We denote the set $\{a'a : \ a \in A \ and \ a' \in A^*\}$ and $\{aa' : \ a \in A \ and \ a' \in A^*\}$ by A^*A and AA^*, respectively. It is clear that these two sets are subsets of A^*. Assume that A has a BAI. If the equality $A^*A = A^*$ $(AA^* = A^*)$ holds, then we say that A^* factors on the left (right). If both equalities $A^*A = AA^* = A^*$ hold, then we say that A^* factors on both sides.

It is well-known that (see for instance [1]), the second dual A^{**} of A endowed with the either Arens multiplications is a Banach algebra. The constructions of the two Arens multiplications in A^{**} lead us to the definition of topological centers for A^{**} with respect to Arens multiplications. The topological centers of Banach algebras, module actions and applications of them have been introduced and discussed in many papers such as [7, 8, 10, 11, 12]. The extension of bilinear maps on normed spaces and the concept of regularity of bilinear maps have taken a great deal of attention by many researchers (see for example [1, 2, 5, 7, 12]). We commence by recalling some definitions as follows.

Let X, Y and Z be normed spaces and $m : X \times Y \to Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions m^{***} and m^{t***t} of m from $X^{**} \times Y^{**}$ into Z^{**} as follows
1. $m^* : Z^* \times X \to Y^*$, given by $\langle m^*(z', x), y\rangle = \langle z', m(x, y)\rangle$ where $x \in X$, $y \in Y$, $z' \in Z^*$,
2. $m^{**} : Y^{**} \times Z^* \to X^*$, given by $\langle m^{**}(y'', z'), x\rangle = \langle y'', m^*(z', x)\rangle$ where $x \in X$, $y'' \in Y^{**}$, $z' \in Z^*$,
3. $m^{***} : X^{**} \times Y^{**} \to Z^{**}$, given by $\langle m^{***}(x'', y''), z'\rangle$ $= \langle x'', m^{**}(y'', z')\rangle$ where $x'' \in X^{**}$, $y'' \in Y^{**}$, $z' \in Z^*$.
The mapping m^{***} is the unique extension of m such that $x'' \to m^{***}(x'', y'')$ from X^{**} into Z^{**} is $weak^*$-$weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $y'' \to m^{***}(x'', y'')$ is not in general $weak^*$-$weak^*$ continuous from Y^{**} into Z^{**} unless $x'' \in X$. Hence the first topological center of m may be defined as follows

$$Z_1(m) = \{x'' \in X^{**} : y'' \to m^{***}(x'', y'') \ is \ weak^*\text{-to-}weak^* \ continuous\}.$$

Let now $m^t : Y \times X \to Z$ be the transpose of m defined by $m^t(y, x) = m(x, y)$ for every $x \in X$ and $y \in Y$. Then m^t is a continuous bilinear map from $Y \times X$ to Z, and so it may be extended as above to

$m^{t***}: Y^{**} \times X^{**} \to Z^{**}$. The mapping $m^{t***t}: X^{**} \times Y^{**} \to Z^{**}$ in general is not equal to m^{***}, see [1]. If $m^{***} = m^{t***t}$, then m is called Arens regular. The bounded bilinear mapping m is said to be left (respectively, right) strongly Arens irregular whenever m^{***} and m^{t***t} are equal only on $X \times Y^{**}$ (respectively, $X^{**} \times Y$). The mapping $y'' \to m^{t***t}(x'', y'')$ is *weak**-*weak** continuous for every $y'' \in Y^{**}$, but the mapping $x'' \to m^{t***t}(x'', y'')$ from X^{**} into Z^{**} is not in general $weak^* - weak^*$ continuous for every $y'' \in Y^{**}$. So we define the second topological center of m as

$$Z_2(m) = \{y'' \in Y^{**} :\ x'' \to m^{t***t}(x'', y'') \ \textit{is weak}^*\textit{-weak}^* \textit{ continuous}\}.$$

It is clear that m is Arens regular if and only if $Z_1(m) = X^{**}$ or $Z_2(m) = Y^{**}$. Arens regularity of m is equivalent to the following

$$\lim_i \lim_j \langle z', m(x_i, y_j)\rangle = \lim_j \lim_i \langle z', m(x_i, y_j)\rangle,$$

whenever both limits exist for all bounded sequences $(x_i)_i \subseteq X$, $(y_i)_i \subseteq Y$ and $z' \in Z^*$, see [13].

The mapping m is left strongly Arens irregular if $Z_1(m) = X$ and m is right strongly Arens irregular if $Z_2(m) = Y$.

Let now B be a Banach A-bimodule, and let

$$\pi_\ell :\ A \times B \to B \ \textit{and}\ \pi_r :\ B \times A \to B.$$

be the left and right module actions of A on B, respectively. Then B^{**} is a Banach A^{**}-bimodule with module actions

$$\pi_\ell^{***} :\ A^{**} \times B^{**} \to B^{**} \ \textit{and}\ \pi_r^{***} :\ B^{**} \times A^{**} \to B^{**}.$$

Similarly, B^{**} is a Banach A^{**}-bimodule with module actions

$$\pi_\ell^{t***t} :\ A^{**} \times B^{**} \to B^{**} \ \textit{and}\ \pi_r^{t***t} :\ B^{**} \times A^{**} \to B^{**}.$$

We may therefore define the topological centers of the left and right module actions of A on B as follows:

$$Z_{B^{**}}(A^{**}) = Z(\pi_\ell) = \{a'' \in A^{**} :\ \textit{the map}\ b'' \to \pi_\ell^{***}(a'', b'') : B^{**} \to B^{**} \ \textit{is weak}^*\textit{-weak}^* \textit{ continuous}\}$$

$$Z^t_{B^{**}}(A^{**}) = Z(\pi_r^t) = \{a'' \in A^{**} :\ \textit{the map}\ b'' \to \pi_r^{t***}(a'', b'') : B^{**} \to B^{**} \ \textit{is weak}^*\textit{-weak}^* \textit{ continuous}\}$$

$$Z_{A^{**}}(B^{**}) = Z(\pi_r) = \{b'' \in B^{**} :\ \textit{the map}\ a'' \to \pi_r^{***}(b'', a'') : A^{**} \to B^{**} \ \textit{is weak}^*\textit{-weak}^* \textit{ continuous}\}$$

$$Z^t_{A^{**}}(B^{**}) = Z(\pi_\ell^t) = \{b'' \in B^{**} : \textit{ the map } a'' \to \pi_\ell^{t***}(b'', a'') : A^{**} \to B^{**} \textit{ is weak}^*\textit{-weak}^* \textit{ continuous}\}$$

One can also see ready that if B is a left(respectively, right) Banach A-module and $\pi_\ell : \ A \times B \to B$ (respectively, $\pi_r : \ B \times A \to B$) is left (respectively, right) module action of A on B, then B^* is a right (respectively, left) Banach A-module.
We write

$$ab = \pi_\ell(a, b), ba = \pi_r(b, a), \pi_\ell(a_1 a_2, b) = \pi_\ell(a_1, a_2 b),$$
$$\pi_r(b, a_1 a_2) = \pi_r(ba_1, a_2), \ \pi_\ell^*(a_1 b', a_2) = \pi_\ell^*(b', a_2 a_1),$$
$$\pi_r^*(b' a, b) = \pi_r^*(b', ab),$$

for all $a_1, a_2, a \in A$, $b \in B$ and $b' \in B^*$ when there is no confusion.
Regarding A as a Banach A-bimodule, the operation $\pi : A \times A \to A$ extends to π^{***} and π^{t***t} defined on $A^{**} \times A^{**}$. These extensions are known, respectively, as the first (respectively, left) and the second (respectively, right) Arens products, and with each of them, the second dual space A^{**} becomes a Banach algebra. In this situation, we shall also simplify our notations. So the first (respectively, left) Arens product of $a'', b'' \in A^{**}$ shall be simply indicated by $a''b''$ and defined by the three steps:

$$\langle a'a, b\rangle = \langle a', ab\rangle,$$
$$\langle a''a', a\rangle = \langle a'', a'a\rangle,$$
$$\langle a''b'', a'\rangle = \langle a'', b''a'\rangle.$$

for every $a, b \in A$ and $a' \in A^*$. Similarly, the second (respectively, right) Arens product of $a'', b'' \in A^{**}$ shall be indicated by $a''ob''$ and defined by :

$$\langle aoa', b\rangle = \langle a', ba\rangle,$$
$$\langle a'oa'', a\rangle = \langle a'', aoa'\rangle,$$
$$\langle a''ob'', a'\rangle = \langle b'', a'oa''\rangle.$$

for all $a, b \in A$ and $a' \in A^*$.
The regularity of a normed algebra A is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. Let a'' and b'' be elements of A^{**}, the second dual of A. By Goldstine's Theorem ([6], page 424), there are bounded nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in A such that $a'' = weak^*\text{-}\lim_\alpha a_\alpha$ and $b'' = weak^*\text{-}\lim_\beta b_\beta$. So it is easy to see that for all $a' \in A^*$,

$$\lim_\alpha \lim_\beta \langle a', \pi(a_\alpha, b_\beta)\rangle = \langle a''b'', a'\rangle$$

and

$$\lim_{\beta}\lim_{\alpha}\langle a', \pi(a_\alpha, b_\beta)\rangle = \langle a''ob'', a'\rangle,$$

where $a''b''$ and $a''ob''$ are the first and second Arens products of A^{**}, respectively, see [5, 11, 12].

We find the usual first and second topological center of A^{**}, which are

$$Z_{A^{**}}(A^{**}) = Z(\pi) = \{a'' \in A^{**} : b'' \to a''b'' \text{ is weak}^*\text{-weak}^* \text{ continuous}\},$$

$$Z^t_{A^{**}}(A^{**}) = Z(\pi^t) = \{a'' \in A^{**} : a'' \to a''ob'' \text{ is weak}^*\text{-weak}^* \text{ continuous}\}.$$

An element e'' of A^{**} is said to be a mixed unit if e'' is a right unit for the first Arens multiplication and a left unit for the second Arens multiplication. That is, e'' is a mixed unit if and only if, for each $a'' \in A^{**}$, $a''e'' = e''oa'' = a''$. By ([3], page 146), an element e'' of A^{**} is a mixed unit if and only if it is a *weak** cluster point of some BAI $(e_\alpha)_{\alpha\in I}$ in A.

A functional a' in A^* is said to be *wap* (weakly almost periodic) on A if the mapping $a \to a'a$ from A into A^* is weakly compact. In [13], Pym showed that this definition is equivalent to the following condition.

For every two nets $(a_i)_i$ and $(b_j)_j$ in $A_1 = \{a \in A : \parallel a \parallel \leq 1\}$, we have

$$\lim_i \lim_j \langle a', a_i b_j\rangle = \lim_j \lim_i \langle a', a_i b_j\rangle,$$

whenever both iterated limits exist. The collection of all *wap* functionals on A is denoted by $wap(A)$. Also we have $a' \in wap(A)$ if and only if $\langle a''b'', a'\rangle = \langle a''ob'', a'\rangle$ for every a'', $b'' \in A^{**}$.

Throughout the paper, for two normed spaces A and B, $\mathbf{B}(A, B)$ is the set of all bounded linear operators from A into B.

In the next section, we investigate the relationships between the Arens regularity of some bilinear mappings and Banach algebras, particulary, under some conditions, for bounded bilinear mappings $m : A \times B \to B$, $m' : A \times A \to B$, we show that m or m' are Arens regular (respectively, irregular) if and only if A or B are Arens regular (respectively, irregular). We also give some results in special Banach algebras such as $L^1(G)$, $M(G)$, $L^\infty(X)$ and $C(X)$ whenever G is a locally compact group and X is a semigroup. In Section 3, we extend some results from [11] into module actions along with some new results.

2. **Arens regularity of some bilinear forms**

In Theorem 2.1 and Theorem 2.6 of this section, we introduce some special bilinear mappings from $A\times B$ or $A\times A$ into B. We establish some relationships between the Arens regularity of these bilinear mappings and Arens regularity of A or B with some applications in group algebras.

Theorem 2.1. *Let A be a normed space and B be a Banach algebra. Let $T \in \mathbf{B}(A, B)$ and m be the bilinear mapping from $A \times B$ into B such that for every $a \in A$ and $b \in B$ we have $m(a, b) = T(a)b$. Then the following statements are true.*
a) If B is Arens regular, then m is Arens regular.
b) If T is surjective, then we have
i) B is Arens regular if and only if m is Arens regular.
ii) If m is left strongly Arens irregular, then B is left strongly Arens irregular.
iii) If T is injective, then B is left strongly Arens irregular if and only if m is left strongly Arens irregular.

Proof. a) By the definition of m^{***}, we have $m^{***}(a'', b'') = T^{**}(a'')b''$ and $m^{t***t}(a'', b'') = T^{**}(a'')ob''$ where $a'' \in A^{**}$ and $b'' \in B^{**}$. Since $Z_1(B^{**}) = B^{**}$, the mapping $b'' \to T^{**}(a'')b'' = m^{***}(a'', b'')$ is $weak^*$-$weak^*$ continuous for all $a'' \in A^{**}$. Also since $Z_2(B^{**}) = B^{**}$, the mapping $a'' \to T^{**}(a'')ob'' = m^{t***t}(a'', b'')$ is $weak^*$-$weak^*$ continuous for all $b'' \in B^{**}$. Hence m is Arens regular.
b) i) Let m be Arens regular. Then $Z_1(m) = A^{**}$. Let $b''_1, b''_2 \in B^{**}$ and $(b''_\alpha)_\alpha \in B^{**}$ such that $b''_\alpha \overset{w^*}{\to} b''_2$. Since T is surjective, there is an $a'' \in A^{**}$ such that $T^{**}(a'') = b''_1$. Then we have

$$b''_1 b''_2 = T^{**}(a'')b''_2 = m^{***}(a'', b''_2) = weak^*\text{-}\lim_\alpha m^{***}(a'', b''_\alpha)$$

$$= weak^*\text{-}\lim_\alpha T^{**}(a'')b''_\alpha = weak^*\text{-}\lim_\alpha b''_1 b''_\alpha.$$

Hence $Z_1(B^{**}) = B^{**}$ consequently B is Arens regular.
b) ii) Let m be left strongly Arens irregular. Then $Z_1(m) = A$. For $b''_1 \in Z_1(B^{**})$ the mapping $b''_2 \to b''_1 b''_2$ is $weak^*$-$weak^*$ continuous. Also since T is surjective, there exists $a'' \in A^{**}$ such that $T^{**}(a'') = b''_1$ and the mapping $b''_2 \to T^{**}(a'')b''_2 = m^{***}(a'', b''_2)$ is $weak^*$-$weak^*$ continuous. Hence $a'' \in Z_1(m) = A$. Consequently we have $b''_1 = T^{**}(a'') \in B$. It follows that $Z_1(B^{**}) = B$.
b) iii) Let B be left strongly Arens irregular. So $Z_1(B^{**}) = B$. For $a'' \in Z_1(m)$ the mapping $b'' \to m^{***}(a'', b'')$ is $weak^*$-$weak^*$ continuous

and consequently $T^{**}(a'') \in Z_1(B^{**}) = B$. Since T is bijective, $a'' \in A$. We conclude $Z_1(m) = A$. $\square$

In Theorem 2.1, if we replace the left strongly Arens irregularity of A, B and m with the right strongly Arens irregularity of them, then the results hold.
The following definition, introduced by Ülger [16], has an important role in showing some sufficient conditions for the Arens regularity of tensor product $A\hat{\otimes}B$ where A and B are Banach algebra.
Suppose that A and B are Banach algebras. We recall that a bilinear form $m : A \times B \to \mathbb{C}$ is biregular if for any two pairs of sequences $(a_i)_i$, $(\tilde{a}_j)_j$ in A_1 and $(b_i)_i$, $(\tilde{b}_j)_j$ in B_1, we have

$$\lim_i \lim_j m(a_i\tilde{a}_j, b_i\tilde{b}_j) = \lim_j \lim_i m(a_i\tilde{a}_j, b_i\tilde{b}_j)$$

provided that these limits exist.

Corollary 2.2. *Let B be a unital Banach algebra and suppose that A is subalgebra of B. If A is not Arens regular, then $A\hat{\otimes}B$ is not Arens regular.*

Proof. Let $m : A \times B \to \mathbb{C}$ be the bilinear form that was introduced in Theorem 2.1 where $T : A \to B$ is natural inclusion. Since A is not Arens regular, m is not biregular. Consequently by ([16], Theorem 3.4), $A\hat{\otimes}B$ is not Arens regular. $\square$

Example 2.3. *Let $C(X)$ be the Banach algebra of all continuous bounded functions on $X = [0, 1]$ with the supremum norm and the convolution as multiplication defined by*

$$f * g(x) = \int_0^x f(x - t)g(t)dt \text{ where } 0 \leq x \leq 1.$$

Let $T : C(X) \to L^\infty(X)$ be the natural inclusion and $m : C(X) \times L^\infty(X) \to L^\infty(X)$ be defined by $m(f, g) = f * g$ where $f \in C(X)$ and $g \in L^\infty(X)$. By [2], $L^\infty(X)$ is Arens regular and by Theorem 2.1, we conclude that m is Arens regular.
Similarly since c_0 is Arens regular, see [1, 5], by using Theorem 2.1, it is clear that the bounded bilinear mapping $(f, g) \to f * g$ from $\ell^1 \times c_0$ into c_0 is Arens regular.

For a Banach algebra A, we recall that a bounded linear operator $T : A \to A$ is said to be a left (respectively, right) multiplier if for all $a, b \in A$, $T(ab) = T(a)b$ (respectively, $T(ab) = aT(b)$). We denote

by $LM(A)$ (respectively, $RM(A)$) the set of all left (respectively, right) multipliers of A. The set $LM(A)$ (respectively, $RM(A)$) is a normed subalgebra of the algebra $L(A)$ of bounded linear operator on A.
Now, we define a new concept which is an extension of the left (respectively, right) multiplier from a Banach algebra to module actions. We show that some relationships hold between this concept and Arens regularity of some bilinear mappings.

Definition 2.4. *Let B be a left Banach (respectively, right) A-module and $T \in \mathbf{B}(A, B)$. Then T is called extended left (respectively, right) multiplier if*

$$T(a_1a_2) = \pi_r(T(a_1), a_2)\ (resp.\ T(a_1a_2) = \pi_\ell(a_1, T(a_2)))for\ all\ a_1, a_2 \in A$$

We denote by $LM(A, B)$ [respectively, $RM(A, B)$] the set of all left (respectively, right) multiplier extension from A into B.

Example 2.5. *Let $a' \in A^*$. Then the mapping $T_{a'} : a \to a'a$ (respectively $R_{a'}\ a \to aa'$) from A into A^* is a left (respectively, right) multiplier, that is, $T_{a'} \in LM(A, A^*)$ ($R_{a'} \in RM(A, A^*)$). It is also clear that $T_{a'}$ is weakly compact if and only if $a' \in wap(A)$.*

Theorem 2.6. *Let B be a left (respectively, right) Banach A-module and $T \in \mathbf{B}(A, B)$ be a continuous map. Assume that $m : A \times A \to B$ is the bilinear mapping such that $m(a_1, a_2) = T(a_1a_2)$. Then we have the following assertions*
i) If A is Arens regular, then m is Arens regular.
ii) If m is left (right) strongly Arens irregular, then A is left (respectively, right) strongly Arens irregular.
iii) If $T \in LM(A, B)$, then $T^{**}(Z_1(m)) \subseteq Z_{A^{**}}(B^{**})$.
iv) If $T \in LM(A, B)$, then $T^{**} \in LM(A^{**}, B^{**})$.
v) Suppose that B is Banach algebra and T is an embedding. Then, B is Arens regular if and only if m is Arens regular.

Proof. i) An easy calculation shows that

$$m^{***}(a''_1, a''_2) = T^{**}(a''_1a''_2)\ ,\ m^{t***t}(a''_1, a''_2) = T^{**}(a''_1oa''_2).$$

Since A is Arens regular, the mapping $a''_2 \to a''_1a''_2$ is $weak^*$-$weak^*$ continuous for all $a''_1 \in A^{**}$. Also the mapping $a''_1 \to a''_1oa''_2$ is $weak^*$-$weak^*$ continuous for all $a''_2 \in A^{**}$. Hence both mappings $a''_2 \to T^{**}(a''_1a''_2) = m^{***}(a''_1, a''_2)$ and $a''_1 \to T^{**}(a''_1oa''_2) = m^{t***t}(a''_1, a''_2)$ are $weak^*$-$weak^*$ continuous for all $a''_1 \in A^{**}$ and $a''_2 \in A^{**}$, respectively. We conclude that $Z_1(m) = Z_2(m) = A^{**}$.

ii) Let $a_1'' \in Z_1(A^{**})$. Then the mapping $a_2'' \to a_1''a_2''$ is $weak^*$-$weak^*$ continuous. Consequently, the mapping $a_2'' \to T^{**}(a_1''a_2'') = m^{***}(a_1'', a_2'')$ is $weak^*$-$weak^*$ continuous. Hence $a_1'' \in Z_1(m) = A$.
iii) Let $a_1'' \in Z_1(m)$. Then the mapping

$$a_2'' \to m^{***}(a_1'', a_2'') = T^{**}(a_1'')a_2''$$

is $weak^*$-$weak^*$ continuous from A^{**} into B^{**}. It follows that $T^{**}(a_1'') \in Z_{A^{**}}(B^{**})$.
iv) If we set $m(a_1, a_2) = T(a_1a_2)$ (*respectively*, $= T(a_1)a_2$) for all $a_1, a_2 \in A$, then $m^{***}(a_1'', a_2'') = T^{**}(a_1''a_2'')$ (*respectively*, $= T^{**}(a_1'')a_2''$) for all $a_1'', a_2'' \in A^{**}$. Thus, we conclude that $T^{**}(a_1''a_2'') = T^{**}(a_1'')a_2''$ for all $a_1'', a_2'' \in A^{**}$.
v) Let m be Arens regular and $b_1'', b_2'' \in B^{**}$ and let $(b_\alpha'')_\alpha \in B^{**}$ such that $b_\alpha'' \overset{w^*}{\to} b_2''$. We set $a_1'', a_2'' \in A^{**}$ and $(a_\alpha'')_\alpha \in A^{**}$ such that $T^{**}(a_1'') = b_1''$, $T^{**}(a_2'') = b_2''$ and $T^{**}(a_\alpha'') = b_\alpha''$. Then

$$\begin{aligned} b_1''b_2'' &= T^{**}(a_1'')T^{**}(a_2'') = T^{**}(a_1''a_2'') = m^{***}(a_1'', a_2'') \\ &= weak^*\text{-}\lim_\alpha m^{***}(a_1'', a_\alpha'') = weak^*\text{-}\lim_\alpha T^{**}(a_1''a_\alpha'') \\ &= weak^*\text{-}\lim_\alpha T^{**}(a_1'')T^{**}(a_\alpha'') = weak^*\text{-}\lim_\alpha b_1''b_\alpha'', \end{aligned}$$

where by the open mapping theorem, we have $a_\alpha'' \overset{w^*}{\to} a_2''$. Consequently $Z_1(B^{**}) = B^{**}$.
Conversely, let B be Arens regular and $a_1'', a_2'' \in A^{**}$ and $(a_\alpha)_\alpha \in A^{**}$ such that $a_\alpha'' \overset{w^*}{\to} a_2''$. Then

$$\begin{aligned} m^{***}(a_1'', a_2'') &= T^{**}(a_1''a_2'') = weak^* - \lim_\alpha T^{**}(a_1''a_\alpha'') \\ &= weak^*\text{-}\lim_\alpha m^{***}(a_1'', a_\alpha''). \end{aligned}$$

It follow that $Z_1(m) = A^{**}$. Thus m is Arens regular. □

Example 2.7. *Assume that $T : c_0 \to \ell^\infty$ is the natural inclusion map and $m : c_0 \times c_0 \to \ell^\infty$ be the bilinear mapping such that $m(f, g) = f * g$, for all $f, g \in c_0$. Since c_0 is Arens regular, m is Arens regular. Similarly the bilinear mapping $m : C(G) \times C(G) \to L^\infty(G)$ defined by formula $(f, g) \to f * g$, for all $f, g \in C(G)$ is Arens regular whenever G is compact.*

For normed spaces X, Y, Z, W let $m_1 : X \times Y \to Z$ and $m_2 : X \times W \to Z$ be bounded bilinear mappings. If $h : Y \to W$ is a continuous linear mapping such that $m_1(x, y) = m_2(x, h(y))$ for all $x \in X$ and $y \in Y$, then we say that m_1 factors through m_2, see [2]. We say that the continuous bilinear mapping $m : X \times Y \to Z$ factors if m is onto Z, see [7].

Theorem 2.8. *Let A and B be Banach algebras and B be a Banach A-bimodule. Let $T \in \mathbf{B}(A, B)$ be a continuous homomorphism. If T is weakly compact, then the bilinear mapping $m(a_1, a_2) = T(a_1a_2)$ from $A \times A$ into B is Arens regular.*

Proof. Let m' be the bilinear mapping that we introduced in Theorem 2.1. Then $m(a_1, a_2) = m'(a_1, T(a_2))$ for all $a_1, a_2 \in A$. Consequently m factors through m'. So by ([2], Theorem 2), we conclude that m is Arens regular. □

Example 2.9. *Suppose that $T : L^1(G) \to M(G)$ is the natural inclusion. Then the bilinear mapping $m : L^1(G) \times L^1(G) \to M(G)$ defined by $m(f, g) = f * g$ for all $f, g \in L^1(G)$ is Arens regular whenever G is finite, see* [18]. *Also the left strongly Arens irregularity of m implies that $L^1(G)$ is also left strongly Arens irregular, see* [10, 11].

3. Unital A-modules and module actions

In [11], Lau and Ülger showed that for a Banach algebra A, A^* factors on the left if and only if A^{**} is unital with respect to the first Arens product. In this section we extend this result to module actions.

We say that A^{**} has a *weak* bounded left approximate identity*, abbrevaited as W^*BLAI with respect to the first Arens product, if there is a bounded net as $(e_\alpha)_\alpha \subseteq A$ such that for all $a'' \in A^{**}$ and $a' \in A^*$, we have $\langle e_\alpha a'', a' \rangle \to \langle a'', a' \rangle$. The definition of W^*RBAI is similar and if A^{**} has both W^*LBAI and W^*RBAI, then we say that A^{**} has W^*BAI.

Assume that B is a Banach A-bimodule. Then we define the set AB as the linear span of the set $\{ab : \ a \in A, \ b \in B\}$. We say that B factors on the left (right) if $B = BA$ ($B = AB$) and B factors on both sides, if $B = BA = AB$.

Definition 3.1. *Let B be a left Banach $A-$module and e be a left unit element of A. We say that e is a left unit (respectively, weak left unit) for A-module B if $\pi_\ell(e, b) = b$ (respectively, $\langle b', \pi_\ell(e, b)\rangle = \langle b', b\rangle$ for all $b' \in B^*$) where $b \in B$. The definition of right unit (respectively, weak right unit) is similar.*

We say that a Banach A-bimodule B is a unital as an A-module if B has the same left and right unit. Thus in this case, we say that B is

unital as A-module.
Let B be a left Banach A-module and $(e_\alpha)_\alpha \subseteq A$ be a LAI (respectively, weakly left approximate identity, abbrevaited as (WLAI)) for A. Then $(e_\alpha)_\alpha$ is said to be a left approximate identity, abbrevaited as (LAI) (respectively, weakly left approximate identity, abbrevaited as ($WLAI$)) for B, if for each $b \in B$, we have $\pi_\ell(e_\alpha, b) \to b$ (respectively, $\pi_\ell(e_\alpha, b) \overset{w}{\to} b$). The definition of the right approximate identity (= RAI) (respectively, weakly right approximate identity, abbrevaited as ($WRAI$)) is similar. We say that $(e_\alpha)_\alpha$ is an approximate identity, abbrevaited as (AI) (respectively, weakly approximate identity, abbrevaited as (WAI)) for B, if B has the same left and right approximate identity (respectively, weakly left and right approximate identity).
Let $(e_\alpha)_\alpha \subseteq A$ be weak* left approximate identity for A^{**}. Then $(e_\alpha)_\alpha$ is weak* left approximate identity (W^*LAI) as A^{**}-module for B^{**}, if for all $b'' \in B^{**}$, we have $\pi_\ell^{***}(e_\alpha, b'') \overset{w^*}{\to} b''$. The definition of the weak* right approximate identity, abbrevaited as (W^*RAI) is similar. We say that $(e_\alpha)_\alpha$ is a weak* approximate identity, abbrevaited as (W^*AI) for B^{**}, if B^{**} has the same weak* left and right approximate identity.

Example 3.2. *Let G be a locally compact group. We know that $L^p(G)$, for $1 \le p \le \infty$, is a Banach $M(G)-bimodule$ under the convolution as multiplication. Thus $L^p(G)$ is a unital $M(G)-bimodule$.*

Theorem 3.3. *Assume that A is a Banach algebra and A has a BAI $(e_\alpha)_\alpha$. Then we have the following assertions.*
i) Let B be a right Banach A-module. Then B factors on the left if and only if B has a $WRAI$.
ii) Let B be a left Banach A-module. Then B factors on the right if and only if B has a $WLAI$.
iii) B factors on both side if and only if B has a WAI.

Proof. i) Suppose that $B = BA$. Let $b \in B$ and $b' \in B^*$. Then there are $x \in B$ and $a \in A$ such that $b = xa$. Therefore

$$\begin{aligned}\langle b', \pi_r(b, e_\alpha)\rangle &= \langle b', \pi_r(xa, e_\alpha)\rangle = \langle \pi_r^*(b', x), ae_\alpha\rangle \to \langle \pi_r^*(b', x), a\rangle \\ &= \langle b', \pi_r(x, a)\rangle = \langle b', b\rangle.\end{aligned}$$

It follows that $\pi_r(b, e_\alpha) \overset{w}{\to} b$, and so B has a $WRAI$.
For the converse, since BA is a weakly closed subspace of B, so by Cohen factorization theorem, see [5], the result is immediate.
ii) The proof is similar to that of (i).
iii) This is clear. □

In Theorem 3.3, if we set $B = A$, then we obtain Lemma 2.1 from [11].

Theorem 3.4. *Assume that B is a left Banach A-module and A^{**} has a right unit e''. Then, B factors on the left if and only if e'' is a right unit for A^{**}-module B^{**}.*

Proof. Since A^{**} has a right unit e'', there is a $BRAI$ $(e_\alpha)_\alpha$ for A such that $e_\alpha \overset{w^*}{\to} e''$, see [3]. Let $BA = B$ and $b \in B$. Thus, there is $x \in B$ and $a \in A$ such that $b = xa$. Then for all $b' \in B^*$, we have

$$\langle \pi_r^{**}(e'', b'), b\rangle = \langle e'', \pi_r^{*}(b', b)\rangle = \lim_\alpha \langle e_\alpha, \pi_r^{*}(b', b)\rangle$$

$$= \lim_\alpha \langle \pi_r^{*}(b', b), e_\alpha\rangle = \lim_\alpha \langle b', \pi_r(b, e_\alpha)\rangle$$

$$= \lim_\alpha \langle b', \pi_r(xa, e_\alpha)\rangle = \lim_\alpha \langle \pi_r^{*}(b', x), ae_\alpha\rangle$$

$$= \langle \pi_r^{*}(b', x), a\rangle = \langle b', \pi_r(x, a) = \langle b', b\rangle.$$

Thus $\pi_r^{**}(e'', b') = b'$. Now let $b'' \in B^{**}$. Then we have

$$\langle \pi_r^{***}(b'', e''), b'\rangle = \langle b'', \pi_r^{**}(e'', b')\rangle = \langle b'', b'\rangle.$$

We conclude that $\pi_r^{***}(b'', e'') = b''$. Hence it follows that B^{**} has a right unit.
Conversely, assume that e'' is a right unit for B^{**}. Let $b \in B$ and $b' \in B^*$. Then we have

$$\langle b', \pi_r(b, e_\alpha)\rangle = \langle \pi_r^{*}(b', b), e_\alpha)\rangle \to \langle \pi_r^{*}(b', b), e''\rangle = \langle e'', \pi_r^{*}(b', b)\rangle$$

$$= \langle \pi_r^{**}(e'', b'), b\rangle = \langle b, \pi_r^{**}(e'', b')\rangle = \langle \pi_r^{***}(b, e''), b'\rangle$$

$$= \langle b', b\rangle.$$

Consequently $\pi_r(b, e_\alpha) \overset{w}{\to} \pi_r(b, e'') = b$, and so $b \in \overline{BA}^w$. Since BA is a *weakly* closed subspace of B, so by the Cohen factorization theorem, $b \in BA$. □

Definition 3.5. *Let B be a Banach A-bimodule and $a'' \in A^{**}$. We define the sets $Z^t_{a''}(B^{**})$ (or $Z^t_{a''}(\pi_\ell^t)$) and $Z_{a''}(B^{**})$ (or $Z_{a''}(\pi_r^t)$) respectively, as follows*

$$Z^t_{a''}(B^{**}) = Z^t_{a''}(\pi_\ell^t) = \{b'' \in B^{**} :\ \pi_\ell^{t***t}(a'', b'') = \pi_\ell^{***}(a'', b'')\},$$

$$Z_{a''}(B^{**}) = Z_{a''}(\pi_r^t) = \{b'' \in B^{**} :\ \pi_r^{t***t}(b'', a'') = \pi_r^{***}(b'', a'')\}.$$

It is clear that

$$\bigcap_{a''\in A^{**}} Z^t_{a''}(B^{**}) = Z^t_{A^{**}}(B^{**}) = Z(\pi^t_\ell),$$

$$\bigcap_{a''\in A^{**}} Z_{a''}(B^{**}) = Z_{A^{**}}(B^{**}) = Z(\pi_r).$$

Theorem 3.6. *Assume that A is a Banach algebra and A^{**} has a mixed unit e''. Then we have the following assertions.*
i) Let B be a left Banach A-module. Then, B^* factors on the left if and only if B^{**} has a left unit e''.
ii) Let B be a right Banach A-module and $Z_{e''}(\pi^t_r) = B^{**}$. Then, B^* factors on the right if and only if B^{**} has a right unit e''.
iii) Let B be a Banach A-bimodule and $Z_{e''}(\pi^t_r) = B^{**}$. Then, B^* factors on both sides if and only if B^{**} has a unit e''.

Proof. i) Let $(e_\alpha)_\alpha \subseteq A$ be a BAI for A such that $e_\alpha \overset{w^*}{\to} e''$. Suppose that $B^*A = B^*$. Thus for all $b' \in B^*$, there are $a \in A$ and $x' \in B^*$ such that $x'a = b'$. Then for all $b'' \in B^{**}$, we have

$$\langle \pi^{***}_\ell(e'', b''), b'\rangle = \langle e'', \pi^{**}_\ell(b'', b')\rangle = \lim_\alpha \langle \pi^{**}_\ell(b'', b'), e_\alpha\rangle$$

$$= \lim_\alpha \langle b'', \pi^*_\ell(b', e_\alpha)\rangle = \lim_\alpha \langle b'', \pi^*_\ell(x'a, e_\alpha)\rangle$$

$$= \lim_\alpha \langle b'', \pi^*_\ell(x', ae_\alpha)\rangle = \lim_\alpha \langle \pi^{**}_\ell(b'', x'), ae_\alpha\rangle$$

$$= \langle \pi^{**}_\ell(b'', x'), a\rangle = \langle b'', \pi^*_\ell(x', a)\rangle = \langle b'', b'\rangle.$$

Thus $\pi^{***}_\ell(e'', b'') = b''$. Consequently B^{**} has left unit.
Conversely, Let e'' be a left unit for B^{**} and $b' \in B^*$. Then for all $b'' \in B^{**}$, we have

$$\langle b'', b'\rangle = \langle \pi^{***}_\ell(e'', b''), b'\rangle = \langle e'', \pi^{**}_\ell(b'', b')\rangle$$

$$= \lim_\alpha \langle \pi^{**}_\ell(b'', b'), e_\alpha\rangle = \lim_\alpha \langle b'', \pi^*_\ell(b', e_\alpha)\rangle.$$

Thus we conclude that $weak\text{-}\lim_\alpha \pi^*_\ell(b', e_\alpha) = b'$. So by the Cohen factorization Theorem, the proof is immediate.
ii) Suppose that $AB^* = B^*$. Thus for all $b' \in B^*$, there are $a \in A$ and $x' \in B^*$ such that $ax' = b'$. Assume $(e_\alpha)_\alpha \subseteq A$ is a BAI for A such that

$e_\alpha \xrightarrow{w^*} e''$. Let $b'' \in B^{**}$ and $(b_\beta)_\beta \subseteq B$ such that $b_\beta \xrightarrow{w^*} b''$. Then

$$\begin{aligned}\langle \pi_r^{***}(b'', e''), b'\rangle &= \lim_\beta \langle \pi_r^{***}(b_\beta, e''), b'\rangle = \lim_\beta \lim_\alpha \langle b', \pi_r(b_\beta, e_\alpha)\rangle \\ &= \lim_\beta \lim_\alpha \langle ax', \pi_r(b_\beta, e_\alpha)\rangle = \lim_\beta \lim_\alpha \langle x', \pi_r(b_\beta, e_\alpha)a\rangle \\ &= \lim_\beta \lim_\alpha \langle x', \pi_r(b_\beta, e_\alpha a)\rangle = \lim_\beta \lim_\alpha \langle \pi_r^*(x', b_\beta), e_\alpha a)\rangle \\ &= \lim_\beta \langle \pi_r^*(x', b_\beta), a)\rangle = \langle b'', b'\rangle.\end{aligned}$$

We conclude that

$$\pi_r^{***}(b'', e'') = b''$$

for all $b'' \in B^{**}$.
Conversely, suppose that $\pi_r^{***}(b'', e'') = b''$ where $b'' \in B^{**}$ and $(b_\beta)_\beta \subseteq B$ such that $b_\beta \xrightarrow{w^*} b''$. Let $(e_\alpha)_\alpha \subseteq A$ be a BAI for A such that $e_\alpha \xrightarrow{w^*} e''$. Since $Z_{e''}(\pi_r^t) = B^{**}$, for all $b' \in B^*$, we have

$$\begin{aligned}\langle b'', b'\rangle &= \langle \pi_r^{***}(b'', e''), b'\rangle = \langle b'', \pi_r^{**}(e'', b')\rangle = \lim_\beta \langle \pi_r^{**}(e'', b'), b_\beta\rangle \\ &= \lim_\beta \langle e'', \pi_r^*(b', b_\beta)\rangle = \lim_\beta \lim_\alpha \langle \pi_r^*(b', b_\beta), e_\alpha\rangle \\ &= \lim_\beta \lim_\alpha \langle \pi_r^*(b', b_\beta), e_\alpha\rangle = \lim_\beta \lim_\alpha \langle b', \pi_r(b_\beta, e_\alpha)\rangle \\ &= \lim_\alpha \lim_\beta \langle b', \pi_r(b_\beta, e_\alpha)\rangle = \lim_\alpha \lim_\beta \langle \pi_r^{***}(b_\beta, e_\alpha), b'\rangle \\ &= \lim_\alpha \lim_\beta \langle b_\beta, \pi_r^{**}(e_\alpha, b')\rangle = \lim_\alpha \langle b'', \pi_r^{**}(e_\alpha, b')\rangle.\end{aligned}$$

It follows that *weak*- $\lim_\alpha \pi_r^{**}(e_\alpha, b') = b'$. So by the Cohen factorization theorem, we are done.
iii) This is clear. □

In Theorem 3.6, if we set $B = A$, then we obtain Proposition 2.10 from [11].

Corollary 3.7. *Let B be a Banach A-bimodule and A^{**} has a mixed unit e''.*
a) Let $Z_{e''}(\pi_r^t) = B^{**}$. Then we have the following assertions
i) If B^* factors on the left but not on the right, then $\pi_\ell \neq \pi_r^t$.
ii) If B^* factors on the left and $\pi_\ell = \pi_r^t$, then B^* factors on the right .
iii) e'' is a right unit for B^{**} if and only if $(e_\alpha)_\alpha$ is a W^*RAI for B^{**} whenever $e_\alpha \xrightarrow{w^*} e''$.
b) Let $Z^t_{e''}(\pi_\ell^t) = B^{**}$. Then we have the following assertions

i) If B^* factors on the right but not on the left, then $\pi_r \neq \pi_\ell^t$.
ii) If B^* factors on the right and $\pi_r = \pi_\ell^t$, then B^* factors on the left .
iii) e'' is a left unit for B^{**} if and only if $(e_\alpha)_\alpha$ is a W^*LAI for B^{**} whenever $e_\alpha \stackrel{w^*}{\to} e''$.
c) Let $Z^\ell_{e''}(\pi_\ell) = Z^r_{e''}(\pi_r) = B^{**}$. Then we have the following assertions
i) If B^* factors on the one side, but not on the other side, then $\pi_r \neq \pi_\ell^t$ and $\pi_\ell \neq \pi_r^t$.
ii) e'' is a unit for B^{**} if and only if $(e_\alpha)_\alpha$ is a W^*AI for B^{**} whenever $e_\alpha \stackrel{w^*}{\to} e''$.

Proof. a) i) Let B^* factor on the right but not on the left . By Theorem 3.5, e'' is a right unit for B^{**}. Thus we have $\pi_r^{***}(b'', e'') = b''$ for all $b'' \in B^{**}$. If we set $\pi_\ell = \pi_r^t$, then $\pi_\ell^{***}(e'', b'') = \pi_r^{t***}(e'', b'') = \pi_r^{t***t}(b'', e'') = \pi_r^{***}(b'', e'') = b''$ for all $b'' \in B^{**}$. Consequently, e'' is left unit for B^{**}. Then by Theorem 3.5, B^* factors on the left which is impossible.
ii) The proof similar to that of (i).
iii) Since $e_\alpha \stackrel{w^*}{\to} e''$, $weak^*\text{-}\lim_\alpha \pi_r^{***}(b'', e_\alpha) = \pi_r^{***}(b'', e'')$ for all $b'' \in B^{**}$. Hence the proof is complete.
The proofs of (b) and (c) are the same and are immediately followed. □

Assume that $Z^t_{e''}(\pi_\ell^t) = Z_{e''}(\pi_r) = B^{**}$. Let $\pi_r = \pi_\ell^t$ and $\pi_\ell = \pi_r^t$. Using Corollary 3.7, we know that if B^* factors on the one side, then B^* factors on the other side. On the other hand, if we set $\pi_\ell = \pi_r^t$ and $Z_{e''}(\pi_r) = B^{**}$ where e'' is a mixed unit for A^{**}, then, by using this corollary, B^* factors on the right if and only if B factors on the left.

Questions.
Suppose that B is a Banach A-bimodule. Under which conditions the following results hold?
i) B factors on the left if and only if B^{**} has a left unit.
ii) B factors on the one side if and only if B^* factors on the same side.

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(Kazem Haghnejad Azar) Department of Mathematics, University of Mohghegh Ardabili, Ardabil, Iran

E-mail address: haghnejad@uma.ac.ir